Inclusive hadronic decay rate of the τ lepton from lattice QCD

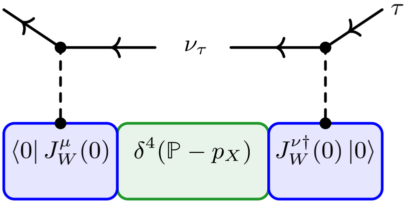
Antonio Evangelista

Lattice QCD and its phenomenological applications

Centre Paul Langevin, Aussois

TOR VERGATA
UNIVERSITY OF ROME
School of Mathematics, Physical and Natural Sciences

- Decay–amplitude of τ–lepton in the fg–flavored channel



$$\mathcal{A}_X = \frac{G_F}{\sqrt{2}} V_{fg} \langle \tau | J^\mu_{W,\ell}(0) J^\mu_{W,H}(0) | \nu, X_{fg} \rangle = \\ = \frac{G_F}{\sqrt{2}} V_{fg} \Big\{ \langle \tau | J^\mu_{W,\ell}(0) | \nu \rangle \Big\}_{\text{QED}} \Big\{ \langle 0 | J^\mu_{W,H}(0) | X_{fg} \rangle \Big\}_{\text{QCD}} , \quad (1)$$

- The process is completely factorazible if radiative corrections are neglected

- $S_{EW} = 1 + \mathcal{O}(\alpha_{EM})$ take into account the short–distance elettroweak correction at the decay rate

$$\Gamma_{fg} = \frac{G_F^2 S_{EW}}{4m_\tau} |V_{gf}|^2 \int \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2E_\nu} \mathrm{d}^4 p_X \delta^{(4)} \left(p_\tau - p_\nu - p_X\right) \mathcal{L}^{\mu\nu}(p_\tau, p_\nu) \mathcal{H}_{\mu\nu}(p_X) \,, \tag{2}$$

- The lepton tensor can be calculated with pertubation theory

$$\mathcal{L}^{\mu\nu}(p_\tau, p_\nu) = 4 \left\{ p_\tau^\mu p_\nu^\nu + p_\tau^\nu p_\nu^\mu - g^{\mu\nu} p_\tau \cdot p_\nu \right\} - 4i\epsilon^{\mu\nu\alpha\beta} p_{\tau,\alpha} p_{\nu_\tau,\beta} \,. \tag{3}$$

- By using the Lorentz covariance and the fact that it depends only on the impulse p_X the hadronic tensor reads

$$\begin{aligned} \mathcal{H}^{\mu\nu}(p_X) &\equiv \langle 0 | \, J_W^\mu(0) (2\pi)^4 \delta^{(4)} \left(\hat{P} - p_X \right) J_W^{\nu\dagger}(0) \, | 0 \rangle = \\ &= p_X^\mu p_X^\nu \, \rho_L(p_X^2) + \left[p_X^\mu p_X^\nu - g^{\mu\nu} p_X^2 \right] \rho_T(p_X^2) \end{aligned} \tag{4}$$

- By relying on Lorentz invariance we choose the reference frame with τ at rest $p_\tau = (m_\tau, \vec{0})$
- In this frame $p_X^2 = (p_\tau - p_\nu)^2 = m_\tau^2(1 - 2\frac{E_\nu}{m_\tau}) \equiv m_\tau^2 \omega^2$

$$R_{fg} \equiv \frac{\Gamma_{fg}}{\Gamma_e} = 12\pi\, S_{EW} |V_{gf}|^2 \int_{r_{fg}}^{1} \mathrm{d}\omega\, \omega \left(1-\omega^2\right)^2 \left\{\rho_L(\omega^2) + \rho_T(\omega^2)\left(1+2\omega^2\right)\right\} \tag{5}$$

where $r_{fg} = \frac{m_{fg}}{m_\tau}$ with m_{fg} the mass of the lightest final hadronic state.

$$\rho_L(\omega^2) = -\frac{\mathcal{H}^{00}}{m_\tau^2\omega^2} \equiv \frac{\mathcal{H}^L}{m_\tau^2\omega^2}\,, \qquad \rho_T(\omega^2) = \frac{\frac{1}{3}\sum_i \mathcal{H}^{ii}}{m_\tau^2\omega^2} \equiv \frac{\mathcal{H}^T}{m_\tau^2\omega^2}\,, \tag{6}$$

Note that for $\omega < r_{fg}$ the spectral densities are vanishing.

