# Tetraquarks in the Steiner tree model of confinement

available at *http://lpsc.in2p3.fr/theorie/Richard/SemConf/Talks.html*

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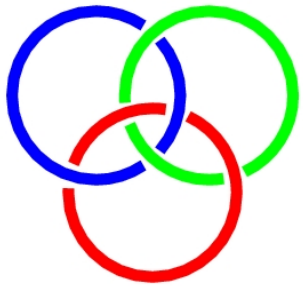


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# Introduction

## Speculations on multiquarks

- low mass of $(q\bar{q})$, hence $(qq\bar{q}\bar{q})$ in S-wave perhaps lighter than $(q\bar{q})$ in P-wave. Applied to the problem of scalar mesons.
- Peculiar features of chiral dynamics. Speculations on the late pentaquarks made of light or strange quarks or antiquarks,
- Coherences in the hyperfine interaction $\longrightarrow$ see next section,
- Properties of the mass dependence in a flavour-independent potential, $\longrightarrow$ below
- Favourable 4-body interaction in QCD $\longrightarrow$ below.

# Symmetry breaking and tetraquarks-1

Consider

$$H = H_0(\textit{even}) + \lambda H_1(\textit{odd}).$$

Then for the ground state, with $\psi_0(H_0)$ as trial w.f, $\langle\psi_0|H_1|\psi_0\rangle = 0$

$$E(H) \leq E(H_0),$$

i.e., *H benefits* of symmetry breaking.
For instance $E(p^2 + x^2 + \lambda x) = 1 - \lambda^2/4$.

This is very general.
Starting, e.g., from a symmetrical four-body system $(\mu, \mu, \bar{\mu}, \bar{\mu})$ breaking particle identity or charge conjugation lowers the ground state, but has different consequences on stability

# Breaking particle identity

$H(M, m, M, m)$, where V does not change if $M$ and/or $m$ is modified, can be rewritten as

$$H = \underbrace{\left(\frac{1}{4M} + \frac{1}{4m}\right)\left[\boldsymbol{p}_1^2 + \cdots + \boldsymbol{p}_4^2\right] + V}_{H_0} + \underbrace{\left(\frac{1}{4M} - \frac{1}{4m}\right)\left[\boldsymbol{p}_1^2 - \boldsymbol{p}_2^2 + \boldsymbol{p}_3^2 - \boldsymbol{p}_4^2\right]}_{H_1}$$

Thus $E(H) \leq E(H_0)$. But in general, the threshold *also* benefits from this symmetry breaking, and actually benefits more, so that four-body binding deteriorates.

For instance, in atomic physics $(e^+, e^+, e^-, e^-)$ and any equal-mass $(\mu^+, \mu^+, \mu^-, \mu^-)$ weakly bound below the atom–atom threshold, but $(M^+, m^+, M^-, m^-)$ unstable for $M/m \gtrsim 2.2$, see Dario.

Then: breaking the symmetry of identical particles does not help

# Breaking charge conjugation

$H(M, M, m, m)$ written as

$$H = \underbrace{\left(\frac{1}{4M} + \frac{1}{4m}\right)\left[\boldsymbol{p}_1^2 + \cdots + \boldsymbol{p}_4^2\right] + V}_{H_0} + \underbrace{\left(\frac{1}{4M} - \frac{1}{4m}\right)\left[\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2 - \boldsymbol{p}_3^2 - \boldsymbol{p}_4^2\right]}_{H_1}$$

still benefits to the four-body system, $E(H) \leq E(H_0)$, but $H$ and $H_0$ have the same threshold $(M^+, m^-) + (M^+, m^-)$. Hence binding improves. Indeed, $\mathrm{H}_2$ more bound than $\mathrm{Ps}_2$ and has even a rich spectrum of excitations.

# Quark model analogs

For a central, flavour-independent, confining interaction $V$,

- Equal mass case $(q, q, \bar{q}, \bar{q})$ hardly bound
- Hidden-flavour case $(Q, q, \bar{Q}, \bar{q})$ even farther from binding,
- $(QQ\bar{q}\bar{q})$ with flavour = 2 bound if $M/m$ large enough
  See Ader et al. (then at CERN), Heller et al. (Los Alamos), Zouzou et al. (Grenoble), D. Brink et al. (Oxford), Rosina et al. (Slovenia), Lipkin, Vijande et al., etc.

