Three-body Scattering Processes in a framework of Faddeev Approach

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Outline

- Faddeev equations in coordinate space
- Scattering calculations for 2 ’2, 3 processes
- S-matrix and resonance calculations

Three-body, theory formalism

Faddeev Equations: Algebraic scheme

Equation $H\Psi = E\Psi$ is equivalent to the Lippman-Schwinger eq.

$$H = H_0 + V$$

$$V = V_1 + V_2 + V_3$$

$$\Psi = -G_0(E)V\Psi \equiv -G_0(E)\sum_{\alpha=1}^{3} V_\alpha \Psi$$

We introduce the vectors

$$\Phi_\alpha = -G_0(E)V_\alpha\Psi$$ *Definition of the Faddeev components*

and note,

$$\sum_{\alpha=1}^{3}\Phi_\alpha \equiv \Psi$$

Meanwhile, applying $(H_0 - E)$ to both sides of vectors definition one obtains

$$(H_0 - E)\Phi_\alpha = -V_\alpha\Psi \equiv -V_\alpha \sum_{\beta=1}^{3}\Phi_\beta$$

Or, after transfer of Φ_α from r.h.s. to l.h.s.:

$$(H_0 + V_\alpha - E)\Phi_\alpha = -V_\alpha \sum_{\beta\neq\alpha}\Phi_\beta$$ *Faddeev equations*

In the form $\Phi_\alpha = -(H_0 + V_\alpha - E)^{-1} V_\alpha \sum_{\beta\neq\alpha}\Phi_\beta$ they were introduced by L.D.Faddeev in 1960

Faddeev Equations

$$(H_0 + V_\alpha - E)\Phi_\alpha = -V_\alpha \sum_{\beta \neq \alpha} \Phi_\beta$$

can be written in the matrix form

$$\begin{pmatrix} H_0 + V_1 & V_1 & V_1 \\ V_2 & H_0 + V_2 & V_2 \\ V_3 & V_3 & H_0 + V_3 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = E \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}$$

Even the number of equation became tripled, in many respects these equations are more convenient then the initial equation

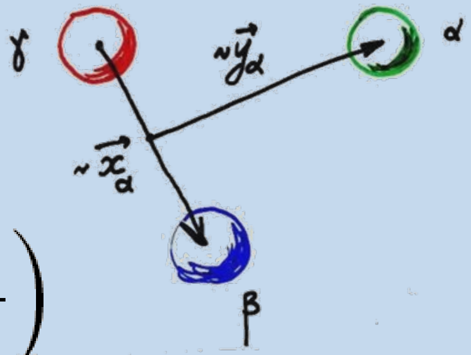
$$H\Psi = E\Psi$$

In case of a three-body problem this is especially true in the scattering case since the Faddeev operator decouples two-body channels

In describing the three-body system we use the standard Jacobi coordinates [4] $\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha$, $\alpha = 1, 2, 3$, expressed in terms of the position vectors of the particles $\boldsymbol{r}_i \in \mathbb{R}^3$ and their masses m_i,

$$\boldsymbol{x}_\alpha = \left[\frac{2m_\beta m_\gamma}{m_\beta + m_\gamma}\right]^{1/2} (\boldsymbol{r}_\beta - \boldsymbol{r}_\gamma)$$

$$\boldsymbol{y}_\alpha = \left[\frac{2m_\alpha(m_\beta + m_\gamma)}{m_\alpha + m_\beta + m_\gamma}\right]^{1/2} \left(\boldsymbol{r}_\alpha - \frac{m_\beta \boldsymbol{r}_\beta + m_\gamma \boldsymbol{r}_\gamma}{m_\beta + m_\gamma}\right)$$



where (α, β, γ) stands for a cyclic permutation of the indices $(1, 2, 3)$. The coordinates $\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha$ fix the six-dimensional vector $X \equiv (\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha) \in \mathbb{R}^6$. The vectors $\boldsymbol{x}_\beta, \boldsymbol{y}_\beta$ corresponding to the same point X as the pair $\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha$ are obtained using the transformations

$$\boldsymbol{x}_\beta = \boldsymbol{c}_{\beta\alpha}\boldsymbol{x}_\alpha + \boldsymbol{s}_{\beta\alpha}\boldsymbol{y}_\alpha \qquad \boldsymbol{y}_\beta = -\boldsymbol{s}_{\beta\alpha}\boldsymbol{x}_\alpha + \boldsymbol{c}_{\beta\alpha}\boldsymbol{y}_\alpha$$

where the coefficients $\boldsymbol{c}_{\beta\alpha}$ and $\boldsymbol{s}_{\beta\alpha}$ fulfil the conditions $-1 < \boldsymbol{c}_{\beta\alpha} < +1$ and $\boldsymbol{s}^2_{\beta\alpha} = 1 - \boldsymbol{c}^2_{\beta\alpha}$ with $\boldsymbol{c}_{\alpha\beta} = \boldsymbol{c}_{\beta\alpha}$, $\boldsymbol{s}_{\alpha\beta} = -\boldsymbol{s}_{\beta\alpha}$, $\beta \neq \alpha$ and depend only on the particle masses [4]. For equal masses $\boldsymbol{c}_{\beta\alpha} = -\frac{1}{2}$.

[4] - L.D.Faddeev,S.P.Merkuriev, 1993, *Quantum scattering theory for several particles*

Let $\Psi(X)=\sum_{\alpha=1}^{3}\Phi_\alpha(X)^6, X\equiv(\boldsymbol{x}_\alpha,\boldsymbol{y}_\beta)\in \mathrm{P}^6$ be the three-body wave function corresponding to a (2'2,3) process, where in the initial state the pair subsystem β is bound in a state $\psi_\beta(x_\beta)$ with energy ε_β. and the complementary particles asymptotically free, the relative momentum being p_β. The Faddeev components satisfy the differential equations

$$(H_0+V_\alpha-E)\Phi_\alpha(X)=-V_\alpha\sum_{\beta\neq\alpha}\Phi_\beta(X)$$

and have asymptotic behavior [4]

$$\Phi_\alpha(X)\underset{X\to\infty}{=}\delta_{\alpha\beta}\chi_\beta(X)+\psi_\alpha(\boldsymbol{x}_\alpha)\frac{\exp(\pm i\sqrt{E-\varepsilon_\alpha}\,|\boldsymbol{y}_\alpha|)}{|\boldsymbol{y}_\alpha|}a_\alpha(\hat{\boldsymbol{y}}_\alpha)$$

$$V(r)\to\frac{1}{r^{3+\varepsilon}}\qquad\qquad +\frac{\exp(\pm i\sqrt{E}\,|X|)}{|X|^{5/2}}A_\alpha(\hat{X})$$

