Lattice QCD
and its phenomenological applications

Introduction to Lattice QCD - 1

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lattice virtual academy

A sketchy (and subjective) timeline

1930: Quantum Electrodynamics

1940: Feynman diagrams, path integral

1950: Renormalization, deal with divergencies

1954: Yang-Mills theories

1973: Quantum Chromodynamics

1976: Ken Wilson, Cargese lectures
“present methods for solving field theories do not work for strong coupling”

1970+ : non-perturbative methods, lattice field theory

1977: Giorgio Parisi, Cargese lectures
“ we do not know yet how to get correct answers,
but we begin to understand which are the right questions to ask “

1980+: Complexity

1990+: String theories, CFT

Examples:

- Hadron structure
- The enigma of the muon g-2
- The strong coupling constant
- Flavor Physics for BSM searches

“Lattice 88”

by Paul Mackenzie

The ability to understand the properties of the strongly interacting particles from first principles is a 40-year-old dream which is now approaching reality. Following the development of quantum chromodynamics (QCD) in the early 1970s, honest calculations of the masses and other properties of hadrons were made possible by Ken Wilson’s inventions of lattice gauge theory and renormalization group methods. Lattice gauge theory became a major industry around 1980, when Monte Carlo methods were introduced, and the first prototype calculations of the hadron spectrum yielded qualitatively reasonable results. This past year has seen the most powerful attacks yet on the theory of the strong interactions and the richest variety of physics results.

(Fermilab photograph 88-961-9)

Peter Hasenfratz discusses the bounding of the mass of the Higgs.

(Fermilab photograph 88-962-22)

A. A. Migdal shown here "putting strings on the lattice."

QCD - why a numerical approach

QED vs QCD

- Photons do not carry charge
- Free electrons and free photons exist
- Interactions are strong at short distance - Coulomb force

- Gluons are charged
- Free quarks and gluons do not exist: confinement?
- Interactions are faible at short distance: asymptotic freedom

A theory with only photons
Is free

A theory with gluons only
is interacting and Interesting

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f_{abc} A_\mu^b A_\nu^c$$

QED vs QCD vs Yang-Mills

- Photons do not carry charge
- Free electrons and free photons exist
- Interactions are strong at short distance - Coulomb force

A theory with only photons
Is free

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Gluons are charged
- Free quarks and gluons do not exist: confinement?
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A theory with gluons only (Yang-Mills)
is interacting and Interesting

$$F^{(a)}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$$

From QED to Yang-Mills theories

Electrodynamics:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\nabla_\mu - m)\psi$$

Infinite mass

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\nabla_\mu - m)\psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Free photons

Yang-Mills

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F^{(a)}_{\mu\nu}F^{(a)\mu\nu}$$

Gluons

$$F^{(a)}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$$

Self-interacting gluons

QCD : motivation for a on-perturbative approach

Confinement: quarks and gluons are not observed as asymptotic states

Breaking of chiral symmetry: due to the coupling becoming large at large distance

Topological properties: non-existent at any order in perturbation theory

Calculational schemes
from real to imaginary time

Correlators, correlation length, masses

General calculations scheme:

Rotate to imaginary time $x_0 \equiv t \rightarrow -ix_4 \equiv -i\tau$

$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, e^{-S}$$

<— note: Euclidean space time

$$\mathcal{S} = \int d^4x \, \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \overline{\psi} M \psi\right) .$$

A RFT in d space dimensions becomes a statistical field theory in d+1 dimensions

Integrate out fermions

$$Z = \int \mathcal{D}A_\mu \, \mathrm{det} M \, e^{\int d^4x \, (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu})} .$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S} .$$

$$S = S_{gauge} + S_{quarks} = \int d^4x \, \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) - \sum_i \log(\mathrm{Det} M_i)$$

Minkowski —> Euclidean

Green functions —> Correlation functions

In many cases correlation functions decay exponentially at large distance:

$$\lim_{t\to\infty} < O(t)O(0) > \propto e^{-t/t_0}$$ t_0 correlation length

Back to Minkwoski

$$\int dt e^{ip_0 t} \frac{e^{-t/t_0}}{2t_0} = \frac{1}{p_0^2+\frac{1}{t_0^2}}$$

$$\to_{p0\to iE} = \frac{1}{1/t_0^2-E^2}$$

Mass = inverse correlation length

Minkowski —> Euclidean

Green functions —> Correlation functions

$$\lim_{t\to\infty} < O(t)O(0) > \propto e^{-t\,M}$$

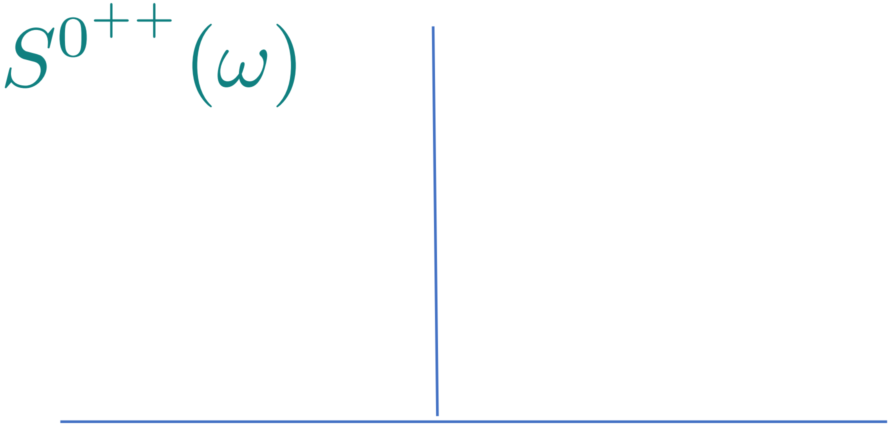
M = lowest excitation in the channel which couples to *O*

From real time to real frequency space:

In imaginary time G(t) $$G(t) = \int \delta(M-\omega)e^{-\omega t} \propto e^{-Mt}$$