- In order to compute the decay rate we need to know the spectral densities $\rho_L(\omega^2)$ and $\rho_T(\omega^2)$
- From parity and Poincarèe symmetries we can relate Euclidean correlators to the hadronic tensor

$$
\begin{aligned}
C^{\mu\nu}(t) &= \int \mathrm{d}^3x \, \langle 0| \, J^\mu_W(0) e^{-\hat{H}t} e^{i\hat{\boldsymbol{P}}\cdot\boldsymbol{x}} J^{\nu\dagger}_W(0) \, |0\rangle = \\
&= \int_0^\infty \mathrm{d}\omega \, m_\tau (2\pi)^3 \, \langle 0| \, J^\mu_W(0) e^{-\omega m_\tau t} \delta^{(4)} \left(\hat{P} - p_X\right) J^{\nu\dagger}_W(0) \, |0\rangle \\
&= \frac{m_\tau}{2\pi} \int_0^\infty \mathrm{d}\omega \, \mathcal{H}^{\mu\nu}(\omega) e^{-\omega m_\tau t} \qquad \text{for} \quad t > 0 \, .
\end{aligned} \tag{7}
$$

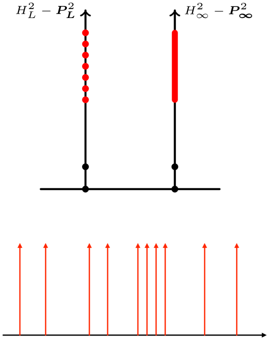
$$\downarrow$$

The spectral densities can be extracted from the lattice correlators by performing an inverse laplace transform

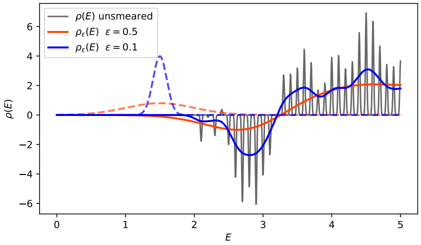
Ill–posed numerical problem, due to lattice correlators uncertainties

- Suppose that we have found a method to regularize numerically the problem
- In the finite volume the spectral densities are distributions since the Hamiltonian has a discrete spectrum

$$\rho_L(\omega) = \langle 0| \, J(0)(2\pi)^4 \delta^{(3)}(\boldsymbol{P}) \delta\left(m_\tau \omega - H_L\right) J^\dagger(0) \, |0\rangle_L = \sum_n c_n(L) \delta(\omega - \omega_n(L))$$



- Smeared integral of spectral densities can be studied at finite volumes



- The smeared integrals are smooth functions and studying their infinte volume limit is a well posed problem

$$\downarrow$$

We can use kinematic factors as the kernel entering in the inverse problem

Gambino and Hashimoto – PRL 125 (2020) 3

$$R_{fg} \propto \int_{r_{fg}}^{1} \mathrm{d}\omega\, \mathcal{H}(\omega) \cdot \text{kinematic} \longrightarrow \int_{r_{fg}}^{\infty} \mathrm{d}\omega\, \mathcal{H}(\omega)\theta(1-\omega) \cdot \text{kinematic} \tag{8}$$