$(QQ\bar{q}\bar{q})$ expected at least in the limit of large or very large $M/m$.
As compared to the "colour-chemistry" (late 70's and early 80's)

- no exotic colour configuration
- for large $M/m$, almost pure $3 \to \bar{3}$ for $(QQ)$ as in every baryon,
- and then $\bar{3} \times \bar{3} \times \bar{3} \to 1$ for $[(QQ) - \bar{q}\bar{q}]$ as in every antibaryon: well probed colour structure.

# The additive model of tetraquark confinement

Questions:

- What is $V$ for tetraquarks?
- Even earlier: what is the link from mesons to baryons?

The additive model By analogy with QED,

$$V(1,2,\ldots) = -\frac{3}{16}\sum_{i<j}\tilde{\lambda}_i^{(c)}.\tilde{\lambda}_j^{(c)}v(r_{ij})\ ,$$

- $v(r)$ is the quarkonium potential fitted to mesons,
- $\lambda^{(c)}$ is the non-abelian colour operator

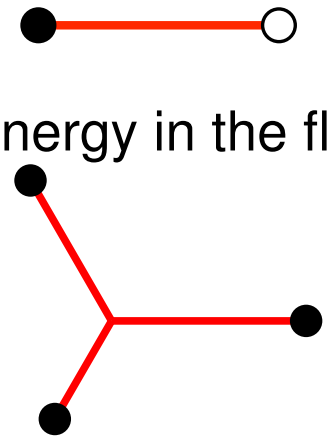
Consequences

- A reasonable simultaneous phenomenology of baryon and meson spectra
- Multiquarks unbound, except $(QQ\bar{q}\bar{q})$ with large $M/m$.
- Hence multiquark binding was based on other mechanisms: chrmomagnetism (Jaffe, Lipjkin, Gignoux et al.), chiral dynamics (cf. the late pentaquark), etc.

# The Steiner-tree model of baryons

*Y*-shape potential:

- Proposed by Artru, Dosch, Merkuriev, etc., proposed a better ansatz, often verified and rediscovered (strong coupling, adiabatic bag model (Kuti et al.), flux tube, lattice QCD, etc.)
- The linear $q - \bar{q}$ potential of mesons interpreted as minimising the gluon energy in the flux tube limit



- The $q - q - q$ potential of baryons is with the junction optimised, i.e., fulfilling the conditions of the well-known Fermat-Torricelli problem.

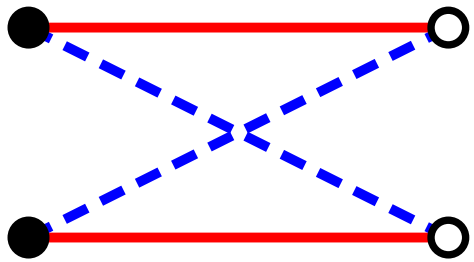
This potential was used for baryons (Taxil et al., Semay et al., Kogut et al.), but it does not make much difference as compared with the additive ansatz.

# The Steiner tree model of tetraquarks

A tempting generalisation to tetraquarks is the combination $V_4 = \min(V_f, V_S)$ of

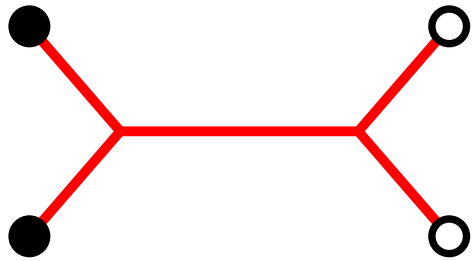
- flip-flop $V_f$ (already used in its quadratic version by Lenz et al.)

$$V_f = \lambda \min(r_{13} + r_{24}, r_{23} + r_{14})$$



- and Steiner-tree $V_S$

$$V_S = \lambda \min_{k,\ell}(r_{1k} + r_{2k} + r_{k\ell} + r_{\ell 3} + r_{\ell 4}) \ .$$