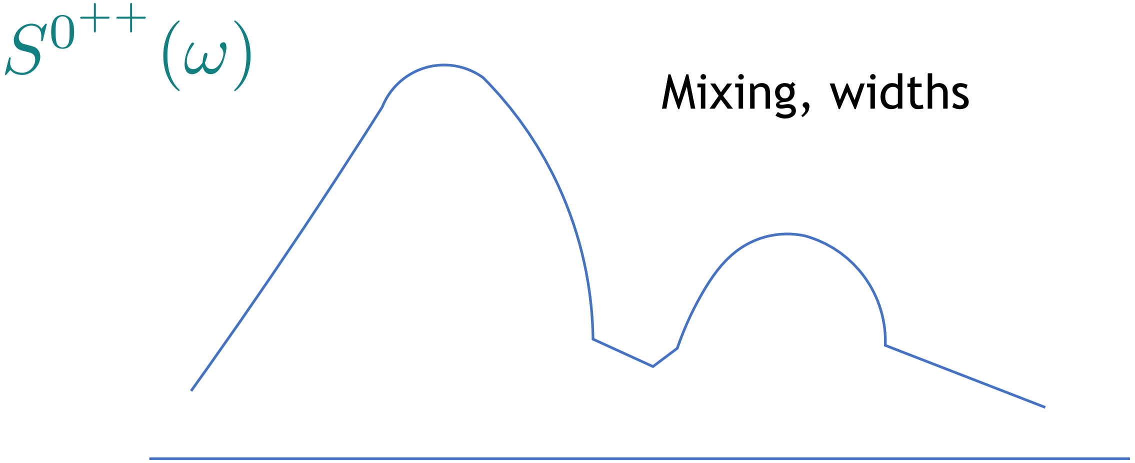
In real frequency space: $$\delta(M-\omega)$$

Spectral functions and two point functions : a challenge for LFT



Yang-Mills

$$G(t) = \int \delta(M-\omega)e^{-\omega t} \propto e^{-Mt}$$



QCD

$$G(t) = \int S(\omega)e^{-\omega t}$$

FT Euclidean space — take home message

Complete equivalence between Minkowski FT in d space dimension with statistical field theory in d+1 dimension

Grand Canonical Partition Function defines all the observables of the theory

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S}$$

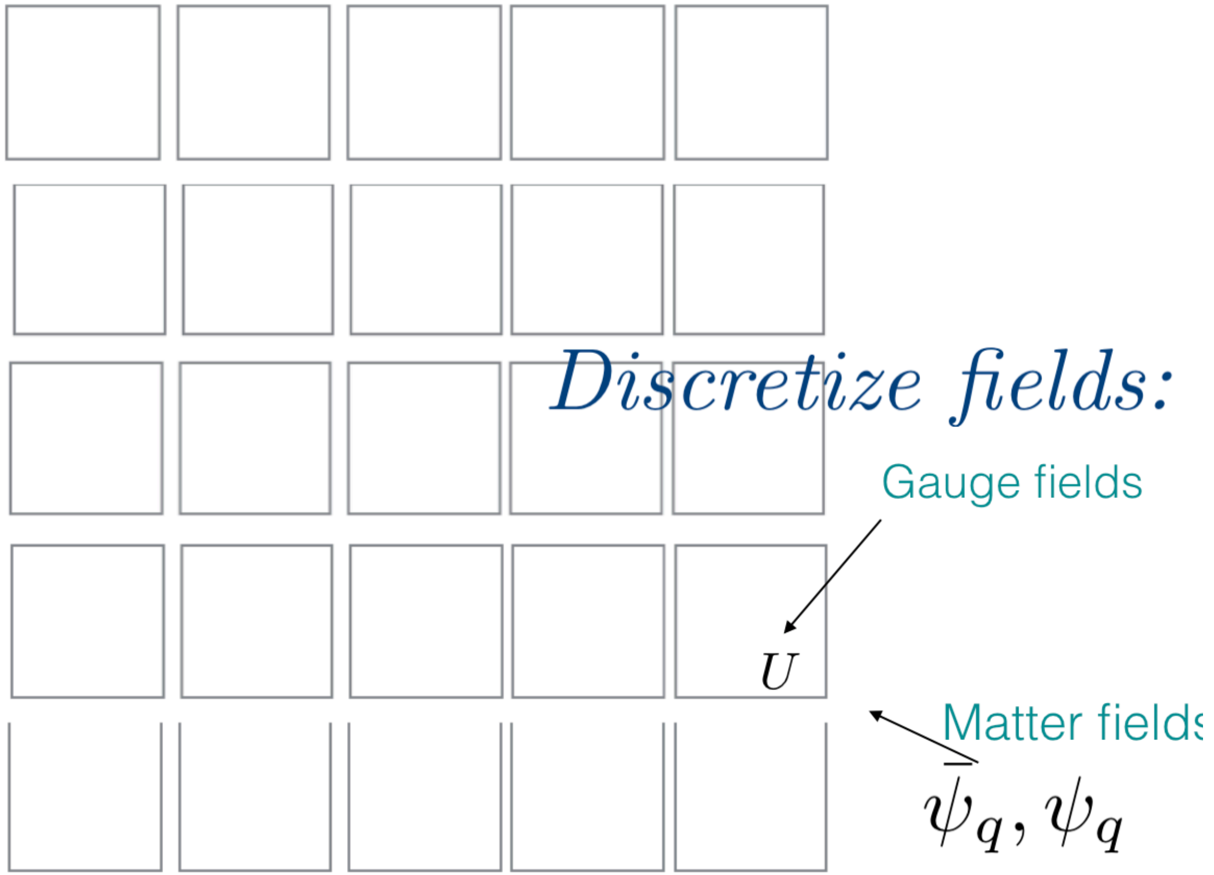
Exponential decays of Euclidean two point functions —> mass of the lowest excitation in that channel

More general functional forms may appear, which require a dedicated analysis

Computational Strategy:

