An introduction to Lattice Gauge Theory

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Outline

- Motivation
- Success and failure of pQCD
- Quest for non-perturbative formulation - The lattice
- Discretized fermions and bosons
- More about fermions
- χ- symmetry
- The true continuum limit
- Simulations
- Computing masses
- Lattice vs Experiment
- Phase diagram of QCD
- Challenges

The success of QCD

- QCD immediately after its birth was very successful
- Renormalizability of non-Abelian gauge theories
- 1999 Nobel Prize awarded to 't Hooft and Veltman
- Asymptotic freedom
- 2004 Nobel Prize awarded to Gross, Politzer and Wilczek

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The failure of PT

- Despite the huge success some phenomena can not be seen in PT
- Confinement and χ-symmetry breaking
- In the physical regime of low energies there is no small expansion parameter

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Salvation comes from the lattice

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(Courtesy of worldofsuperman.blogspot.com)

- initiated by Wegner, Wilson, Creutz, Susskind, Kogut
- gives a rigorous definition of QFT
- allows directly for numerical simulations
- non-perturbative regulator
- minimum wavelength is $\propto a$ and thus max momentum $\propto \pi/a$
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- The phase diagram of QCD
- Hadronic spectroscopy from $\mathcal{L}_{QCD}$
- Topological structure of the QCD vacuum- Confinement- χ-SB
- Physics BSM
- Strongly correlated CM systems-Graphene

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Naive discretization of fermions

- Introduce a 4-dim lattice
- $\Lambda =$ $(n_1, n_2, n_3, n_4)|n_1, n_2, n_3 = 0, \ldots, N-1; n_4 = 0, 1, \ldots, N_T - 1$
- The continuum free action reads
- $S_F = \int d^4x \bar{\psi}(x)(\gamma_\mu \partial_\mu + m)\psi(x)$
- $\int \to \sum$ and $\partial_\mu \to$ finite differences
- $S_F = a^4 \sum_{n \in \Lambda} \bar{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + m\psi(n) \right)$
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- $\psi(n) \rightarrow \psi'(n) = \Omega(n)\psi(n)$ and $\bar{\psi}(n) \rightarrow \bar{\psi}'(n) = \bar{\psi}(n)\Omega^\dagger(n)$
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- $\bar{\psi}(n)\psi(n+\hat{\mu}) \rightarrow \bar{\psi}'(n)\psi'(n+\hat{\mu}) = \bar{\psi}(n)\Omega^\dagger(n)\Omega(n+\hat{\mu})\psi(n+\hat{\mu})$
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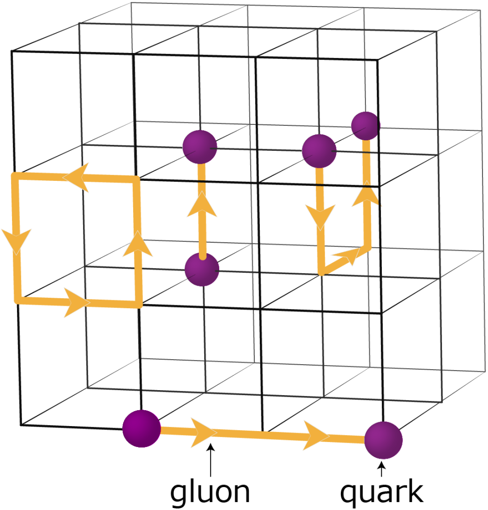
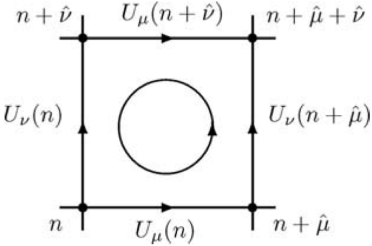
The space time lattice



(Courtesy of JICFus)

The Wilson gauge action

- Construct a LG action for gluons made out of Us which in the continuum yields the YM action
- use the shortest non trivial closed loop: the plaquette



(Courtesy of Gattringer-Lang)