Where $E=\varepsilon_\alpha+p^2$ is energy of the system. For $E>\varepsilon_\alpha,\ a_\alpha(\hat{\boldsymbol{y}}_\alpha)$ is represents the amplitude for the elastic (α=β) or rearrangement (α≠ β) scattering, the functions $A_\alpha(\hat{X})$ provides us with the total breakup amlitude A

[4] - L.D.Faddeev,S.P.Merkuriev, 1993, *Quantum scattering theory for several particles*

When the total angular momentum $\mathcal{L}$ of the system is fixed, the three-body dynamics is constrained onto three-dimensional internal space [5], which can be parametrized by coordinates

$$x_\alpha = |\boldsymbol{x}_\alpha|,\ y_\alpha = |\boldsymbol{y}_\alpha|, z_\alpha = \cos\theta_\alpha = (\hat{\boldsymbol{x}}_\alpha, \hat{\boldsymbol{y}}_\alpha)$$

For zero angular momentum the Faddeev equations in internal space are given by the set of three coupled three-dimensional equations

$$(H_0 + V_\alpha - E)F_\alpha(x_\alpha, y_\alpha, z_\alpha) = -V_\alpha \sum_{\beta\neq\alpha} F_\beta(x_\beta, y_\beta, z_\beta)$$

$$x_\beta = \sqrt{c_{\beta\alpha}^2 x_\alpha{}^2 + s_{\beta\alpha}^2 y_\alpha{}^2 + 2c_{\beta\alpha}s_{\beta\alpha}x_\alpha y_\alpha z_\alpha}$$

$$y_\beta = \sqrt{s_{\beta\alpha}^2 x_\alpha{}^2 + c_{\beta\alpha}^2 y_\alpha{}^2 - 2c_{\beta\alpha}s_{\beta\alpha}x_\alpha y_\alpha z_\alpha}$$

$$x_\beta y_\beta z_\beta = \sqrt{(c_{\beta\alpha}^2 - s_{\beta\alpha}^2)x_\alpha y_\alpha z_\alpha - c_{\beta\alpha}s_{\beta\alpha}(x_\alpha{}^2 - y_\alpha{}^2)}$$

$$H_0 = -\frac{\partial^2}{\partial x_\alpha{}^2} - \frac{\partial^2}{\partial y_\alpha{}^2} - \left(\frac{1}{x_\alpha{}^2} + \frac{1}{y_\alpha{}^2}\right)\frac{\partial}{\partial z_\alpha}(1 - z_\alpha)^{1/2}\frac{\partial}{\partial z_\alpha}$$

or in hyperspherical coordinates $\rho = \sqrt{x_\alpha^2 + y_\alpha^2}, \tan\vartheta_\alpha = y_\alpha / x_\alpha, \eta_\alpha = (\hat{\boldsymbol{x}}_\alpha, \hat{\boldsymbol{y}}_\alpha)$

$$(H_0 + V_\alpha - E)\Phi_\alpha(\rho, \vartheta_\alpha, \eta_\alpha) = -V_\alpha \sum_{\beta\neq\alpha} \Phi_\beta(\rho, \vartheta_\beta, \eta_\beta)$$

$$\Phi(x, y, \eta) \underset{\rho\to\infty}{=} \chi_\beta(x, y) + \psi(x)\exp(ipy)a_0(\eta; E) + \frac{\exp(\pm i\sqrt{E}\rho)}{\rho^{1/2}} A(\vartheta, \eta; E)$$

[5] - V.V.Kostrykin,A.A.Kvitsinsky,S.P.Merkuriev, Few-Body Syst. 6 (1989) 97

For computational purposes, one can reduce the dimension by expanding the Faddeev components into an auxiliary basis, at the expense of dealing with an infinite number of partial equations. Expanding the function F_α in a series of bispherical harmonics

$$F_\alpha(x,y,\theta) = \sum_{l,\lambda} \frac{\Phi^{(\alpha)}_{l\lambda\alpha}(x,y)}{xy} \,|\, l\lambda 0 >$$

One can obtain the partial equation

$$(H_0 + V_\alpha - E)\Phi^{(\alpha)}_{l\lambda}(x_\alpha, y_\alpha) = -V_\alpha \sum_{\beta\neq\alpha} \sum_{l'\lambda'} \int_{-1}^{1} d\eta\, h^{(\alpha\beta)}_{l\lambda l'\lambda'}(x_\alpha, y_\alpha, \eta)\Phi^{(\beta)}_{l'\lambda'}(x_\beta, y_\beta)$$

$$H_0 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{l(l+1)}{x^2} + \frac{\lambda(\lambda+1)}{y^2}$$

$$x_\beta = \sqrt{c^2_{\beta\alpha} x_\alpha{}^2 + s^2_{\beta\alpha} y_\alpha{}^2 + 2c_{\beta\alpha} s_{\beta\alpha} x_\alpha y_\alpha \eta}$$

$$y_\beta = \sqrt{s^2_{\beta\alpha} x_\alpha{}^2 + c^2_{\beta\alpha} y_\alpha{}^2 - 2c_{\beta\alpha} s_{\beta\alpha} x_\alpha y_\alpha \eta}$$

The asymptotic boundary conditions for the partial-wave Faddeev components of the 2’2,3 scattering wave function for and/or reads ρ’∞ and/or *y’∞ reads*

$$\Phi^{(\alpha)}_{l'\lambda'}(x,y) = \delta_{ll'\lambda\lambda}\psi_l(x) j_\lambda(py) + \psi_{l'}(x) h_{\lambda'}(py) a_{l'\lambda'}(p) + \frac{\exp(i\sqrt{E}\rho)}{\sqrt{\rho}} A^{(\alpha)}_{l'\lambda'}(p,\vartheta)$$

Where p is the relative moment conjugate to Jacoby variable y, E is the scattering energy, $E = \varepsilon_\alpha + p^2$, j_λ and h_λ stand for the spherical Bessel and Hankel functions