1) Rotation to imaginary time + discretisation

Lattice Gauge Theory



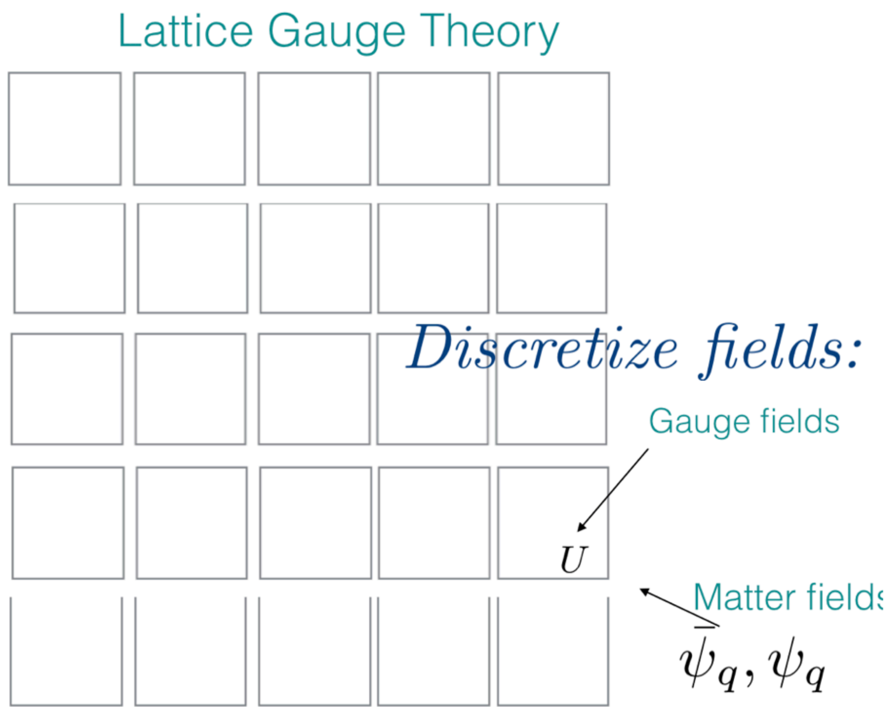
Computational Strategy:

1) Rotation to imaginary time + discretisation

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \; \mathcal{O} \; e^{-S}$$

2) Monte Carlo Simulation

Performing the integration



$$\langle 0|\mathcal{O}|0\rangle = \frac{\sum_\alpha \mathcal{O}_\alpha e^{-S_\alpha}}{\sum_\alpha e^{-S_\alpha}}.$$

Discretising - general issues

—Discretization: from continuum space to a grid

—Why? Two standard motivations:

1. Physical system intrinsically discrete (i.e. spin models)
2. **Make it amenable to a numerical study —> QCD**

— Discretization is in principle trivial:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{f(a)}{2} + \sum_{k=1}^{n-1}\left(f\left(a + k\frac{b-a}{n}\right)\right) + \frac{f(b)}{2}\right)$$

— Already in this simple example:

.Strategies for improvement?

.How to check the ‘**continuum limit?’**

.Suppose a, b $\to \infty$

How to check **convergence to infinite volume?**

— Slightly more complicated: increase the dimensionality, make the function less smooth..

Computational costs??

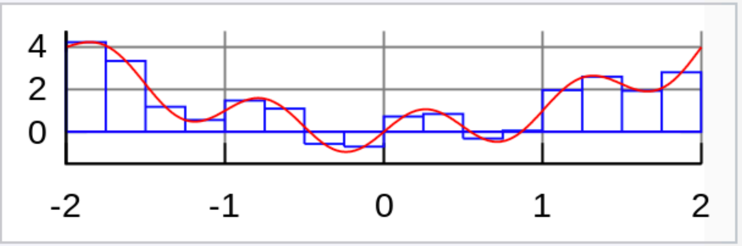


Illustration of the rectangle rule.

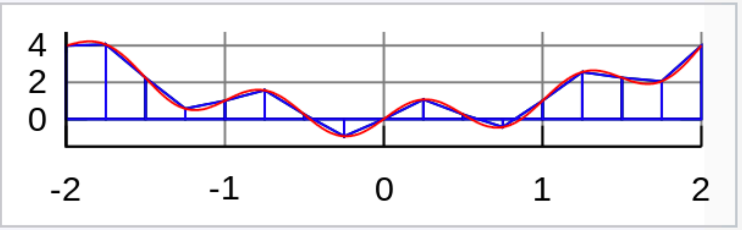


Illustration of the trapezoidal rule.

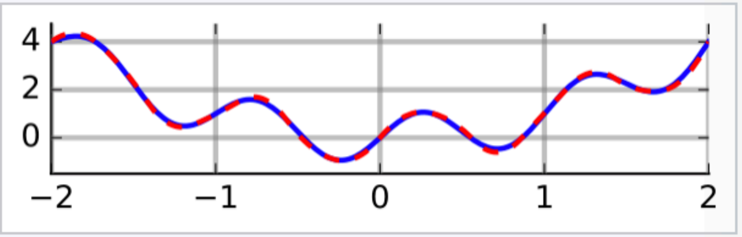


Illustration of Simpson's rule.

Matter (scalar) fields: on sites

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\phi^4.$$

$$S = \sum_n a^4 \left(\frac{1}{2} \sum_{\mu=1}^{4} \left[\frac{\phi(n+1_\mu) - \phi(n-1_\mu)}{2a} \right]^2 + \frac{1}{2}m^2\phi^2(n) + \lambda\phi^4(n) \right)$$