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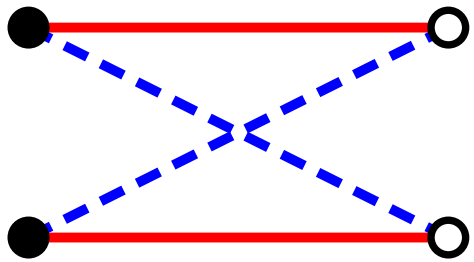

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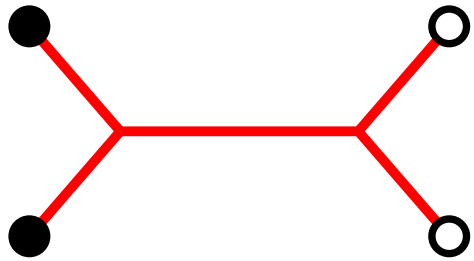
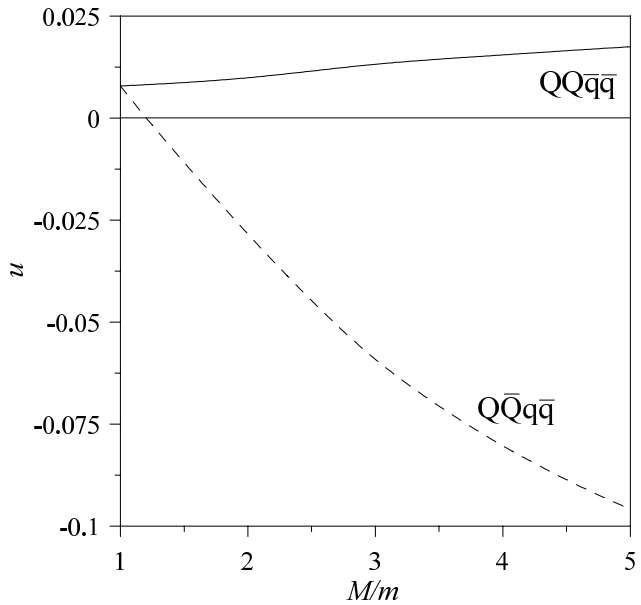


- J. Carlson and V.R. Pandharipande concluded that this potential does not bind, but
- they used too simple trial wave functions for the 4-body problem, and did not consider unequal masses.

# Tetraquarks in the minimal-path model-1

Vijande, Valcarce and R. revisited the calculation of Carlson at al. with a basis of correlated Gaussians (matrix elements painfully calculated numerically), and obtained stability for $(QQ\bar{q}\bar{q})$ even for $M/m = 1$, but better stability for $M/m \gg 1$.

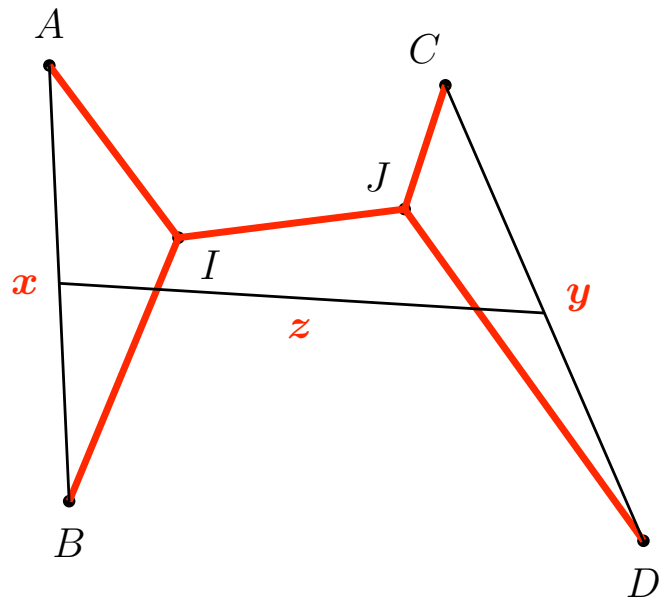



$$u = (E_{th} - E_4)/E_{th}$$

# Tetraquarks in the minimal-path model-2

More recently, Cafer Ay, Hyam Rubinstein (Melbourne) and R.: rigorous proof of stability within the minimal-path model if $M \gg m$.
Obviously,
$V_4 \leq V_S \leq |\boldsymbol{x}| + |\boldsymbol{y}| + |\boldsymbol{z}|$