- $S_G[U] = \frac{2}{g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{tr}\,(1 - U_{\mu\nu}(n))$
- Using the fact that $U_\mu(n) = e^{iaA_\mu(n)}$ and Taylor expanding one gets $U_{\mu\nu}(n) = e^{ia^2F_{\mu\nu}(n)+\mathcal{O}(a^3)}$

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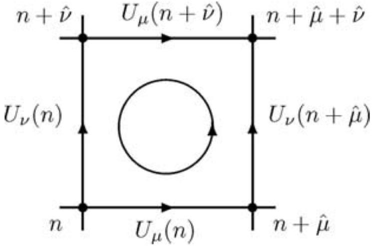

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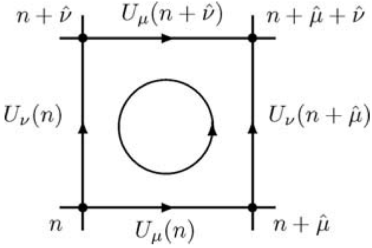

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Formal expression of the QCD PI

- we can compute Euclidean correlators as
- $\langle O_2(t_1) O_1(0) \rangle = \frac{1}{Z} \int D[\psi, \bar{\psi}, U] e^{-S_F - S_G} O_2[\psi, \bar{\psi}, U] O_1[\psi, \bar{\psi}, U]$

Fermions on the lattice- Origin of doublers

- The momentum space propagator (free theory)

$$D(p)|^{-1}_{m=0} = \frac{-ia^{-1}\sum_\mu \gamma_\mu \sin(p_\mu a)}{a^{-2}\sum_\mu \sin(p_\mu a)^2} \xrightarrow{a \to 0} \frac{-i\sum_\mu \gamma_\mu p_\mu}{p^2}$$

- In the continuum one pole at $p = (0,0,0,0)$
- On the lattice additional poles whenever all components are either $p_\mu = 0$ or $p_\mu = \pi/a$
- Our lattice Dirac operator has 15 unphysical poles (doublers) at $p = (\pi/a, 0, 0, 0), (0, \pi/a, 0, 0), ..., (\pi/a, \pi/a, \pi/a, \pi/a)$

A No-go theorem

Nielsen and Ninomiya (1980)

It is not possible to construct a lattice fermion action that is

- Local
- Undoubled
- correct continuum limit
- chirally symmetric $\{D, \gamma_5\} = 0$

Wilson Fermions

Break chiral symmetry explicitly

Wilson (1977)

- add the lattice discretization of the Laplacian $-\frac{a}{2}\partial_\mu\partial_\mu$
- $D(p) = m\mathbf{1} + \frac{i}{a}\sum_{\mu=1}^{4}\gamma_\mu \sin p_\mu a + \mathbf{1}\frac{1}{a}\sum_{\mu=1}^{4}(1-\cos p_\mu a)$
- for components with $p_\mu = 0$ it vanishes
- for each component with $p_\mu = \pi/a$ provides an extra contribution $2/a$
- It acts like an additional "effective" mass term so the total mass of the doublers is $m + 2l/a$
- in the naive continuum limit $a \to 0$ the doublers become very heavy and decouple from the theory

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χ-symmetry-3 sources of breaking

- The massless QCD action in the continuum is invariant under $\mathrm{SU}(N_\mathrm{f})_L \times \mathrm{SU}(N_\mathrm{f})_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$
- Since the nucleon and its parity partner are by far non degenerate we understand this breaking coming from SSB of χ-symmetry
- the order parameter is the $\langle \bar{\psi}(x)\psi(x) \rangle$
- explains lightness of pions
- In the full quantum theory $\mathrm{U}(1)_A$ is broken by the anomaly
- explains why the η' is not light at all
- explicit symmetry breaking by the mass term explains why pions are not exactly massless

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Symmetry Breaking

(Courtesy of http://cp3-origins.dk/a/9471)

χ-symmetry on the lattice

- Ginsparg and Wilson proposed $\{D, \gamma_5\} = aD\gamma_5 D$ as a lattice generalization
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Improvement-Toy example