Gauge fields

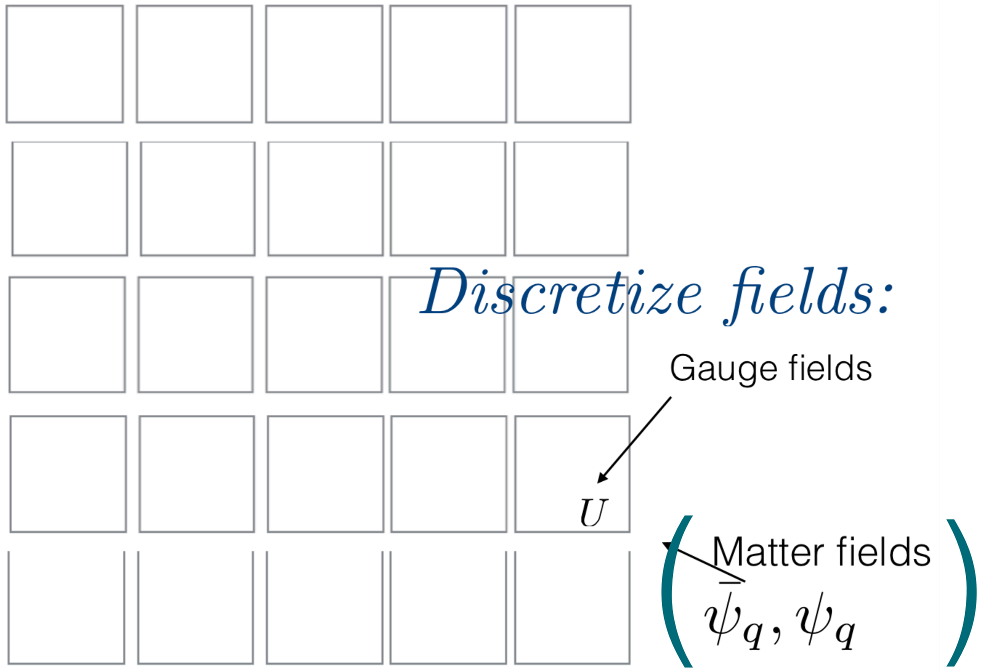
$$U_\mu(n) = \exp(igaT^a A^a_\mu(n))$$

SU(3) Matrix

$$\phi_s(y) \to P(\exp\ ig \int_s dx_\mu A_\mu)\phi(x) \equiv U(y,x)\phi(x)$$

Parallel transport:

Lattice Gauge Theory : gauge invariance 'by fiat'



$$\mathrm{Tr}\ldots U_\mu(x)U_\mu(x+\hat{\mu})\ldots \to \mathrm{Tr}\ldots U_\mu(x)V^\dagger(x+\hat{\mu})V(x+\hat{\mu})U_\mu(x+\hat{\mu})\ldots.$$

Gauge invariant

Build the Action 'by guessing'..

$$S = \frac{2}{g^2} \sum_x \sum_{\mu>\nu} \mathrm{Re}\ \mathrm{Tr}\ (1 - U_{P_{\mu\nu}}(x)).$$

? Does it work?

$$U_{P_{\mu\nu}}(x) = U_\mu(x) U_\nu(x + \hat{\mu}a) U_\mu^\dagger(x + \hat{\nu}a) U_\nu^\dagger(x)$$

Check continuum limit a-> 0

$$A_\mu(x + \hat{\nu}a) = A_\mu(x) + a\partial_\nu A_\mu(x) + \dots$$

$$U_P(x) = 1 + ia^2 F_{\mu\nu} + \dots$$

$$\beta = 2N/g^2$$

Continuum limit OK

Continuum limit

Coming back to correlation functions:

$$\lim_{t\to\infty} < O(t)O(0) > \propto e^{-t\,M}$$

M = dimensionless quantity, expressed in lattice units = $1/\xi$

M = M_{phys} * a = $1/\xi$

$$a \to 0, \xi \to \infty$$

Continuum limit: singularity !

‘g’ in the Lattice Lagrangian is the coupling at the scale ‘a’

$$a\Lambda_L = \left(\frac{1}{b_0 g^2}\right)^{b_1/2b_0^2} e^{-1/2b_0 g^2}$$

Physical scale — dimensional transmutation

$$b_0 = \left(\tfrac{11}{3}N_c - \tfrac{2}{3}N_f\right)/16\pi^2 \text{ and } b_1 = \left(\tfrac{34}{3}N_c^2 - \left(\tfrac{10}{3}N_c + \tfrac{N_c^2-1}{N_c}\right)N_f\right)/(16\pi^2)^2.$$

QCD Asymptotic freedom allows a rigorous continuum limit

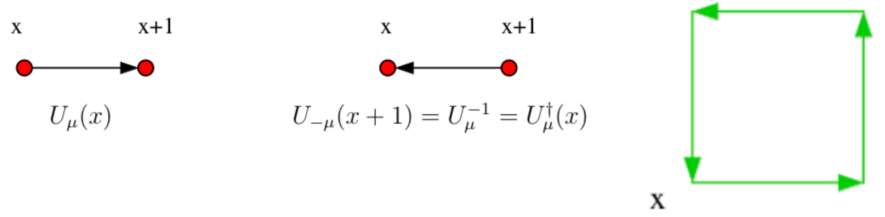
In perturbation theory, a(g) is known.

Yang-Mills, continuum and lattice

$$S_{\rm cont} = \int d^4x \frac{1}{4g^2} \,{\rm Tr}\, F_{\mu\nu}F^{\mu\nu}$$

$$S_{\rm latt} = \beta \sum_p \left(1 - \frac{1}{3}{\rm Re}\left\{{\rm Tr}\, U_p\right\}\right); \quad \beta = \frac{6}{g^2}.$$

Not unique — improvement



U : SU(3) matrix

Det = 1
U^{-1} = U*

a: 0.1 fm? 0.003fm? 1cm???