- Given a smooth function $K(\omega)$ such that $K(\omega) \sim 0$ as $\omega \to \infty$

$$K(\omega) \simeq \sum_{t=1}^{\infty} g(t) e^{-t m_\tau \omega} \tag{9}$$

- By introducing a a smooth representation of the θ–function: $\Theta_\sigma(\omega)$

$$K^L_\sigma(\omega) = \frac{(1-\omega^2)^2}{\omega}\Theta_\sigma(1-\omega)\,, \qquad K^T_\sigma(\omega) = \frac{(1-\omega^2)^2(1+2\omega^2)}{\omega}\Theta_\sigma(1-\omega)\,. \tag{10}$$

- We can trade the integral with a sum over the correlators

$$R_{fg}(\sigma) \equiv \frac{12\pi}{m_\tau^2} S_{EW} \, |V_{fg}|^2 \int_{r_{fg}}^{\infty} \mathrm{d}\omega \left\{ \mathcal{H}^T(\omega) K_\sigma^T(\omega) - \mathcal{H}^L(\omega) K_\sigma^L(\omega) \right\} =$$
$$= 12\pi S_{EW} \, |V_{fg}|^2 \sum_t \frac{2\pi}{m_\tau^3} \left\{ g^T(\sigma,t) C^T(t) + g^L(\sigma,t) C^L(t) \right\} \tag{11}$$

$$C^L(t) = -C^{00}(t)\,, \qquad C^T(t) = \frac{1}{3} \sum_i C^{ii}(t) \tag{12}$$

- The correlators are known at finite volume

$$R_{fg} \equiv \lim_{\sigma \to 0} \lim_{L \to \infty} R_{fg}^{\text{lattice}}(\sigma) \tag{13}$$

The order of the limit can not be inverted!!!

$$W[\boldsymbol{g};\lambda,\kappa] = A_\alpha[\boldsymbol{g}] + \lambda\kappa B[\boldsymbol{g}] \tag{14}$$

- The coefficient $\boldsymbol{g}$ necessary for the reconstruction can be obtained by minimizing the functional w.r.t. the trade off parameter λ for some choice of κ.
- $A_\alpha[\boldsymbol{g}]$ measures the difference between the true and the reconstructed kernel

$$A_\alpha[\boldsymbol{g}] = \frac{\int_{E_0}^{E_{\text{max}}} \mathrm{d}\omega\, e^{\alpha m_\tau \omega} \left| \sum_{t=1}^{t_{\text{max}}} g(t) e^{-t\omega m_\tau} - K(\omega) \right|^2}{\int_{E_0}^{E_{\text{max}}} \mathrm{d}\omega\, e^{\alpha m_\tau \omega} \left| K(\omega) \right|^2}$$

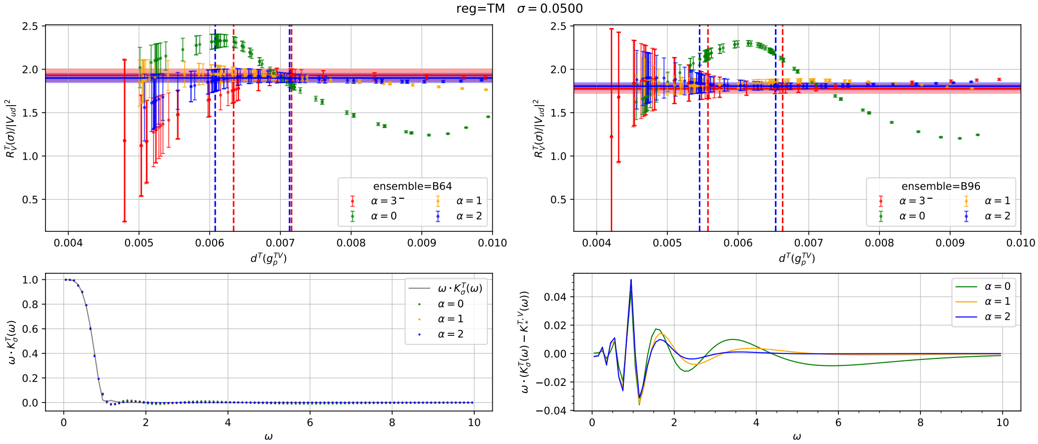
- $B\,[\boldsymbol{g}]$ is the regulator using the covariance matrix of the lattice correlators $\text{Cov}\,(t_1, t_2)$

$$B\,[\boldsymbol{g}] \propto \boldsymbol{g}^T \text{Cov}\,\big(C(t)\big)\, \boldsymbol{g}$$