- start from a discretized version of the quantity of interest
- $\frac{f(x+a)-f(x-a)}{2a} = f'(x) + a^2 C^{(2)}(x) + a^4 C^{(4)}(x) + \mathcal{O}(a^6)$
- identify the correction term in continuum language
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- One needs to improve the action as well as the operators
- follow the toy approach in the case of the action
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- after using the EOM and redefinition of the bare m, g
- $S_{imp} = S_W + c_{SW} a^5 \sum_{n \in \Lambda} \sum_{\mu < \nu} \bar{\psi}(n) \frac{1}{2} \sigma_{\mu\nu} \hat{F}_{\mu\nu}(n) \psi(n)$
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The true continuum limit

- couplings and masses entering the action are bare parameters
- when removing the lattice cutoff physical observables must agree with their experimental values
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- the vev of an observable in pure gauge theory is given by $\langle O \rangle = \frac{1}{\mathcal{Z}} \int DU e^{-S_g} O[U]$ with $\mathcal{Z} = \int DU e^{-S_g}$
- a MC simulation approximates the integral by an average of N sample gauge field conf U_m distributed with probability $e^{-S[U_m]}$
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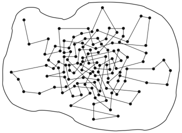
- the change of a field configuration is called an update and should satisfy
- ergodicity
- detailed balance



Sketch of a Markov chain in the space of all configurations

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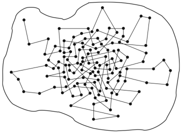

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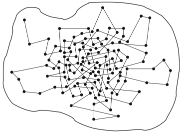

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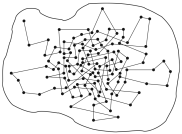

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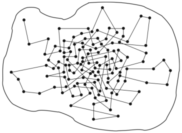
- 

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the Metropolis algorithm

- applied to the Ising model where
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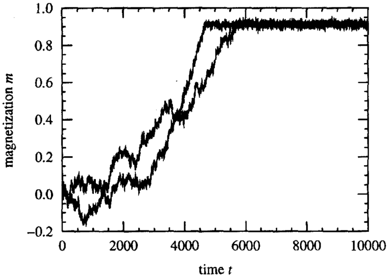
- begin measurement when equilibrium has been reached
- watch out for autocorrelations



- the magnetization of a 100x100 Ising model as a function of MC steps. The two simulations started off in two different ($T = \infty$) random spin states. At $t = 6000$ the two simulations have converged to the same magnetization within statistical errors due to fluctuations. (Courtesy of Newman-Barkema)

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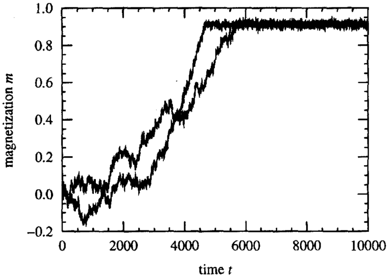

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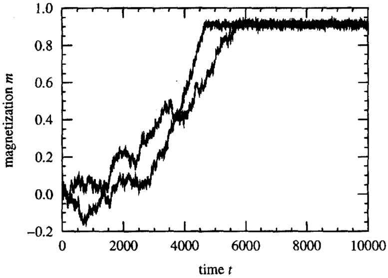

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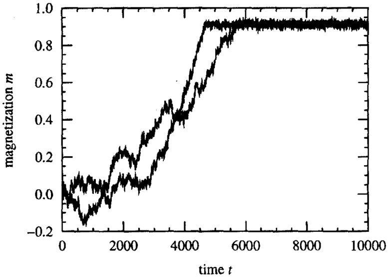

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Extracting the masses

- look at a 2 pt correlator $\langle O_p(t) O_p^\dagger(0)\rangle_T = \frac{1}{\mathcal{Z}_T}\mathrm{tr}\,[e^{-(T-t)\mathcal{H}} O_p e^{-t\mathcal{H}} O_p^\dagger]$ where $\mathcal{Z}_T = \mathrm{tr}\, e^{-T\mathcal{H}}$
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$$\begin{aligned}\langle O_p(t) O_p^\dagger(0)\rangle_T &= \frac{1}{\mathcal{Z}_T}\sum_{n,m}\langle m|e^{-(T-t)\mathcal{H}} O_p|n\rangle\langle n|e^{-t\mathcal{H}} O_p^\dagger|m\rangle \\ &= \frac{1}{\mathcal{Z}_T}\sum_{n,m} e^{-(T-t)E_m}\langle m|O_p|n\rangle e^{-tE_n}\langle n|O_p^\dagger|m\rangle\end{aligned}$$