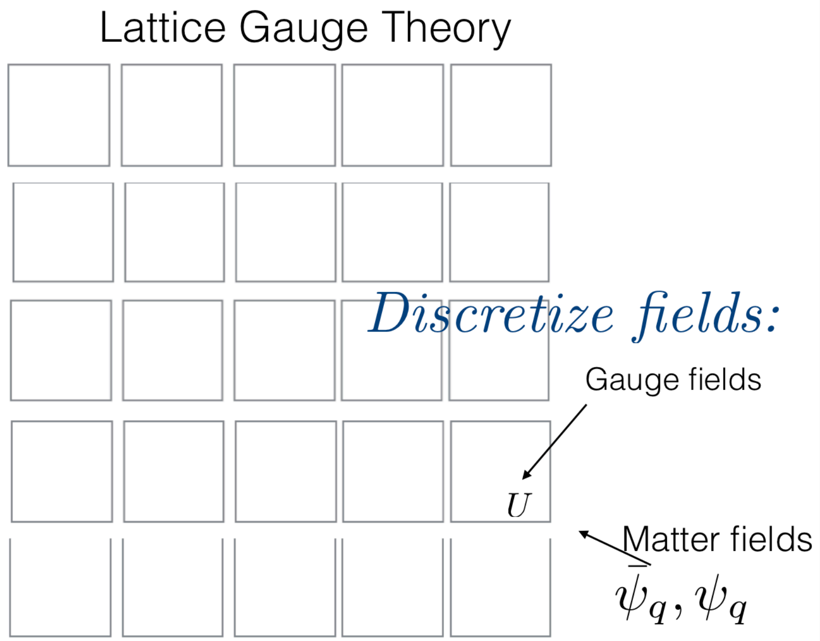
g only parameter. —> where is the spacing?

After discretising -

Back to the continuum

Planning a simulation

Parameters: $N_\sigma\,, N_\tau\,, g$



Assume we have the lattice results for some masses

$M1_{Lat}$
$M2_{Lat}$
$M3_{Lat}$

How do we get results in physical units?

Issues

— Scale setting : one physical value needed as input!

—Scaling : how strong are the discretisation effects?

—Asymptotic scaling : are we sensitive to the g=0 singularity?

An example of scale setting

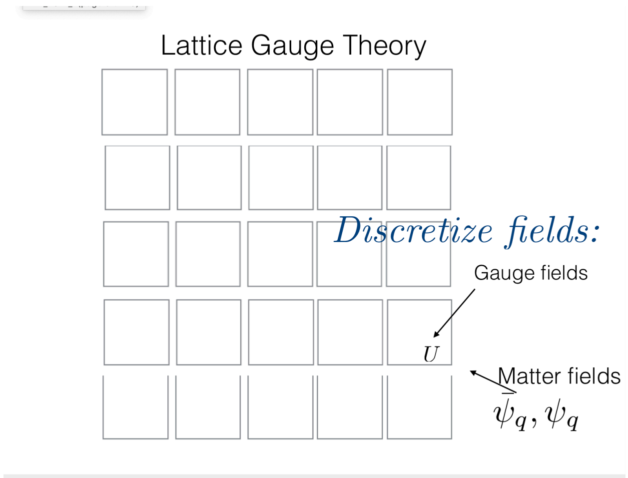
Suppose Mp=M1Latt= 0.25 is the proton mass in lattice units

g=0.2

a*Mp = 0.25 Mp = 970 MeV (approx.)

a = 0.25 *970 Mev**(-1) = 0.29*970/197 = 1fm

Knowing a, M2 and M3 can be computed



Scelte correnti : a circa 0.05 fm, Ns, Nt > 48

Scaling : repeat for different couplings and check consistencies of results - or, which is the same, check that dimensionless ratios do not depend on g

Asymptotic scaling : repeat for different couplings, and check consistency with the two-loop universal scaling — implies scaling, but much harder to get

$$a\Lambda_L = \left(\frac{1}{b_0 g^2}\right)^{b_1/2b_0^2} \mathrm{e}^{-1/2b_0 g^2}$$

Improvement: in general, a program aimed at controlling lattice artifacts, so to reach faster and with more confidence scaling (hence continuum limit)

The functional integral

Sampling the phase space

Task:

$$\langle \mathcal{O} \rangle \ = \ \frac{1}{Z} \int \mathcal{D}A_\mu \ \mathcal{O} \ e^{-S}$$

from Mike Creutz:

A direct evaluation of such an integral has pitfalls. At first sight, the basic size of the calculation is overwhelming. Considering a 10^4 lattice, small by today standards, there are 40,000 links. For each is an $SU(3)$ matrix, parametrized by 8 numbers. Thus we have a $10^4 \times 4 \times 8 = 320,000$ dimensional integral. One might try to replace this with a discrete sum over values of the integrand. If we make the extreme approximation of using only two points per dimension, this gives a sum with

$$2^{320,000} = 3.8 \times 10^{96,329} \tag{6}$$

terms! Of course, computers are getting pretty fast, but one should remember that the age of universe is only $\sim 10^{27}$ nanoseconds.

!!!

Monte Carlo methods:

Create a sample of configurations distributed according to

$$e^{-S}$$

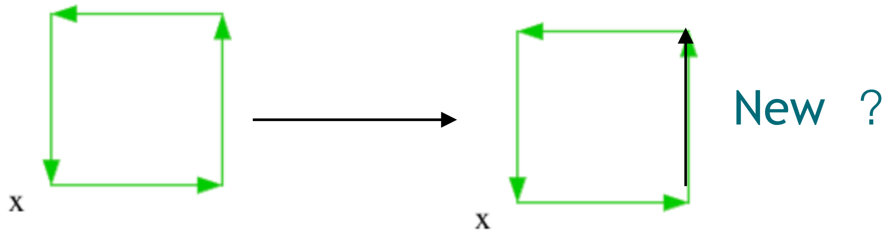
The functional integral may then be traded with an average over configurations

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S}$$

—->

$$\boxed{\frac{\sum_\alpha \mathcal{O}_\alpha e^{-S_\alpha}}{\sum_\alpha e^{-S_\alpha}}}$$

Monte Carlo time ‘evolution’



Different methods use different strategies for choosing the new link

Crucial point: positive Action!

