Hadronic vacuum polarisation contributions to anomalous muon magnetic moment

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Anomalous muon magnetic moment

The anomalous magnetic moment of the muon is defined as $a_\mu = (g-2)/2$, where g is the Landé g-factor, proportional to the muon's intrinsic magnetic moment. In Dirac's relativistic quantum theory g = 2 exactly, but in the Standard Model (SM) of particle physics it gets tiny corrections from the electromagnetic, strong, and weak interactions. Experimental results of magnetic moment from Brookhaven national lab is 3.7σ larger than the standard model prediction and the new result from Fermi lab has larger discrepancy of 4.2σ with the Standard Model.

Reasons

1. The reason for discrepancy between experiment and theory lies in the computation of the quantum loop corrections – loops with photons and leptons, electroweak contribution and hadronic contribution.
2. Hadronic contribution can be categorised into hadronic vacuum polarization (HVP) and light-by-light scattering.
3. In HVP, a virtual photon briefly explodes into a ”hadronic blob", before being reabsorbed, while the real photon (corresponding to magnetic field) is simultaneously absorbed by the muon.

Hadronic vacuum polarization

The total HVP contribution to a_μ comes from both connected and disconnected-quark line diagrams for each flavor of quark in nature. The u, d quark-connected contributions are by far the largest, thus requiring the highest precision

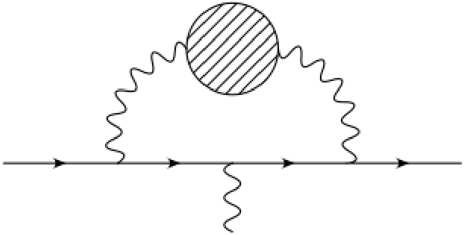


Figure 1: The quark connected diagram contributing to the hadronic vacuum polarization contribution to the muon anomaly.

We begin with the computation of loop graph in perturbation theory of the vertex function without hadronic contribution. After applying Feynman rules and taking the limit $q^2 \to 0$, the a_μ from the contribution of the diagram in Fig (1) reads

$$a_\mu = e^2 i \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[(p-k)^2 + m_\mu^2 - i\epsilon\right]^2} \frac{1}{k^2 - i\epsilon} \left(\frac{16(p\cdot k)^2}{3m_\mu^2} + \frac{4}{3}k^2 + 4(p\cdot k) \right) \tag{1}$$

Next step is to do Wick rotation into Euclidean spacetime and performing the angular integrations and analytically continuing $p^2 \to -m_\mu^2$ on shell for the external momenta,

$$a_\mu = \frac{\alpha}{\pi} \int_0^\infty dK^2 \frac{m_\mu^2 K^2 Z^3 \left(1 - K^2 Z\right)}{1 + m_\mu^2 K^2 Z^2} = \frac{\alpha}{\pi} \int_0^\infty dK^2 f\left(K^2\right) \qquad (2)$$

where,

$$Z = -\frac{K^2 - \sqrt{K^4 + 4m_\mu^2 K^2}}{2m_\mu^2 K^2}$$

The loop does not affect the rest of the integral and also it depends only on K^2, we can insert its contribution into eq(2) to obtain the $\mathcal{O}(\alpha^2)$ hadronic contribution.

$$a_\mu^{\mathrm{HVP}} = 4\alpha^2 \int_0^\infty dq^2 f\left(q^2\right) \hat{\Pi}\left(q^2\right) \tag{3}$$

$f\left(q^2\right)$ is known, $\hat{\Pi}\left(q^2\right)$ is subtracted HVP, $\hat{\Pi}\left(q^2\right) = \Pi\left(q^2\right) - \Pi(0)$, computed directly on Euclidean space-time lattice from the Fourier transform of the two-point current function

$$\Pi^{\mu\nu}(q) = \int e^{iqx} \langle j^\mu(x) j^\nu(0)\rangle \quad = \Pi\left(q^2\right)\left(q^\mu q^\nu - q^2 \delta^{\mu\nu}\right) \tag{4}$$

writing everything in time-momentum representation -

$$\Pi\left(q^2\right) - \Pi(0) = \sum_t \left(\frac{\cos qt - 1}{q^2} + \frac{1}{2}t^2\right) C(t) \tag{5}$$