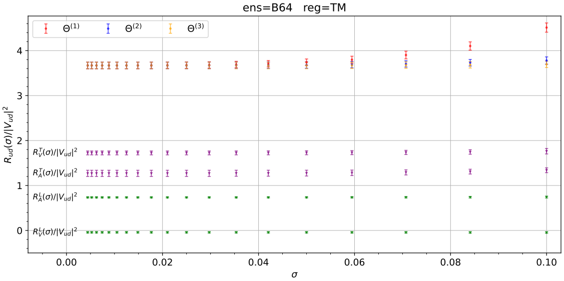
$$\begin{cases} A_\alpha\,[\boldsymbol{g}^\star] = 10^4 B\,[\boldsymbol{g}^\star] \\ A_\alpha\,[\boldsymbol{g}^{\star\star}] = 10^3 B\,[\boldsymbol{g}^{\star\star}] \end{cases} \qquad \longrightarrow \qquad \Delta^{sys} = \left| R_\sigma^{(10^4)} - R_\sigma^{(10^3)} \right| \cdot \text{erf}\left(\frac{\left| R_\sigma^{(10^4)} - R_\sigma^{(10^3)} \right|}{\sqrt{2}\Delta_{\text{stat}}^{(10^3)}} \right)$$

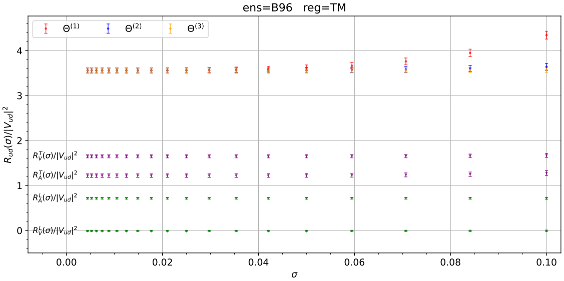
- With $E_{\text{max}} = \infty \quad \longrightarrow \quad \alpha < 2 \qquad$ or $\alpha > 2$ are allowed with E_{max} finite

- At increasing α the stability improves $\longrightarrow$ central values are less oscillation exploring $d(\boldsymbol{g}) \equiv \sqrt{A_0\,[\boldsymbol{g}]}$



- Partial and total results are very stable in σ.
  All the different representations of the θ–function approach to the same results



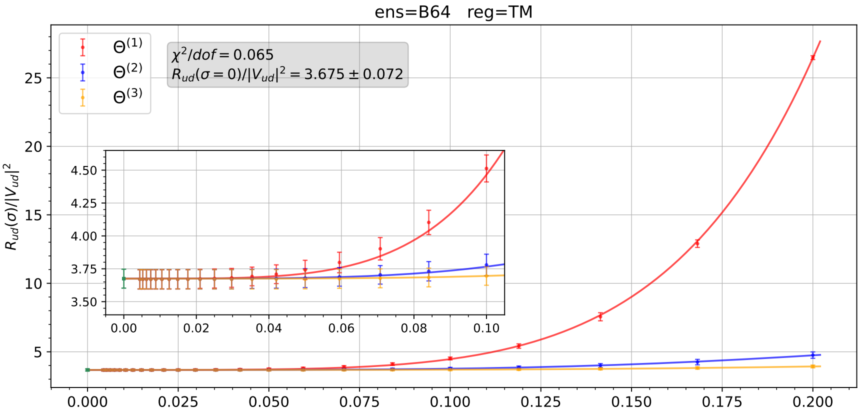


- We used three different regularization of the Heaviside step–function

$$\Theta_\sigma^{(1)}(\omega) = \frac{1}{1+e^{-\omega/\sigma}}\,, \qquad \Theta_\sigma^{(2)}(\omega) = \frac{1}{1+e^{-\sinh\omega/\sigma}}\,, \qquad \Theta_\sigma^{(3)}(\omega) = \frac{1+\mathrm{Erf}\,(\omega/\sigma)}{2}\,. \tag{15}$$