1. Metropolis

$$\Delta S = S(\hat{U}_{ji}) - S(U_{ji}) \text{ (all other variables kept fixed)}.$$

If $\Delta S \leq 0$ the move is accepted

Otherwise: pick random number r 0 < r < 1 and accept if

$$r \leq e^{-\Delta S}.$$

Maximise distance between configurations, at the price of high rejection rate

2. Heath Bath

Choose new U with the appropriate weight:

$$\exp\{-\bar{S}(U_{ji}')\}$$

Accept step always satisfied by construction

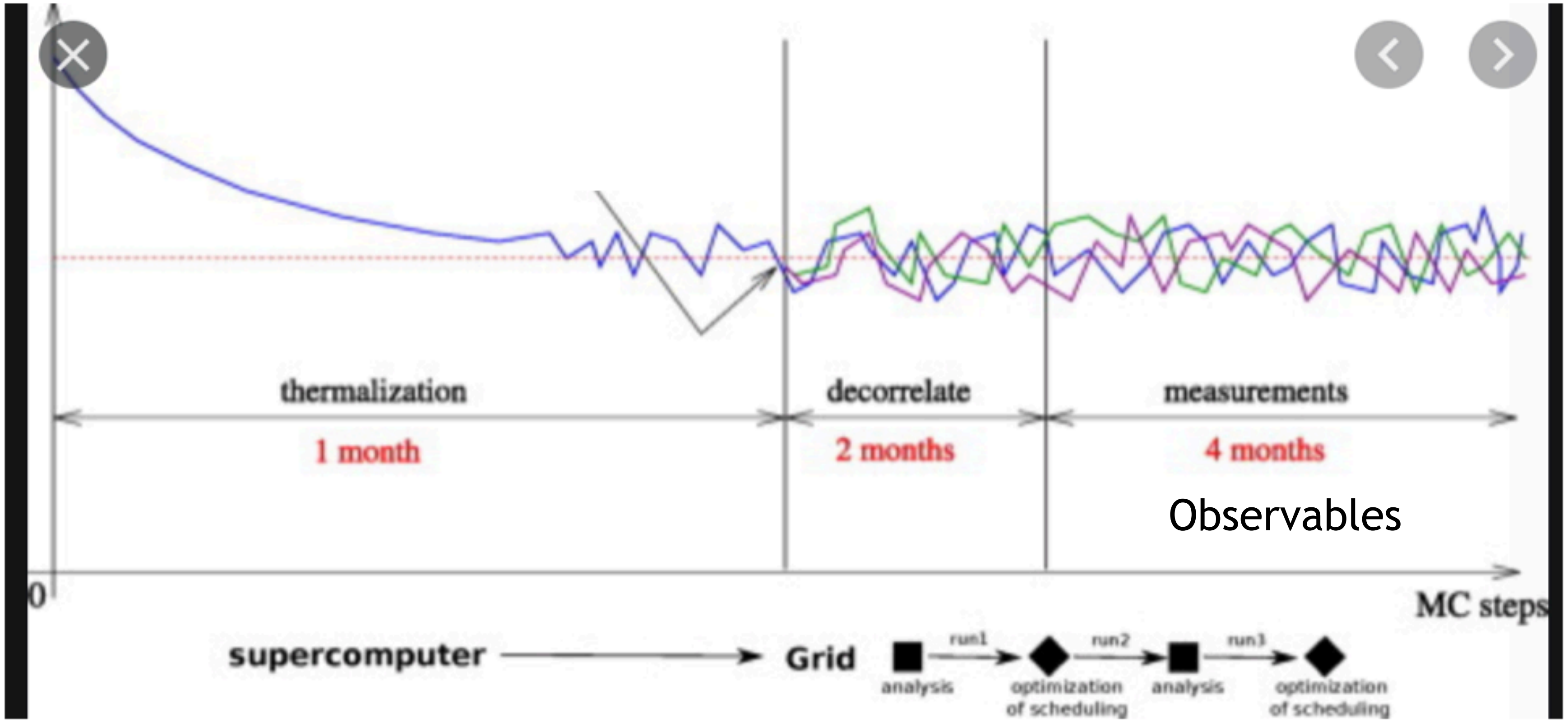
It may be expensive

3. Modified Metropolis

Same as Metropolis, but make several hits
with the same link :
“n upgrading per step”

It ‘interpolates’ between the two previous cases - optimal n to be determined

A typical Monte Carlo simulation:



Observables

Thinking in abstract terms - i.e. let us consider the discretised theory as a statistical system in d+1 dimension – these are basic measurable quantities:

1) Wilson loops

2) Polyakov loop

3) Topological charge

Difficult to draw 'butterly operator' .

$F\tilde{F}$ In the continuum (more later)

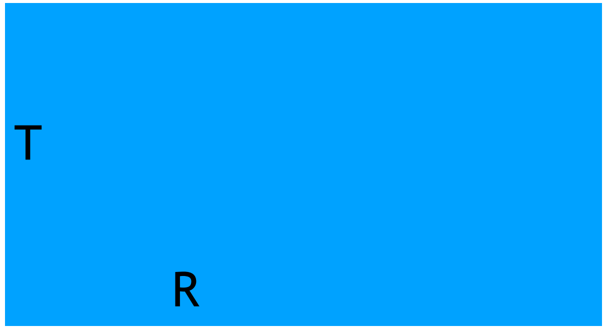
4) Two point functions of any of the above