where $\boldsymbol{x} = \overrightarrow{AB}$ , $\boldsymbol{y} = \overrightarrow{CD}$ ,
and $\boldsymbol{z}$ links the middles.




Then

$$H \leq \left[\frac{\boldsymbol{p}_x^2}{M} + |\boldsymbol{x}|\right] + \left[\frac{\boldsymbol{p}_y^2}{m} + |\boldsymbol{y}|\right] + \left[\frac{\boldsymbol{p}_z^2}{2\mu} + |\boldsymbol{z}|\right]$$

exactly solvable, but not does not demonstrate binding of $(QQ\bar{q}\bar{q})$

# Better bound

- A better bound demonstrates stability for large $M/m$:

$$H \leq \left[\frac{\boldsymbol{p}_x^2}{M} + \frac{\sqrt{3}}{2}|\boldsymbol{x}|\right] + \left[\frac{\boldsymbol{p}_y^2}{m} + \frac{\sqrt{3}}{2}|\boldsymbol{y}|\right] + \left[\frac{\boldsymbol{p}_z^2}{2\mu} + |\boldsymbol{z}|\right]$$

- $p^2 + |\boldsymbol{x}| \quad \Longrightarrow \quad e_0 = 2.3381...$ (Airy function)
- by scaling $\boldsymbol{p}^2/m + \lambda|\boldsymbol{x}| \quad \Longrightarrow \quad e_0\,\lambda^{2/3}\,m^{-1/3}$.
- Threshold $2(Q\bar{q})Q\bar{q})$ at $E_{\rm th} = 2e_0\mu^{-1/3}$, $\mu = Mm/(M+m)$.
- The tetraquark energy has a upper bound

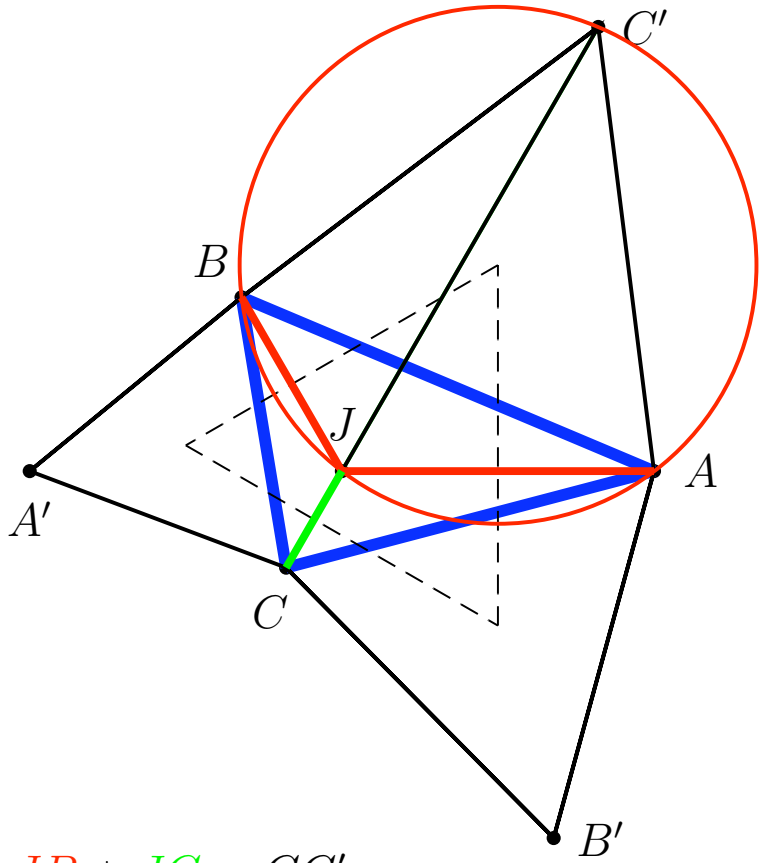
$$E_4 \leq E_4^{\rm up} = e_0\left\{\left(\frac{3}{4}\right)^{1/3}\left[M^{-1/3} + m^{-1/3}\right] + (2\mu)^{-1/3}\right\}$$