On each ensemble and for each regularization, the results corresponding to the three smearing kernels $\Theta_\sigma^{(k)}$ $(k = 1, 2, 3)$ have been fitted by using the following ansatz

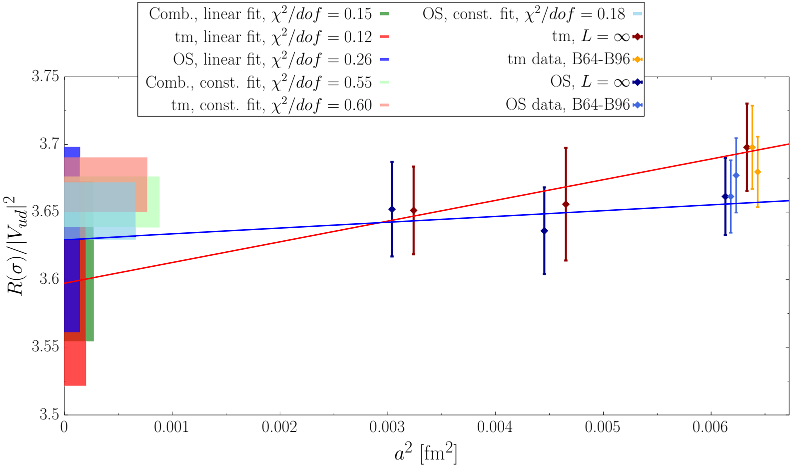
$$R_k(\sigma) = R + c_{1,k}\cdot\sigma^4 + c_{2,k}\cdot\sigma^6\,, \tag{16}$$



- HFLAV – Phys.Rev.D 107 (2023) 052008 and J.C. Hardy and I.S. Towner – Phys. Rev. C 102 (2020) 045501

$$\frac{R}{|V_{ud}|^2} = 3.6615(78)$$

- Our continuum limit at $\sigma = 0.004$



- Study of $|V_{us}|$ analysing correlators with relevant currents us–flavoured, phenomenologically more interesting;
- Study the Strong and Electromagnetic isospin–breaking corrections that in this process could be relevant.

Thank you for the attention!

Backup

$$O_\sigma = \int_0^{+\infty} \mathrm{d}\omega \Theta\left(\frac{\omega^\star - \omega}{\sigma}\right) \rho(\omega) \cdot \mathrm{kin} = \int_0^{+\infty} \mathrm{d}\omega \Theta\left(\frac{\omega^\star - \omega}{\sigma}\right) (\omega^\star - \omega)^m \tilde{\rho}(\omega) \cdot \tag{17}$$

- In our case $m = 2$ and $\omega^\star = 1$

$$\begin{aligned}
\Delta O_\sigma(2) &\equiv \int_0^{+\infty} \mathrm{d}\omega \left\{ \Theta\left(\frac{1-\omega}{\sigma}\right) - \theta(1-\omega) \right\} (1-\omega)^2 \tilde{\rho}(\omega) = \\
&= \sigma^3 \int_{-\infty}^{\frac{1}{\sigma}} \mathrm{d}\omega \left\{ \Theta(\omega) - \theta(\omega) \right\} \omega^2 \tilde{\rho}(1-\sigma\omega) = \\
&= \sigma^3 \left\{ \int_0^{\frac{1}{\sigma}} \mathrm{d}\omega \left\{ \Theta(\omega) - 1 \right\} \omega^2 \tilde{\rho}(1-\sigma\omega) + \int_{-\infty}^{0} \mathrm{d}\omega\, \Theta(\omega) \omega^m \tilde{\rho}(1-\sigma\omega) \right\} = \\
&= \sigma^3 \left\{ \int_0^{+\infty} \mathrm{d}\omega \left\{ \Theta\left(\omega\right) - 1 \right\} \omega^2 \left[\tilde{\rho}(1-\sigma\omega) - \tilde{\rho}(1+\sigma\omega) \right] \right\}
\end{aligned} \tag{18}$$