5) Two point functions of composite fermion operators

Wilson loop

String tension - Interquark potential

$$V(\boldsymbol{R}) = -\lim_{T\to\infty} \frac{1}{T} \ln W(\boldsymbol{R}, T)$$



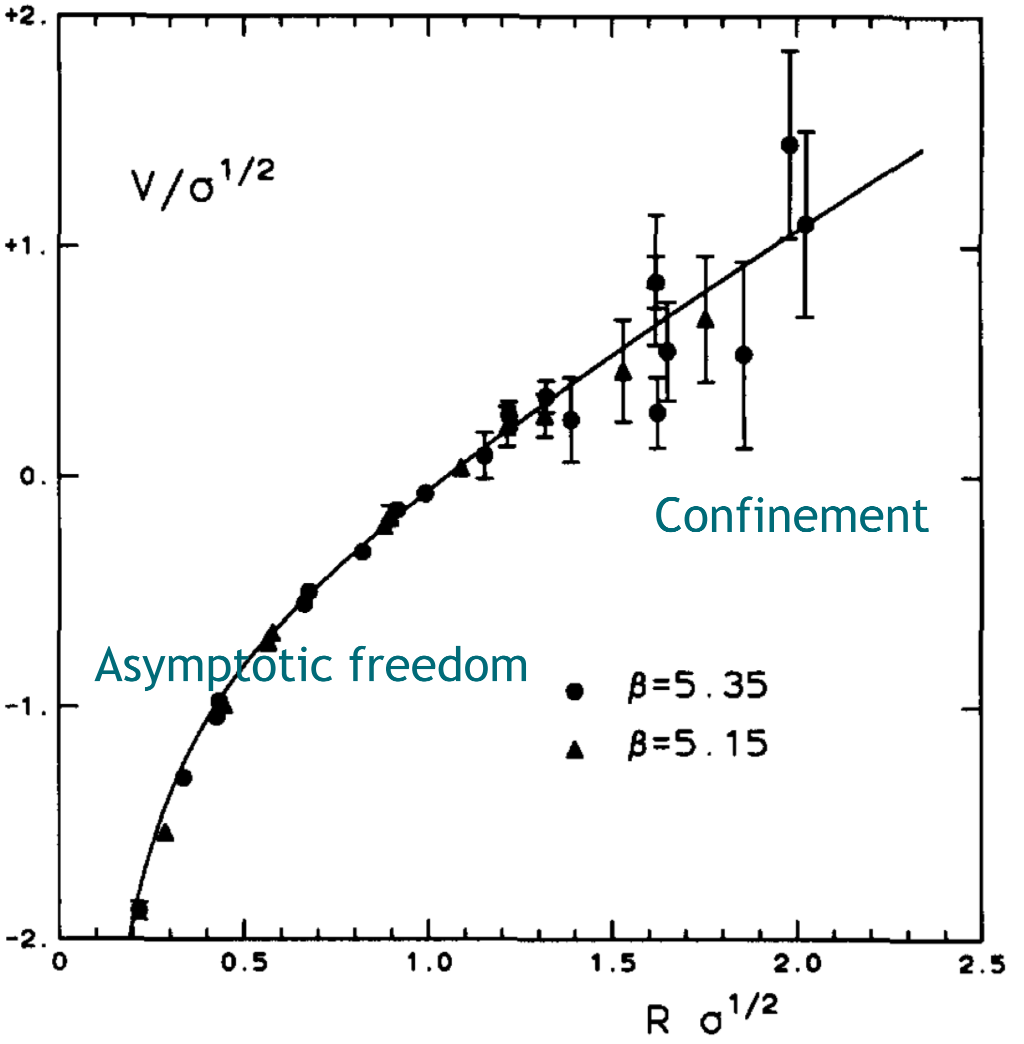


Fig. 1. The interquark potential measured for two values $\beta = 5.15$ and $\beta = 5.35$ and mapped onto each other by setting the scale from the fitted string tension.

$$V_l(R) = c_0 - \frac{e}{R} + \sigma R$$ Cornell form

String tension $\sqrt{\sigma} = 420\,\mathrm{MeV}$, Physical value

Two point functions of simple Wilson loops: Glueballs

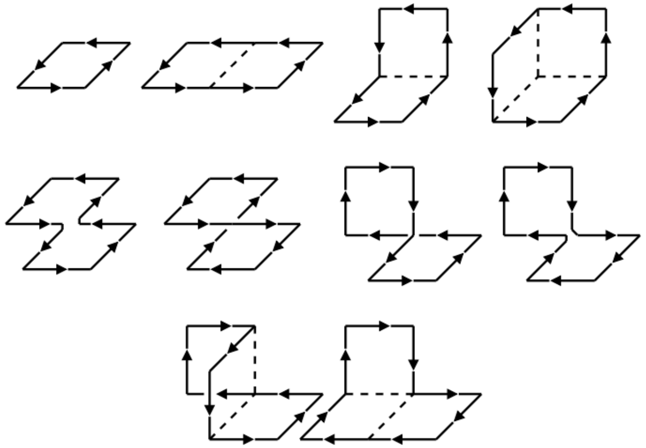


FIG. 1. The Wilson loop shapes used in making the basic glueball operators.

9901004

$$C_{AB}(t) = \sum_{\tau} \langle 0 | \bar{\Phi}_A^{(R)}(\tau+t)\ \bar{\Phi}_B^{(R)}(\tau) | 0 \rangle.$$

$$C_{00}(t) = Z_{00} \left\{ e^{-m_G t} + e^{-m_G (T-t)} \right\},$$

TABLE I. The glueball simulation parameters. Values for the coupling β, input aspect ratio parameter ξ, the mean-link parameter u_s^4, the single-link smearing weight λ_s, the two-link smearing weight λ_f, and the lattice sizes are listed. Results for the hadronic scale r_0 in terms of the lattice spacing a_s are also given. The approximate spatial lattice spacings a_s are determined assuming $r_0^{-1} = 410(20)$ MeV.

β	ξ	u_s^4	λ_s	λ_f	Lattice	r_0/a_s	a_s/r_0	a_s (fm)
1.7	5	0.295	0.1	0.5	$6^3 \times 30$	1.224(1)	0.8169(9)	0.39
1.9	5	0.328	0.1	0.5	$6^3 \times 30$	1.375(2)	0.727(1)	0.35
2.2	5	0.378	0.1	0.5	$8^3 \times 40$	1.761(2)	0.5680(5)	0.27
2.4	5	0.409	0.1	0.5	$8^3 \times 40$	2.180(6)	0.459(1)	0.22
2.5	5	0.424	0.1	0.5	$10^3 \times 50$	2.455(6)	0.407(1)	0.20
3.0	3	0.500	0.4	0.5	$15^3 \times 45$	4.130(24)	0.2421(14)	0.12

c

Spectrum of scalar and pseudoscalar glueballs from functional methods

Markus Q. Huber[a,1], **Christian S. Fischer**[b,1,2], **Hèlios Sanchis-Alepuz**[c,3,4]