NUMERICAL METHOD

The finite-difference approximation in polar coordinates ρ and ϑ has been used to solve this problem. For this, the grid knots were chosen to be the points of intersection of the arcs $\rho = \rho_i$, $i = 1, 2, \ldots, N_\rho$, and the rays $\vartheta = \vartheta_j$, $j = 1, 2, \ldots, N_\vartheta$. The ρ_i points were chosen according to the formulas

$$\rho_i = f(\tau_i)\rho_{N_\rho}, \quad \tau_i = \frac{i}{N_\rho}.$$

The non-linear monotonously increasing function $f(\tau)$, $0 \le \tau \le 1$, satisfying the conditions $f(0) = 0$ and $f(1) = 1$ was chosen in the form

$$f(\tau) = \frac{(1+\mathsf{a})\tau^2}{1+\mathsf{a}\tau}$$

in the case of the ground-state calculations and in the form

$$f(\tau) = \begin{cases} \alpha_0 \tau & , \ \tau \in [0, \tau_0] \\ \alpha_1 \tau + \tau^\nu & , \ \tau \in (\tau_0, 1] \end{cases}.$$

in the case of scattering and excited state calculations. A typical value of the "acceleration" a, $\mathsf{a} \ge 0$, which is satisfactory in ground-state calculations is $\mathsf{a} = 0.4$ (for $\rho_{N_\rho} < 100$ Å). The values of α_0, $\alpha_0 \ge 0$, and α_1, $\alpha_1 \ge 0$, are defined via τ_0 and ν from the continuity condition for $f(\tau)$ and its derivative at the point τ_0. A typical value of τ_0 is 0.2. The value of the power ν depends on the cut-off radius ρ_{N_ρ} =200—600 Å its range being within 3.3—4.75.

The knots ϑ_j for $j = 1, 2, \ldots, N_\rho$ were taken according to $\vartheta_j = \arctan(y_j)$. The rest knots ϑ_j, $j = N_\rho + 1, \ldots, N_\vartheta$, were chosen equidistantly,

$$\vartheta_j = \vartheta_{N_\rho} + \frac{j - N_\rho}{N_\vartheta - N_\rho + 1}\left(\frac{\pi}{2} - \vartheta_{N_\rho}\right).$$

Furthermore, the grid must be constructed in such a manner so that the density of the points is higher where the Faddeev components are important, i. e., for small values of ρ and/or x, and lower in the asymptotic region. Usually we took the same numbers of grid points for both ϑ and ρ, $N_\vartheta = N_\rho$.

In the scattering problem, we firstly, in the component $\Phi(x,y,p) \equiv \Phi_{00}(x,y,p)$ explicitly separate the initial-state wave function $\chi(x,y,p) = \psi_d(x)\sin(py)$. As a result Faddeev equations are reduced to inhomogeneous equations for the remainder $\Phi' = \Phi - \chi$ which differ in form from

$$(H_0 + V(x) - E)\Phi_{aL}(x,y) = -V(x)\sum_{a'}\int_{-1}^{+1} d\eta\, h^L_{aa'}(x,y,\eta)\,\Phi_{a'L}(x',y'), \quad a = \{l,\lambda\} \tag{1}$$

only by the presence on the right-hand side of the inhomogeneous terms

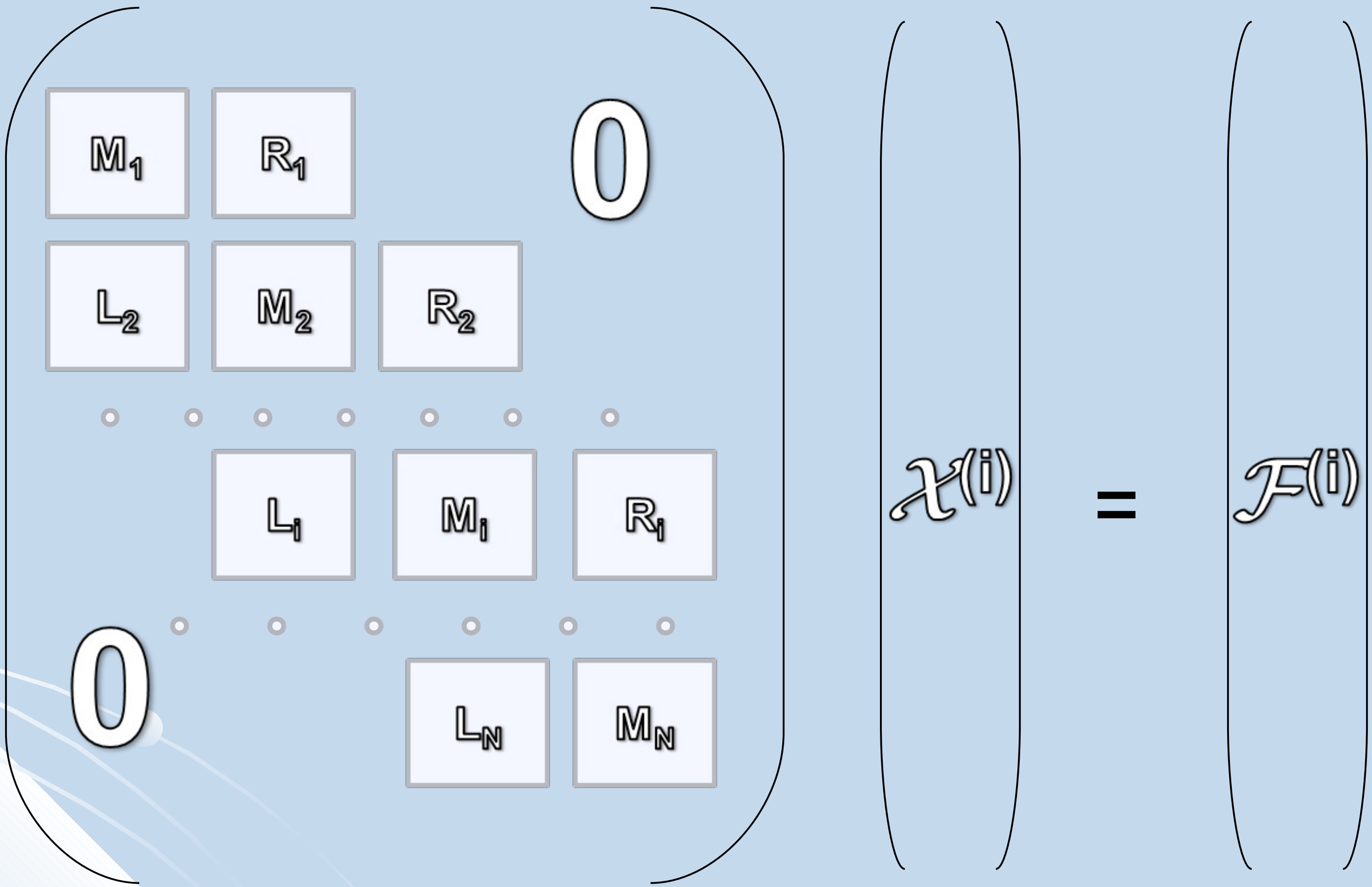
$$F^r(x,y) = -V(x)\int_{-1}^{+1} d\eta\, h^0_{(0,0)(0,0)}(x,y,\eta)\chi(x',y',p) \tag{2}$$