$$\tilde{\rho}(\omega) = \tilde{\rho}_{reg}(\omega) + \tilde{\rho}_{+}(\omega)\theta(1-\omega) + \sum_{d=0}^{N_d} \tilde{\rho}_d \, \partial^d \, \delta(1-\omega)\,. \tag{19}$$

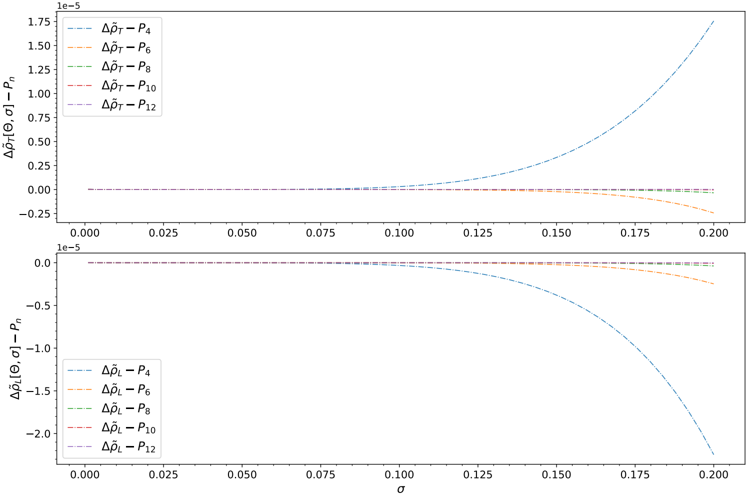
$$C_\Theta^n = \int_0^\infty \left\{\Theta(\omega) - 1\right\} \omega^n \,, \tag{20}$$

$$\Delta^{\text{reg}} O_\sigma(2) = \sum_{n=0}^{\infty} \frac{\tilde{\rho}^{(n)}(1)}{n!} \left[(-1)^n - 1\right] C_\Theta^{n+2} \sigma^{n+3} \qquad \to \quad \mathcal{O}\Big(\sigma^4\Big) \tag{21}$$

$$\Delta^{\Theta} O_\sigma(2) = -\sum_{n=0}^{\infty} \frac{\tilde{\rho}_+^{(n)}(1)}{n!} \sigma^{n+3} C_\Theta^{n+2} \qquad \to \quad \mathcal{O}\Big(\sigma^3\Big) \tag{22}$$

$$\Delta^{\delta} O_\sigma(2) = \sum_{d=0}^{N_d} (-1)^{d+3} \tilde{\rho}_d \, \partial^d \left[\left(\Theta\left(\frac{\omega}{\sigma}\right) - 1\right) \omega^2\right]_{\omega=0} \qquad \to \quad \mathcal{O}\Big(\sigma^0\Big) \tag{23}$$

- By assuming $\rho(\omega)$ regular and trivially equal to 1, we can consider the spectral densities $\begin{cases} \tilde{\rho}_T = \frac{(1+2\omega^2)(1+\omega)^2}{\omega} \\ \tilde{\rho}_L = \frac{(1+\omega)^2}{\omega} \end{cases}$
- $\Delta^{\text{reg}} O_\sigma(2) - P_n$, where P_n is the asymptotic series truncated at order n in σ. $\Theta_\sigma^{(3)}$ case.