- Straightforward to check that $E_4^{\rm up} < E_{\rm th}$ for $M/m < 6403$
- Thus $\boxed{E_4 < E_{\rm th}}$ at large $M/m$ demonstrated rigorously
- Actually $\forall\, M/m$ from solving numerically the 4-boby pb.

# Proof-1

A flavour of the proof. In the 3-body case, Steiner tree linked to Napoleon's theorem.
$JA + JB + JC = CC'$ where $C'$ makes an external equilateral triangle associated to the side $AB$.

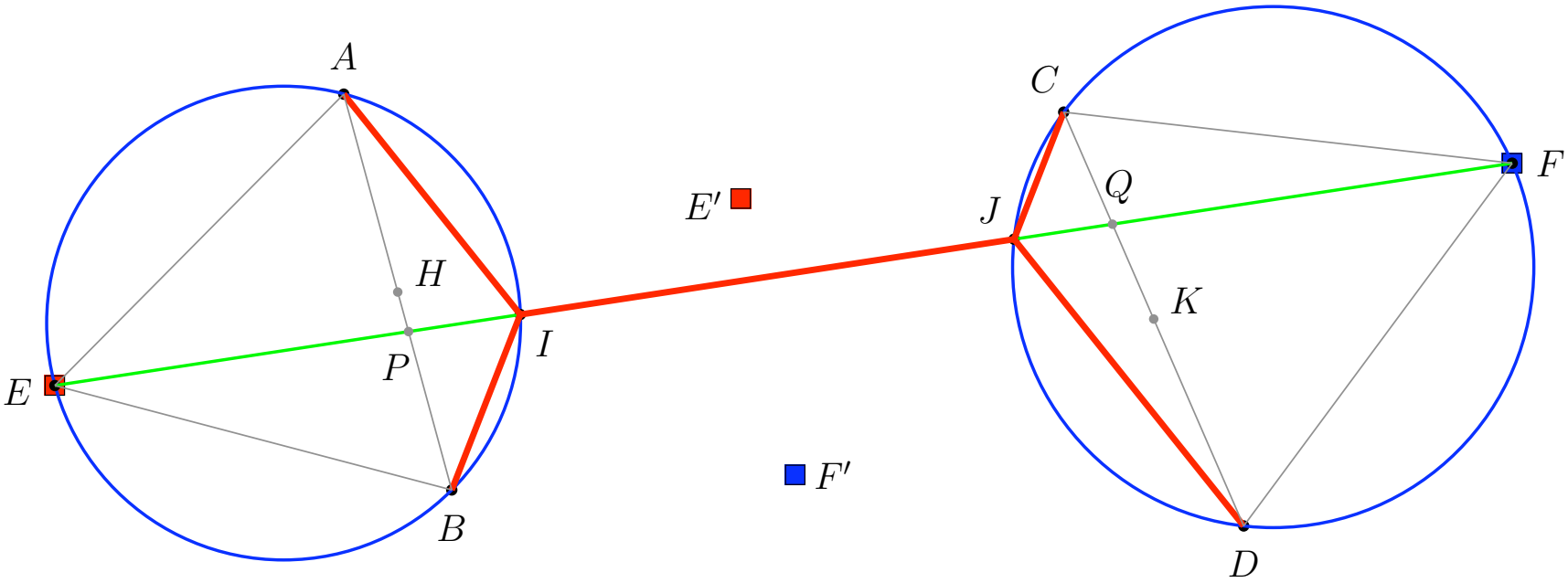
Well-known property of the Fermat-Torricelli problem. ($C'$ belongs to the torroïdal domain associated to $AB$)




$JA + JB + JC = CC'$

# Proof-2

The analogue for the planar tetraquark is




$$V_S = JA + JB + JK + KC + KD = EF$$

The minimal network linking $(A, B, C, D)$ is the maximal distance beween $\{E, E'\}$ and $\{F, F'\}$, which are the torroïdal domains associated to $(A, B)$ and $(C, D)$ (= points completing an equilateral tr.)