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$$\begin{aligned}\lim_{T\to\infty}\langle O_p(t) O_p^\dagger(0)\rangle_T &= \sum_n \langle 0|O_p|n\rangle e^{-tE_n}\langle n|O_p^\dagger|0\rangle e^{-tE_n} \\ &= |\langle p|O_p^\dagger|0\rangle|^2 e^{-tE_p} + |\langle p'|O_p^\dagger|0\rangle|^2 e^{-tE_p'} + \ldots\end{aligned}$$

- for $t >>$ we can pick up the mass of the ground state from the exponential decay of the Euclidean correlator

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- for $t >>$ we can pick up the mass of the ground state from the exponential decay of the Euclidean correlator

Extracting the masses

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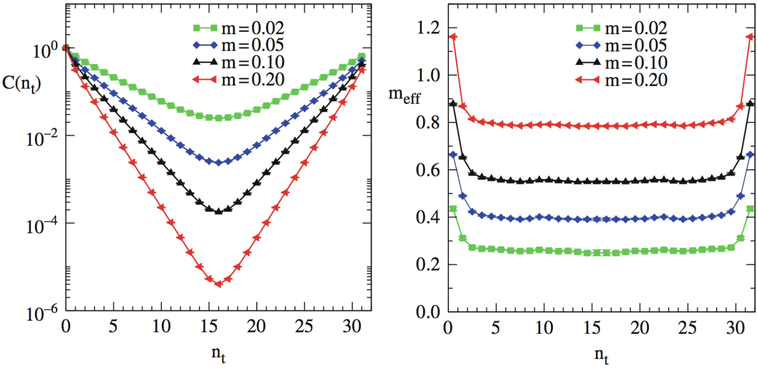
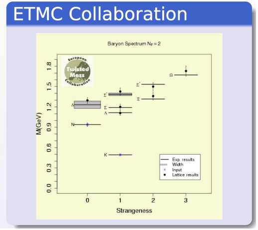
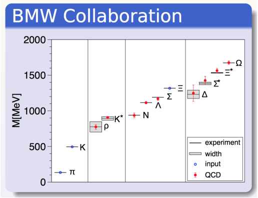


Results of a simulation on a $16^3 \times 32$ lattice with $a = 0.15fm$ LHS: Log-plot of the pion 2-pt function and RHS: effective mass plot in lattice units. Different sets of data correspond to different values of the quark mass (Courtesy of Gattringer-Lang)