On a fixed arc $\rho = \rho_i$ of the polar grid concerned, the values of the function Φ' and inhomogeneous terms (2) form vectors $\mathcal{X}^{(i)} \in \mathbb{C}^{N_\vartheta}$, $\mathcal{F}^{(i)} \in \mathbb{R}^{N_\vartheta}$, having components $\mathcal{X}^{(i)}_j = \Phi'(\rho_i\cos\vartheta_j, \rho_i\sin\vartheta_j)$ and $\mathcal{F}^{(i)}_j = F^r(\rho_i\cos\vartheta_j, \rho_i\sin\vartheta_j)$. The set of vectors $\mathcal{X}^{(i)}$, $\mathcal{F}^{(i)}$, $i = 1,2,\ldots,N_\rho$, determines the vectors $\mathcal{X} \in \mathbb{C}^{N_{\vartheta\rho}}$ and $\mathcal{F} \in \mathbb{R}^{N_{\vartheta\rho}}$, $N_{\vartheta\rho} = N_\vartheta N_\rho$: $\mathcal{X} = \bigoplus_{i=1}^{N_\rho} \mathcal{X}^{(i)}$, $\mathcal{F} = \bigoplus_{i=1}^{N_\rho} \mathcal{F}^{(i)}$.

In such a representation partial Faddeev equations assumed the form

$$\begin{cases} \mathcal{X}^{(0)} = 0, \\ L_i\mathcal{X}^{(i-1)} + (M_i - E\tilde{I}_i)\mathcal{X}^{(i)} + R_i\mathcal{X}^{(i+1)} = \mathcal{F}^{(i)}, \quad i = 1,2,\ldots,N_\rho . \end{cases} \tag{3}$$

Here, L_i, M_i, $\tilde{I}_i$ and R_i are matrices of rank N_ϑ. The matrices L_i and R_i are generated only by the radial part of the Laplacian in (1) and are therefore diagonal. The non-diagonal matrix M_i describes the contribution of the central terms of the radial part of the Laplacian, of its spherical part, the potential, and the integral operator on the arc $\rho = \rho_i$. The matrix $\tilde{I}_i$ differs from the unity one only in a row corresponding to the boundary condition. This row in $\tilde{I}_i$ has zero elements.



$$
\begin{cases}
\mathcal{X}^{(0)} = 0 \\
L_i \mathcal{X}^{(i-1)} + (M_i - E\tilde{I}_i)\mathcal{X}^{(i)} + R_i \mathcal{X}^{(i+1)} = \mathcal{F}^{(i)} \qquad i = 1, 2, \ldots, N_\rho .
\end{cases}
\tag{A1}
$$

The system (A1) includes $N_{\theta\rho}$ equations for $N_{\theta\rho}+N_\theta$ unknowns. An additional relation that selects a unique solution of (A1) follows from the asymptotic conditions (45):

$$\mathcal{X}^{(N_\rho+1)} = B_{N_\rho}\tilde{I}_{N_\rho}\mathcal{X}^{(N_\rho)} + a_0(p)\tilde{I}_{N_\rho}\mathcal{D}^{(N_\rho)} \tag{A2}$$

where $B_{N_\rho} = \mathrm{diag}\{b_1, b_2, \ldots, b_{N_\theta}\}$ is a diagonal matrix with elements

$$b_j = C^+_{N_\rho}[1+\mathrm{o}(\rho_{N_\rho}^{-1/2})], \qquad C^+_{N_\rho} = \sqrt{\frac{\rho_{N_\rho}}{\rho_{N_\rho+1}}}\exp[\mathrm{i}\sqrt{E}(\rho_{N_\rho+1}-\rho_{N_\rho})],$$

and $\mathcal{D}^{(N_\rho)}$, $\mathcal{D}^{(N_\rho)} \in \mathbb{C}^{N_\theta}$, is a vector with components $\mathcal{D}_j^{(N_\rho)} = \chi_1(\rho_{N_\rho+1},\theta_j) - b_j\chi_1(\rho_{N_\rho},\theta_j)$ where $\chi_1(\rho,\theta) = \psi_d(\rho\cos\theta)\exp(\mathrm{i}p\rho\sin\theta)$.

The condition (A2) allows the elimination of $\mathcal{X}^{(N_\rho+1)}$ and reduces the last equation of the system (A1) to

$$L_{N_\rho}\mathcal{X}^{(N_\rho-1)} + (\tilde{M}_{N_\rho} - E\tilde{I}_{N_\rho})\mathcal{X}^{(N_\rho)} = \mathcal{F}^{(N_\rho)} + a_0(p)\tilde{\mathcal{F}}^{(N_\rho)} \tag{A3}$$

where the matrix $\tilde{M}_{N_\rho}$ and the vector $\tilde{\mathcal{F}}^{(N_\rho)}$ are given by $\tilde{M}_{N_\rho} = M_{N_\rho} + R_{N_\rho}B_{N_\rho}\tilde{I}_{N_\rho}$ and $\tilde{\mathcal{F}}^{(N_\rho)} = R_{N_\rho}\tilde{I}_{N_\rho}\mathcal{D}^{(N_\rho)}$.

The system (A1), after replacing its last equation with (A3), can be written in the form

$$(K - E\tilde{I})\mathcal{X} = \mathcal{F} + a_0(p)\mathcal{F}' \tag{A4}$$

The solution of (A4) can be expressed as

$$\mathcal{X} = \mathcal{X}_0 + a_0(p)\mathcal{X}_1, \quad (K - E\tilde{I})\mathcal{X}_0 = \mathcal{F}; \quad (K - E\tilde{I})\mathcal{X}_1 = \mathcal{F}'$$

in which the inhomogeneous terms are known.