$$C(t) = \frac{1}{3}\sum_{x,i} \langle j_i(x) j_i(0)\rangle \tag{6}$$

$$a_\mu^{HVP} = \sum_t w(t) C(t) \tag{7}$$

Correlation function that we use in our calculation is 2-point current-current correlation function

$$\begin{aligned} C(t_x - t_y) &= \sum_{\vec{x},\vec{y}} \langle J_\mu(\vec{x}) J_\nu(\vec{y})\rangle \\ &= \langle j_\mu(x) j_\nu(y)\rangle \\ &= \left\langle [\bar{\psi}\gamma_\mu\psi](x)\bar{\psi}\gamma_\nu\psi](y)\right\rangle \\ &= \left\langle M^{-1}_{y,x}\gamma_\mu M^{-1}_{x,y}\gamma_\nu\right\rangle \end{aligned} \tag{8}$$

The structure of staggered Dirac operator, M is

$$M \begin{pmatrix} n_o \\ n_e \end{pmatrix} = \begin{pmatrix} m & M_{oe} \\ M_{eo} & m \end{pmatrix} \begin{pmatrix} n_o \\ n_e \end{pmatrix} = (m + i\lambda_n) \begin{pmatrix} n_o \\ n_e \end{pmatrix} \tag{9}$$

For a typical size, Dirac operator is a sparse matrix. In order to improve the convergence of conjugate gradient, we used a pre-conditioned operator. In our case, pre-conditioned operator is just $\mathbf{M}^\dagger\mathbf{M}$. This gives us the same set of eigenvectors but the eigenvalues would be squared of the Dirac operator. Preconditioning is useful because it reduces the condition number of the problem

We are interested to compute the inverse of Dirac matrix and our approach is

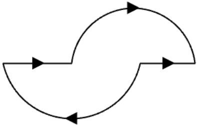
$$M^{-1}S = \psi \Rightarrow \psi = (M^\dagger M)^{-1} M^\dagger S \tag{10}$$

On the lattice the current is not conserved, so we use a point-split current that is exactly conserved,

$$J^{\mu}(x) = \frac{1}{2}\eta_{\mu}(x)(\bar{\chi}(x+\hat{\mu})U^{\dagger}_{\mu}(x)\chi(x) + \bar{\chi}(x)U_{\mu}(x)\chi(x+\hat{\mu})) \tag{11}$$

The 2-point split current-current function becomes

$$4\sum_{\vec{x},\vec{y}}\langle J_{\mu}(x)J_{\nu}(y)\rangle = -\sum_{m,n}\sum_{\vec{x},\vec{y}}\frac{1}{\lambda_m\lambda_n}\Big(\Lambda^{\dagger}_{\mu}(x)_{mn}\Lambda^{\dagger}_{\nu}(y)_{nm} + \Lambda^{\dagger}_{\mu}(x)_{mn}\Lambda_{\nu}(y)_{nm} + \Lambda_{\mu}(x)_{mn}\Lambda^{\dagger}_{\nu}(y)_{nm} + \Lambda_{\mu}(x)_{mn}\Lambda_{\nu}(y)_{nm}\Big) \tag{12}$$



Here λ_n is short for $m \pm i\lambda_n$ with
eigenvector ordering $\lambda_0, -\lambda_0, \lambda_1, -\lambda_1, \ldots, \lambda_{N_{\mathsf{low}}}, -\lambda_{N_{\mathsf{low}}}$,
the sums of labels m and n run up to $2N_{\mathsf{low}}$.

$\left(\Lambda_\mu(t)\right)_{n,m}$ is what we called meson fields,

$$\left(\Lambda_\mu(t)\right)_{n,m} = \sum_{\vec{x}} \langle n \mid x \rangle \eta_\mu(x) U_\mu(x) \langle x + \mu \mid m \rangle (-1)^{(m+n)x+m} \tag{13}$$

where the factor $(-1)^{(m+n)x+m}$ arise from the odd sites sign difference between n_+ and n_-. $\left(\Lambda_\mu(t)\right)_{n,m}$ is the building block for the LMA part of the lattice computation and take the majority of the computation resources. The low-mode eigenvectors are also used to deflate the CG for quark propagators.