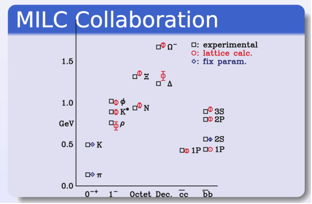
Lattice vs Experiment



Twisted mass Wilson fermions



SW improved Wilson fermions

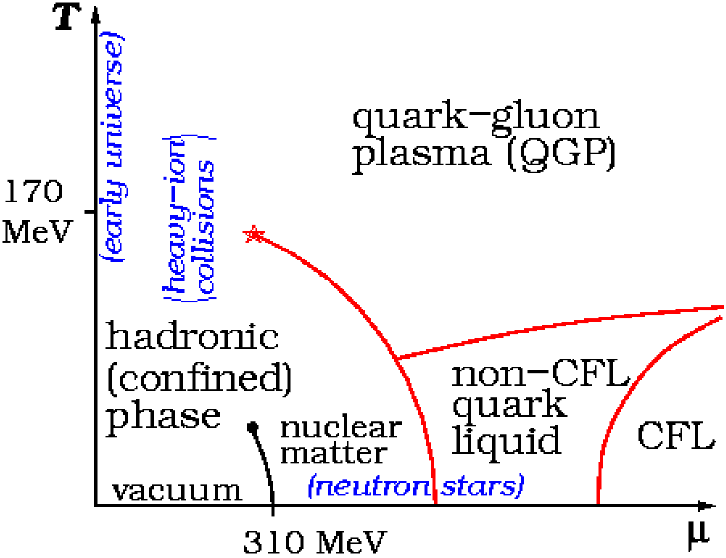


Staggered fermions

The phase diagram of QCD

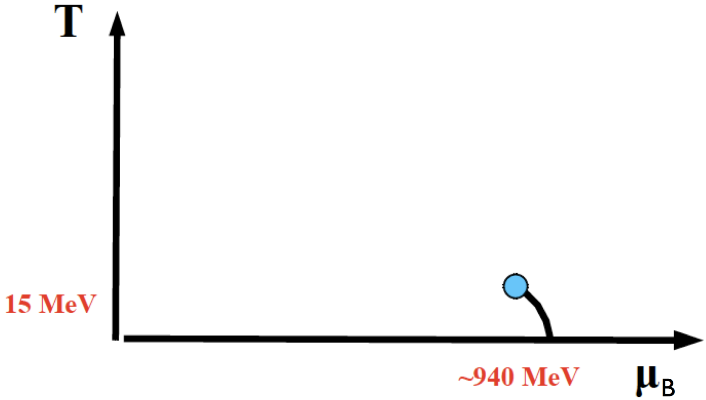
- since the Big Bang the universe has cooled down substantially
- try to understand the QGP which existed shortly after the BB as well as the hadron condensation
- neutron stars-have high density-hadronic matter behaves differently
- explain the results of heavy ion collision experiment (e.g. RHIC)
- understand the deconfinement transition- properties of QCD in the deconfined phase
- but... we always study a system in equilibrium (not the evolution-decay dynamics)

Phase diagram from a theorist's viewpoint



(Courtesy of wikipedia)

Phase diagram from experiment



(Courtesy of Owe Philipsen)

Finite temperature

- the partition function
  $\mathcal{Z}(T) = \mathrm{tr}\, e^{-\beta\mathcal{H}} = \int D\Phi e^{-S_E[\Phi]}$
- where the Euclidean action
  $S_E[\Phi] = \int_0^\beta dt \int_{\mathbb{R}^3} d^3x \mathcal{L}_E$
- now at finite temperature space is infinite but physical extent of time is limited to β.
- Linde problem (arising in the $l+1$ loop contribution to the QCD pressure) - all loop orders contribute to the QCD pressure at $\mathcal{O}(g^6)$
- all loops contribute equally to the coefficient of the chromomagnetic gluon mass
- these statements are true independently on how weak the coupling g is.
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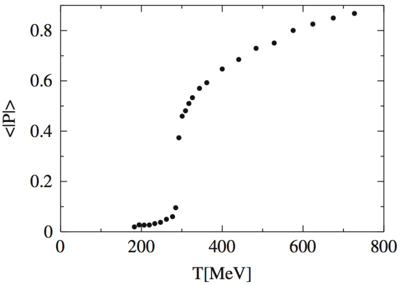
The Polyakov loop

- $P(m) = \mathrm{tr} \prod_{j=0}^{N_T-1} U_4(m,j)$
- $\langle P(m) P^\dagger(n) \rangle = e^{-aN_T F_{\bar{q}q}(a|m-n|)} = e^{-F_{\bar{q}q}/T}$
- $\langle P \rangle = 0 \rightarrow$ confinement
- $\langle P \rangle \neq 0 \rightarrow$ deconfinement



(Courtesy of Gattringer and Lang)