Having determined the vectors $\mathcal{X}_0$ and $\mathcal{X}_1$, we can then proceed, via the asymptotics, to find the elastic scattering amplitude $a_0(p)$

$$a_0(p) = \frac{\left[\mathcal{X}_0^{(N_\rho)}\right]_j}{\chi_1(N_\rho, \vartheta_j) - \left[\mathcal{X}_1^{(N_\rho)}\right]_j}$$

where the index j corresponds to the angles ϑ_j for which $\rho_{N_\rho} \cos\vartheta_j \approx x_0$, $\psi_d(x_0) = \max \psi_d(x)$

Having calculated $a_0(p)$ we can find the vector $\mathcal{X}^{(N_\rho)}$ corresponding to the values of the desired function Φ' on the final arc $\rho = \rho_{N_\rho}$, $\Phi'(\rho_{N_\rho} \cos\vartheta_j, \rho_{N_\rho} \sin\vartheta_j) = \mathcal{X}_j^{(N_\rho)}$, and then determine the Faddeev breakup amplitude

$$A(\vartheta_j) = \left[\mathcal{X}_j^{(N_\rho)} - a_0(p)\chi_1(\rho_{N_\rho}, \vartheta_j)\right] \sqrt{\rho_{N_\rho}} \exp(-i\sqrt{E}\rho_{N_\rho}) \, .$$

Helium trimer
HFD-B potential

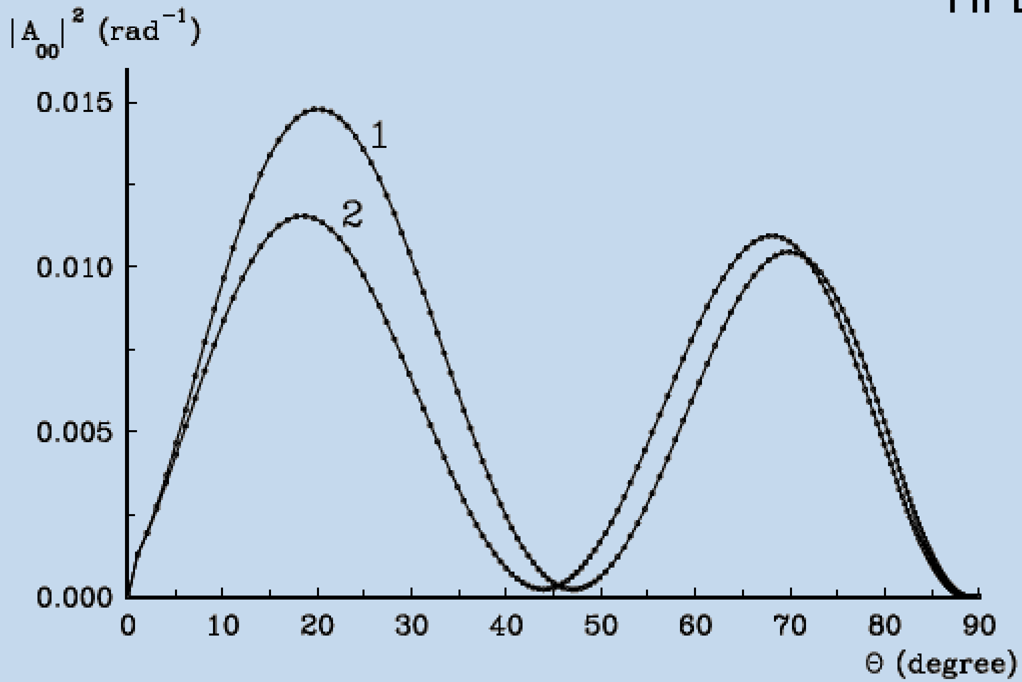


FIG. 1: The square of the modulus of the Faddeev breakup amplitude $A_{00}(\vartheta)$ for HFD-B ^{4}He–^{4}He potential at $E = +1.4$ mK. Curve 1 corresponds to the $L = 0$, $l = \lambda = 0$ partial wave while curve 2 was obtained with the inclusion of the $L = 0$, $l = \lambda = 2$ channel.

$$\mathcal{A}^{[a,\nu]}_{a'L}(\theta) = A^{[a,\nu]}_{a'L}(\theta) + \sum_{a''} \int_{-1}^{1} d\eta \, h^{L}_{a'a''}(x, y, \eta) A^{[a,\nu]}_{a''L}(\theta')$$

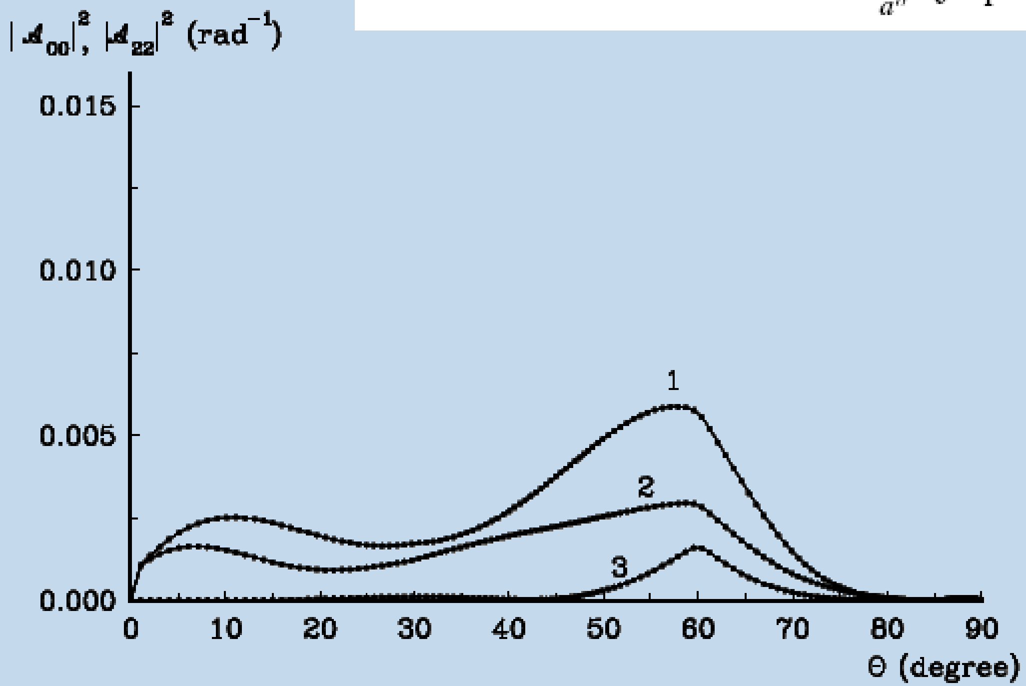


FIG. 2: The squares of the moduli of the physical breakup amplitudes $\mathcal{A}_{00}(\vartheta)$ (curves 1, 2) and $\mathcal{A}_{22}(\vartheta)$ (curve 3) for the HFD-B ^{4}He–^{4}He potential at $E = +1.4$ mK. Curve 1 corresponds to the inclusion of the $L = 0$, $l = \lambda = 0$ channel only, while curves 2 and 3 were obtained with the inclusion of both $l = \lambda = 0$ and $l = \lambda = 2$ partial waves.