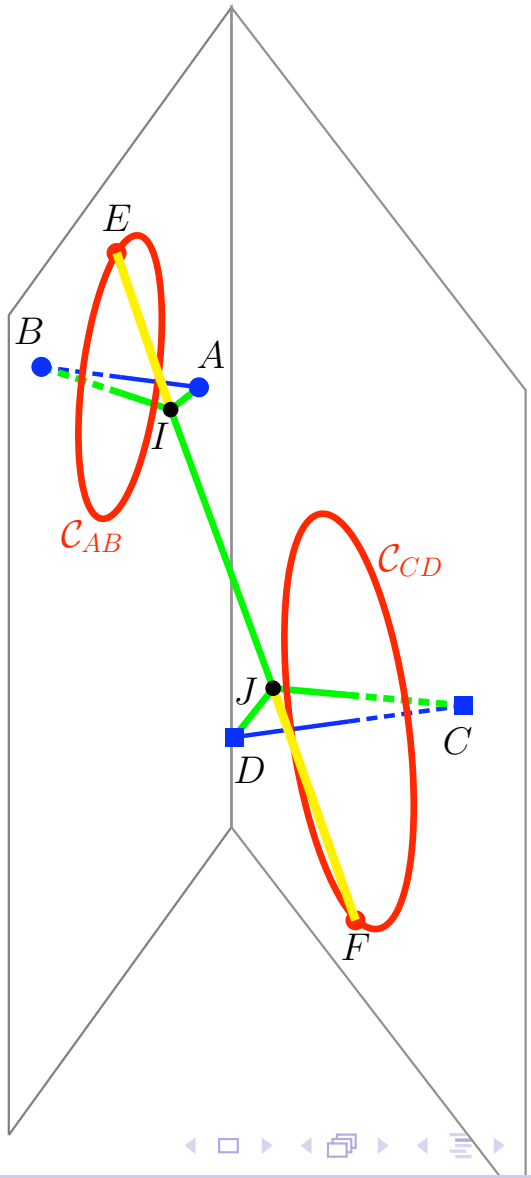
# Proof-3

In space, still

$$V_S = JA + JB + JK + KC + KD = EF$$

where

- $E \in \mathcal{C}_{AB}$= torroïdal domain of quarks $AB$, (equilateral circle)
- $F \in \mathcal{C}_{CD}$= torroïdal domain of antiquarks $CD$,
- $V_S$ is the maximal distance between the circles $\mathcal{C}_{AB}$ and $\mathcal{C}_{CD}$, which is less than the distance between the centres and the sum of radii.



# Conclusions

- New models of confinement beyond naive additive models, in better agreement with QCD in the strong coupling limit
- New inequality on the combined flip-flop and Steiner-tree paths,
- Drastic revision of the four-body spectrum within this model
- Analogous to the Wheeler (1945) – Ore (1946) – Hyllerras & Ore (1947) views on the $Ps_2$ molecule.

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AADNE ORE
*Sloane Physics Laboratory, Yale University, New Haven, Connecticut*
June 10, 1946

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**Binding Energy of the Positronium Molecule**

EGIL A. HYLLERAAS
*Institute of Theoretical Physics, University of Oslo, Oslo, Norway*
AND
AADNE ORE*
*Sloane Physics Laboratory, Yale University, New Haven, Connecticut*
(Received December 26, 1946)

A system of two electrons and two positrons is shown to possess dynamic stability. The variational calculation performed leads to a binding energy of at least 0.11 ev for this cluster. The approximate wave function which yields this value depends on the four electron-positron distances only. Neglect of the two distances between particles of the same kind permits an essential mathematical simplification which might be of interest in other problems.