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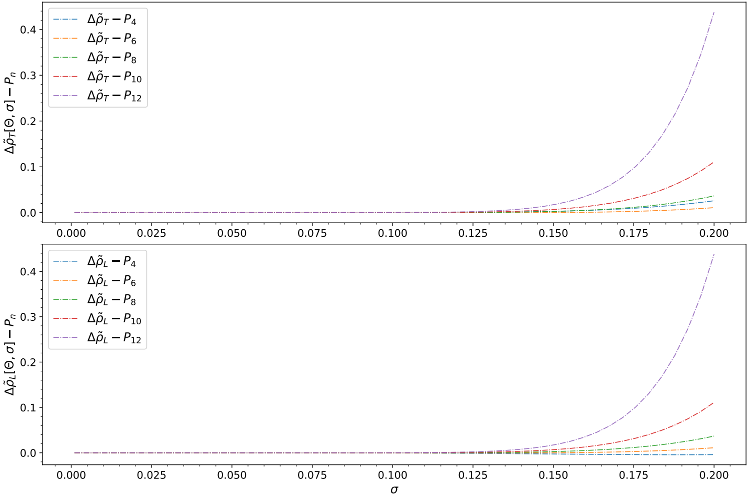


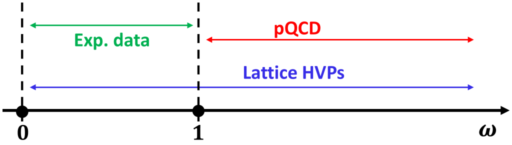
- HVPs $\Pi(Q^2)$ $\longleftrightarrow$ spectral densities ρ (see P. A. Boyle et al. – PhysRevLett.121.202003 for this approach)

$$\Pi^{\mu\nu}(q^2) = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{iqt} C^{\mu\nu}(t) = \frac{m_\tau^2}{2\pi}\int_0^{\infty} \mathrm{d}\omega^2\, \frac{\mathcal{H}^{\mu\nu}(\omega)}{q^2 + m_\tau^2\omega^2} \equiv \left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\Pi_T(q^2) + q^\mu q^\nu \Pi_L(q^2)\,.$$

$$\Pi_J(q^2) - \Pi_J(0) = \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{e^{iqt}-1}{q^2}\, C_J(t) = \frac{m_\tau^2}{2\pi}\int_0^{+\infty} \mathrm{d}\omega^2\, \frac{\rho_J(\omega^2)}{q^2 + \omega^2 m_\tau^2}\,,$$

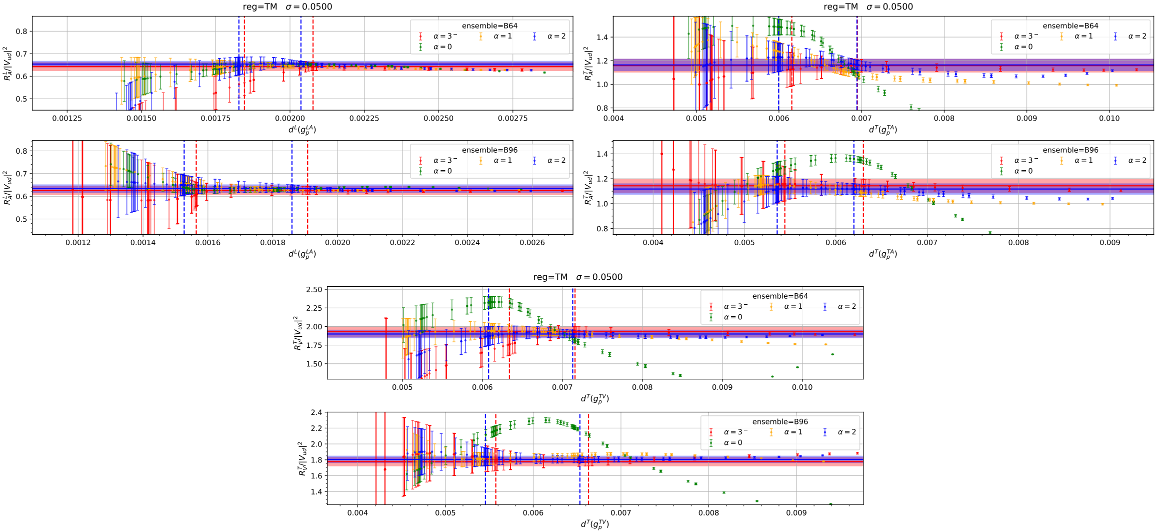
$$\tilde{\rho}_{us}(\omega^2) = \left(1 + 2\omega^2\right)\rho_{us}^{(T)}(\omega^2) + \rho_{us}^{(L)}(\omega^2) \qquad \tilde{\Pi}_{us}(q^2) = \left(1 - 2\frac{q^2}{m_\tau^2}\right)\Pi_{us}^{(T)}(q^2) + \Pi_{us}^{(L)}(q^2)$$