The (2'2) component of the s-wave partial scattering matrix is given by expression

$$S_0 = 1 + 2ia_0(E)$$

Helium trimer
HFD-B potential

$$V(x) = \lambda V_{HFD-B}(x), \quad \lambda < 1$$

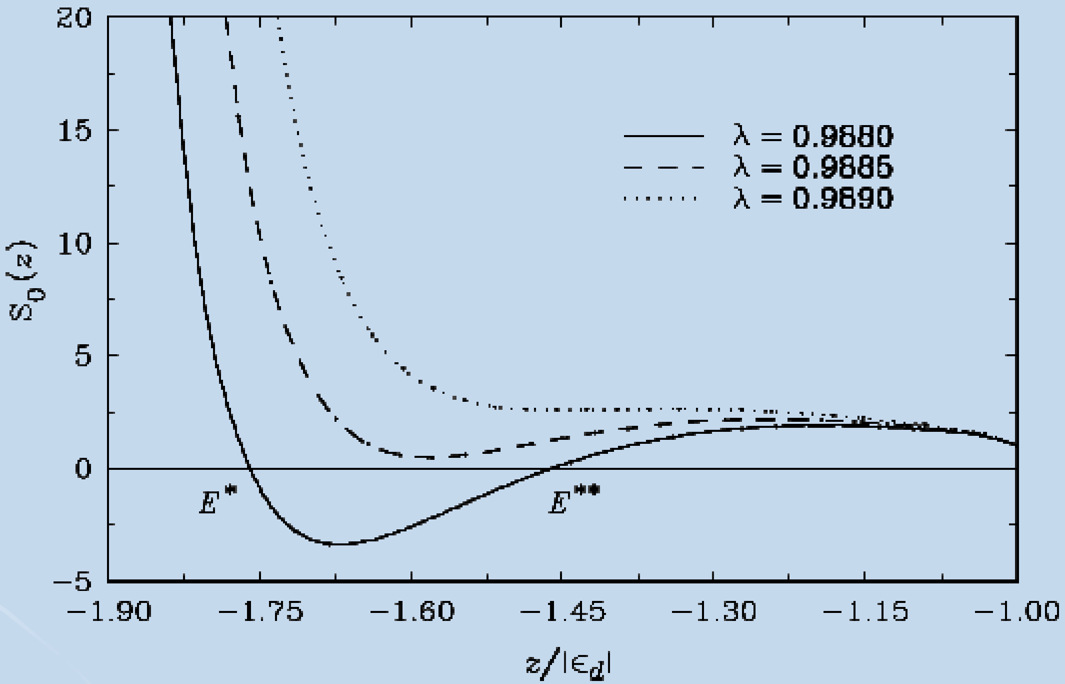
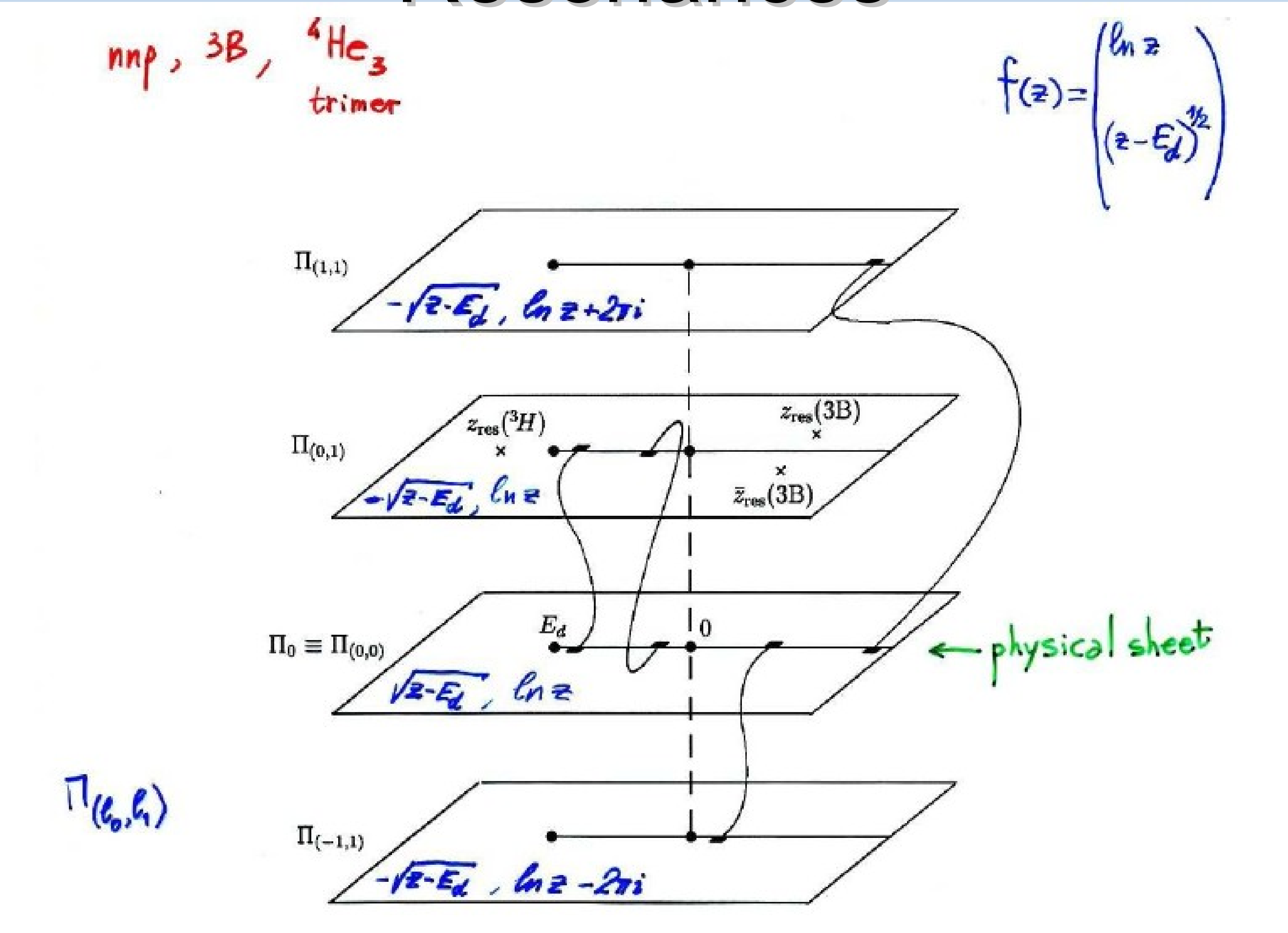


Fig. 1. Graphs of the function $S_0(E)$ at real $E \leqslant \varepsilon_d$ for three values of $\lambda < 1$.