QCD Thermodynamics

- energy density, pressure-fundamental thermodynamic quantities governing the expansion of QGP in the early universe and in heavy ion collisions
- $\epsilon = \frac{T^2}{V} \frac{\partial \log Z}{\partial \beta}$ and $p = T \frac{\partial \log Z}{\partial V}$
- as $T \to \infty$ we have a gas of non-interacting, relativistic gluons and quarks following the SB law $\frac{p}{T^4} = \frac{\pi^2}{90} \left(16 + \frac{7}{8} 12 N_f\right)$
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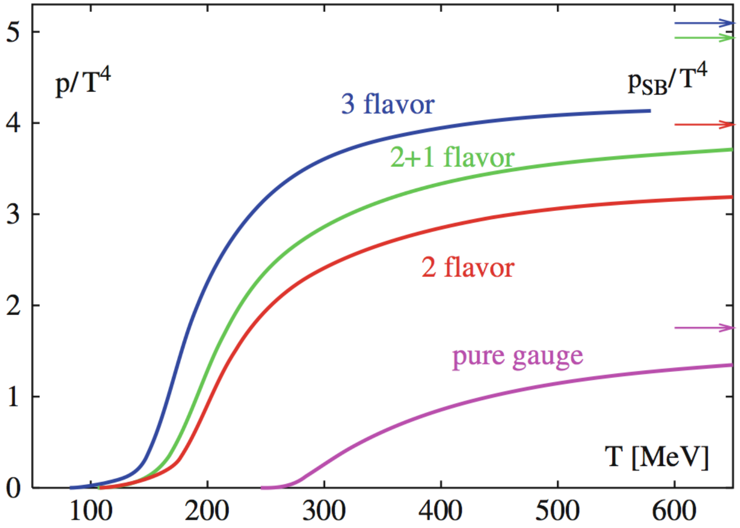


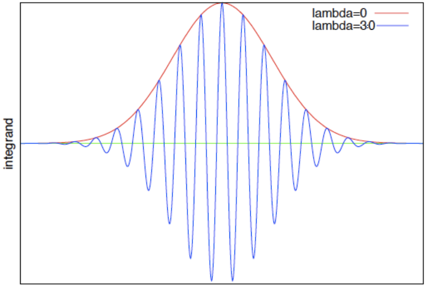
(Courtesy of Karsch et al)

QCD at finite baryon density

- $\mathcal{Z}(T) = \mathrm{tr}\, e^{-(\beta\mathcal{H} - \mu N)}$
- $N = \int d^3x \bar{\psi}(x)\gamma_0\psi(x) = \int d^3x \psi^\dagger(x)\psi(x)$
- $S_F(D(\mu) = S_F(D(0)) + a\mu \sum_x \bar{\psi}(x)\gamma_0\psi(x)$
- this naive implementation introduces an artificial divergence
- instead one needs a la Hasenfratz and Karsch to enhance the forward time propagation by $e^{a\mu}$ and the backward propagation by $e^{-a\mu}$
- we will see that the intro of μ opens a pandora's box...

The sign problem

- The $\det(D + m + \mu\gamma_0)$ becomes complex
- can not be interpreted as a probabilistic weight
- consider $\mathcal{Z}(\lambda) = \int dx e^{-x^2 + i\lambda x}$



(Courtesy of Owe Philipsen)

The failure of the quenched approximation

- quenched limit is the limit where the fermion determinant is ignored in the generation of gauge confs
- for $\mu = 0$ this has been a "reasonable" approximation
- for $\mu \neq 0$ this limit is given by the quenched limit of phase quenched QCD
- phase quenched QCD is QCD at non zero isospin chemical potential
- a phase transition (at low T) to a pion condensation phase occurs @ $\mu = m_\pi/2$
- in full QCD a phase transition (at low T) to a phase with non-zero baryon density occurs @ $\mu = m_N/3$
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NEXT
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The sign problem

- One can look at QCD like theories which are sign-problem free
  - imaginary chemical potential
  - $2-$flavors at finite isospin chemical potential i.e. $\mu_u = -\mu_d = \mu_I$
  - $\mathrm{SU}\,(2)$ QCD
- Note that all these theories have a different phase diagram from QCD
- Existing methods reliable for small μ
  - Reweighting
  - Finite density Taylor expansion
- avoid importance sampling and use stochastic quantization or Langevin algorithms
- for complex actions there is no proof that the Langevin method converges to the correct expectation values
- no successful applications to QCD up to the moment

Challenges

- explain from first principles more experimental data with if possible tiny error bars
- Chiral gauge theories
- Lattice SUSY
- Sign Problem
- Color superconductivity at $\mu >>$
- new algorithms that treat fermions and bosons on the same ground

Thank you.


DON'T WORRY, IT'S GONNA BE COOL... HOMER'S ON THE JOB, MAN!
THIS SITE UNDER CONSTRUCTION