Low mode and all mode averaging

The aim of lattice calculations is to effciently obtain $C(t)$ with as much precision as possible. In HVP calculation, we combine AMA and full-volume LMA as an improved estimator,

$$\langle O \rangle = \langle O \rangle_{\text{exact}} - \langle O \rangle_{\text{appx}} + \frac{1}{N}\sum_i \langle O_i \rangle_{\text{approx}} - \frac{1}{N}\sum_i \langle O_i \rangle_{\mathrm{LM}} + \frac{1}{V}\sum_i \langle O_i \rangle_{\mathrm{LM}}$$

To do this we use the techniques of LMA(low mode average) and AMA(all-mode average)[1]. Both methods rely on the spectral decomposition of the quark propagator in terms of the eigenvectors of the lattice Dirac operator.

1. Schematically, the quark propagator from source point y to sink point x, $S(x,y)$, is given by the inverse of the Dirac operator

$$S(x,y) = D^{-1}(x,y) = \sum_{\lambda} \frac{\langle x|\lambda\rangle \langle \lambda|y\rangle}{\lambda} \tag{14}$$

[1] Blum, Izubuchi, and Shintani, "New class of variance-reduction techniques using lattice symmetries".

2. Then divide the summation into two parts - low modes and high modes

$$\begin{aligned} S(x,y) &= \sum_{\lambda \leq \lambda_{low}} \frac{\langle x|\lambda\rangle \langle\lambda|y\rangle}{\lambda} + \sum_{\lambda > \lambda_{low}} \frac{\langle x|\lambda\rangle \langle\lambda|y\rangle}{\lambda} \\ &= \sum_{\lambda \leq \lambda_{low}} \frac{\langle x|\lambda\rangle \langle\lambda|y\rangle}{\lambda} + D^{-1}(x,y) \left(1 - \sum_{\lambda \leq \lambda_{low}} \frac{\langle x|\lambda\rangle \langle\lambda|y\rangle}{\lambda}\right) \\ &= S_L + S_H \end{aligned} \tag{15}$$

3. And finally $C(t)$ is determined by the sum of four parts,

$$C_{\mu\nu} = \sum_{x,y} \mathrm{Tr}\, \gamma_\mu S(x,y) \gamma_\mu S(y,x) = C_{LL} + C_{LH} + C_{HL} + C_{HH} \quad (16)$$

In practice, the low modes are computed directly, using an effcient version of the Lanczos algorithm and once the low modes are determined, S_H is determined by computing the inverse of the deflated Dirac operator, using the conjugate gradient algorithm

Computational techniques

1. Eigenvectors: The low modes of the preconditioned Dirac operator are computed 8 with the implicitly restarted Lanczos (IRL) algorithm.
2. C_{LL}: The LL part of the correlation function is constructed from products of inner- products of eigenvectors and scales linearly in size of the eigenvectors and quadratically with the number of eigenvectors[2].
3. $C_{HL/LH}$: We calculate the deflated (low-mode pro- jected, or high mode part of the) quark propagator on each low mode on a time slice. We will work to reduce this dominant cost by testing and tuning a new block-conjugate gradient solver and the split-grid technique to speed up the solves.
4. C_{HH}: The high-high part of the correlation function dominates the small Euclidean time part of $C(t)$, which has exponentially smaller errors. Here we simply replicate our previous AMA-style calculation with deflated, approximate, conjugate gradient solves from the quark propagator on source points.

[2] Blum et al., "Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment".

What am I doing currently?

The full volume average of the low modes which shows even smaller fluctuations. This means there is still significant noise in the $C_{HL(LH)}$ part of the correlation function.

1. The low modes of the preconditioned Dirac operator are computed with the Lanczos (IRL) algorithm.
2. Using these eigenvectors we construct sources.
3. Calculate the deflated quark propagator on a time slice.
4. We have results working fine for $4^3 \times 4$ lattice. Hope to get results for lattices ad big as like $96^3 \times 192$

Thank you!