Resonances



nnp, 3B, $^4He_3$ trimer
$f(z) = \begin{pmatrix} \ln z \\ (z-E_d)^{1/2} \end{pmatrix}$
$\Pi_{(1,1)}$
$-\sqrt{z-E_d}$, $\ln z + 2\pi i$
$\Pi_{(0,1)}$
$z_{res}(^3H)$
$z_{res}(3B)$
$\bar{z}_{res}(3B)$
$-\sqrt{z-E_d}$, $\ln z$
E_d
0
$\Pi_0 \equiv \Pi_{(0,0)}$
$\sqrt{z-E_d}$, $\ln z$
← physical sheet
$\Pi_{(l_0,l_1)}$
$\Pi_{(-1,1)}$
$-\sqrt{z-E_d}$, $\ln z - 2\pi i$

Resonances

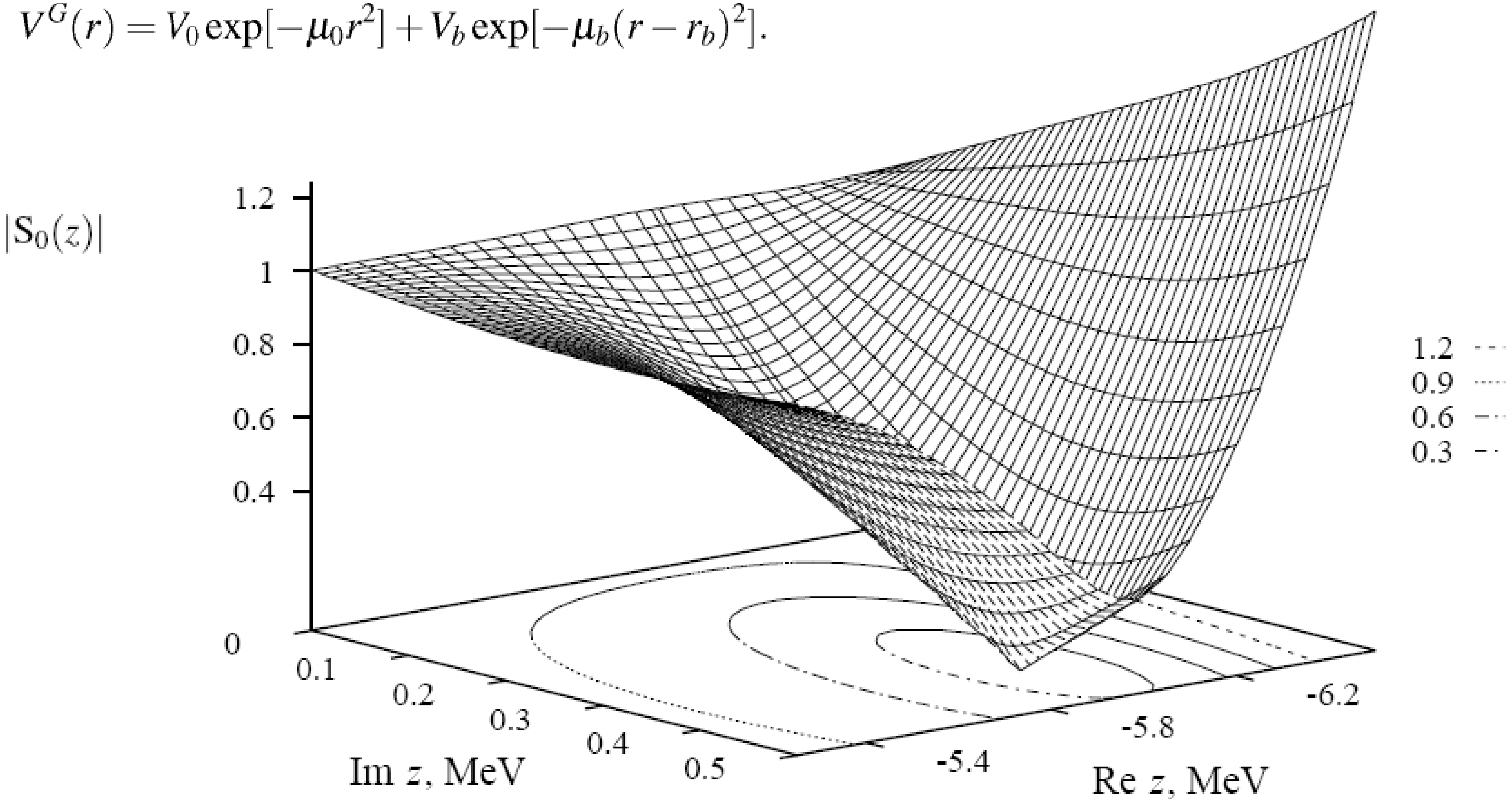
In [6] the explicit representations for analytic continuation of the T-matrix Faddeev components on unphysical sheets have been derived.

As follows from the representations constructed in [6], the nontrivial (i.e. differing from the poles at the discrete spectrum eigenvalues of the three-body Hamiltonian) singularities of the T-matrix, scattering matrices and resolvent situated on an unphysical sheet Π_l, are singularities of the inverse truncated scattering matrix $S_l^{-1}(z)$. Therefore, the resonances on the sheet Π_l considered as poles of the T-matrix, scattering matrix and resolvent continued on Π_l are those values of the energy z for which the matrix $S_l(z)$ has zero as eigenvalue.

Thereby, to search for the resonances situated on a certain unphysical sheet P_l, one can apply any method allowing to compute analytical continuation on the physical sheet of the elastic scattering, rearrangement or breakup amplitudes necessary for construction of respective $S_l(z)$. It is only necessary to go out in this formulation on the complex plane of z including the asymptotical boundary conditions.

[5] - A. K. Motovilov, Mathmatische Nachrichten **187** (1997)147

$$V^G(r) = V_0 \exp[-\mu_0 r^2] + V_b \exp[-\mu_b (r - r_b)^2].$$



Surface of the function $|S_0(z)|$ in the model system of three bosons with the nucleon masses. The potential used with $V^G(r)$ the barrier V_b=1.5 MeV, V_0=55 MeV, μ_0=0.2 fm^{-2},μ_b=0.01fm^{-2},r_b=5 fm. Position of the resonance *(-5.95-i0.403)* corresponds to the zero value of $|S_0(z)|$.