$$\int_0^{\infty} \mathrm{d}\omega^2\, \tilde{\rho}_{us}(\omega^2) w_N(\omega^2) = \frac{2\pi}{m_\tau^2}\sum_{k=1}^{N} \frac{\tilde{\Pi}_{us}(Q_k^2)}{\prod_{j\neq k}(Q_j^2 - Q_k^2)} \equiv \tilde{F}_{w_N}\,, \quad \text{where} \quad w_N(\omega) = \prod_{k=1}^{N} \frac{1}{\omega^2 m_\tau^2 + Q_k^2}$$

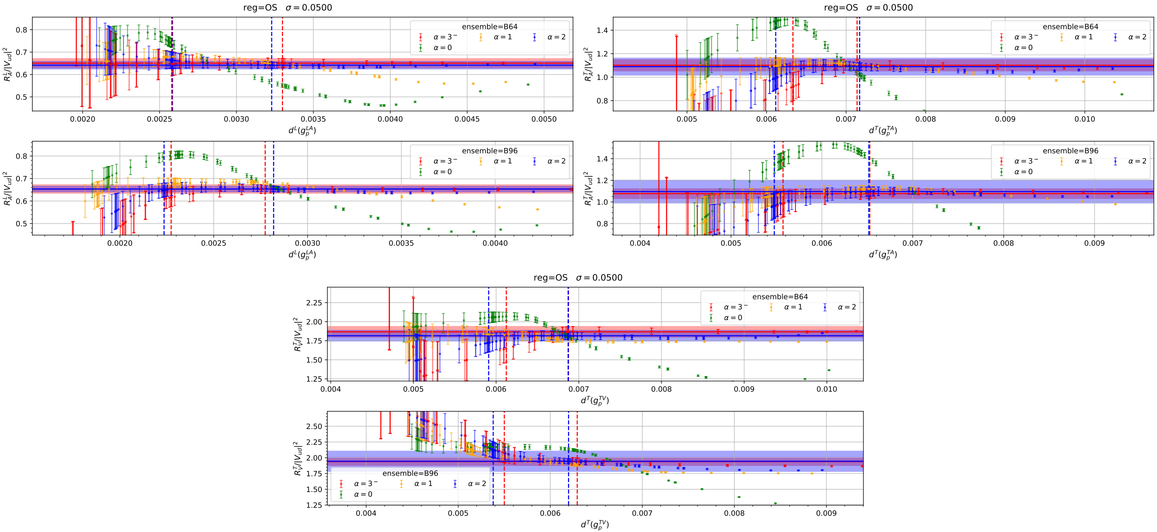


$$|V_{us}|^2 = \frac{R_{us}^{\mathrm{exp}}}{\mathrm{HVPs} - \mathrm{pQCD}}$$

- Stability plots for the results obtained from all axial and vectorial correlators in TM regularization at $\sigma = 0.05$
- $\alpha = 3^-$ with $E_{\max} = 4a^{-1}$ $\longrightarrow$ better understanding required (e.g. independence from $E_{\max}$, central values extraction point, error budget: systematic + statistic)



- Example of stability plots for the results obtained from all axial and vectorial correlators in OS regularization at $\sigma = 0.05$



- Example of stability plots for the results obtained from all axial and vectorial correlators in the two volumes at $\sigma = 0.05$