We start with recalling that the three-body Schrödinger operator H_{3b} reads after the scaling transorm as follows

$$H_{3b}(\vartheta) = U(\vartheta) H_{3b} U(-\vartheta) = -e^{-2\vartheta}\Delta_X + \sum_\alpha v_\alpha(e^\vartheta |x_\alpha|), \qquad \alpha = 1,2,3. \tag{1}$$

where $\vartheta = i\theta$ with $\theta \in \mathbb{R}$ while x_α, y_α are the standard Jacobi variables, $X \equiv (x_\alpha, y_\alpha)$. By Δ_X we understand the 6–dimensional Laplacian and by v_α, the two-body potentials. The corresponding scaled Faddeev equations that we solve have the following form:

$$[-e^{-2\vartheta}\Delta_X + v_\alpha(e^\vartheta|x_\alpha|) - z]\Phi^{(\alpha)}(X) + v_\alpha(e^\vartheta|x_\alpha|)\sum_{\beta\neq\alpha}\Phi^{(\beta)}(X) = f_\alpha(X), \qquad \alpha = 1,2,3. \tag{2}$$

Here $f = (f_1, f_2, f_3)$ is an arbitrary three-component vector with components f_α belonging to the three-body Hilbert space. The partial-wave version of the equations (2) for a system of three identical bosons with $L = 0$ reads

$$e^{-2i\theta} H_0^{(l)} \Phi_l(x,y) - z\,\Phi_l(x,y) + V(e^{i\theta}x)\Psi_l(x,y) = f_l(x,y), \tag{3}$$

Here, $H_0^{(l)}$ denotes the partial kinetic energy operator and Ψ_l, the partial-wave component

For compactly supported inhomogeneus terms $f_l(x,y)$ the partial-wave Faddeev component $\Phi_l(x,y)$ satisfies the asymptotic condition

$$\begin{aligned} \Phi_l(x,y) \; = \; & \delta_{l0}\psi_d(e^{i\theta}x)\exp(i\sqrt{E_t-\epsilon_d}\; e^{i\theta}y)\left[a_0 + o\left(y^{-1/2}\right)\right] \\ & + \frac{\exp(i\sqrt{E_t}\; e^{i\theta}\rho)}{\sqrt{\rho}}\left[A_l(y/x) + o\left(\rho^{-1/2}\right)\right], \end{aligned} \tag{4}$$

[6] – E.Kolganova,A.Motovilov,Y.H.Ho Nucl.Phys.A **684** (2001) 623

In the scaling method a resonance is looked for as the energy z which produces a pole to the function

$$\Phi(\theta, z) = \langle [H_F(\theta) - z]^{-1} f, f \rangle \qquad (5)$$

where $H_F(\theta)$ is the non-selfadjoint operator resulting from the complex-scaling transformation of the Faddeev operator. This is just the operator constituted by the l. h. s. parts of Eqs. (2). The resonance energies should not, of course, depend on the scaling parameter θ and on the choice of the terms $f_l(x, y)$.

[6] – E.Kolganova,A.Motovilov,Y.H.Ho Nucl.Phys.A **684** (2001) 623

θ	z_{res} (MeV)	θ	z_{res} (MeV)
0.25	$-5.9525 - 0.4034\,i$	0.50	$-5.9526 - 0.4032\,i$
0.30	$-5.9526 - 0.4033\,i$	0.60	$-5.9526 - 0.4033\,i$
0.40	$-5.9526 - 0.4032\,i$	0.70	$-5.9526 - 0.4034\,i$

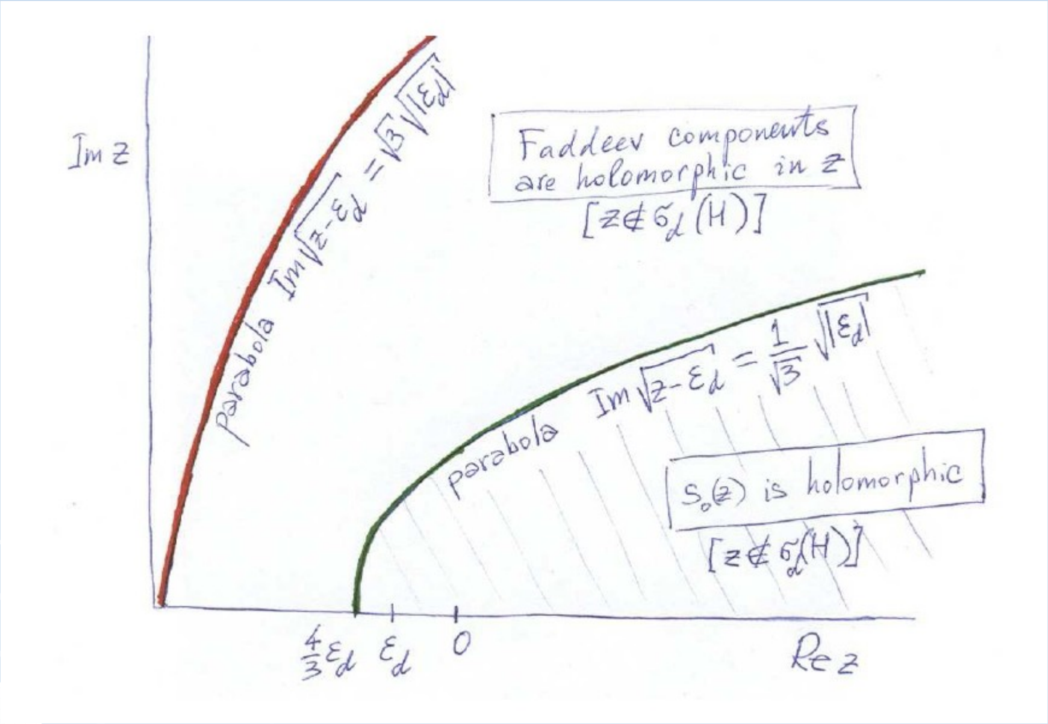
We compare the resonance values of the table to the resonance value $z_{res} = -5.952 - 0.403\,i$ MeV obtained for the same three-boson system with exactly the same potentials

[7] – D.Fedorov,E.Garrido,A.Jensen *Complex Scaling of the Hyper-Spheric Coordinates and Faddeev Equations* FBS **33** (2003) 153

Table 1. The ground-state energy E_0 and the resonance energy E_1 for different scaling angles θ for the model system

θ	E_0, MeV	E_1, MeV
0.30	−37.221	$-5.968 - 0.400i$
0.35	−37.220	$-5.962 - 0.404i$
0.40	−37.221	$-5.963 - 0.401i$

Limits



Im z
parabola Im $\sqrt{z-\varepsilon_d} = \sqrt{3}\sqrt{|\varepsilon_d|}$
Faddeev components are holomorphic in z
$[z \notin \sigma_d(H)]$
parabola Im $\sqrt{z-\varepsilon_d} = \frac{1}{\sqrt{3}}\sqrt{|\varepsilon_d|}$
$S_0(z)$ is holomorphic
$[z \notin \sigma_d(H)]$
$\frac{4}{3}\varepsilon_d$
ε_d
0
Re z

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