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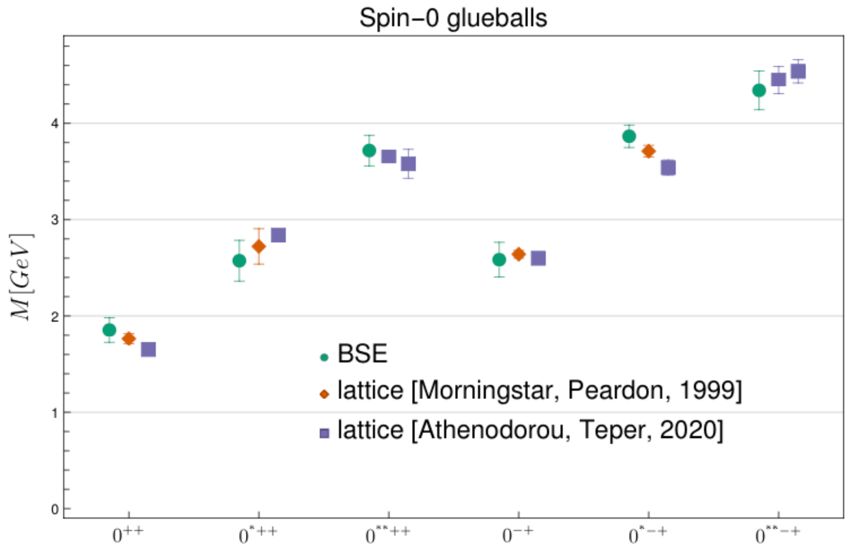
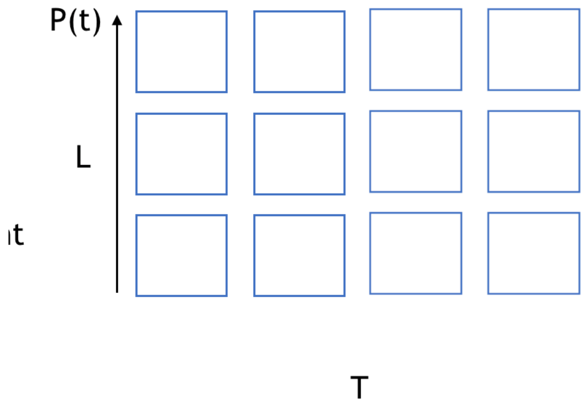


Two point functions of Polyakov loop: alternative extraction of the potential

$$e^{-V(R,T)/T} \propto < P(\vec{0}) P^\dagger(\vec{R}) > \xrightarrow[R \to \infty]{} \propto e^{-\sigma R}$$



As an aside:

Parametro d'ordine per $Z(N_c)$. —> Confinement as a symmetry

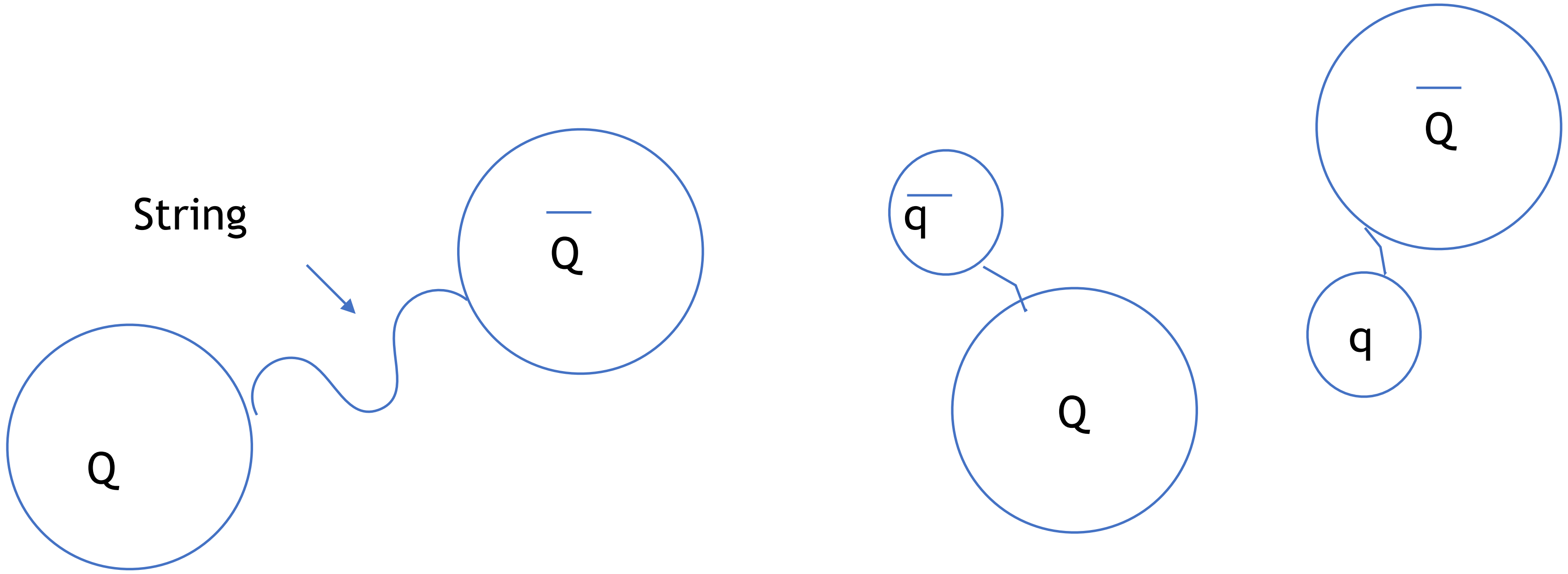
For two colours: Z(2): Ising model!! —> Universality

'Build' Polyakov loop system: it is a cube of spins!! —> dimensional reduction at Tc

Polyakov loop again : YM vs QCD

Used as order parameter for the Yang-Mills transition

No longer an order parameter with matter fields: the string can break due to recombination with light quarks popping out of the vacuum



Two point functions of composite fermions: meson spectrum

Basic :

$$\mathcal{O} = (\overline{\psi}\psi)_y(\overline{\psi}\psi)_x,$$

$$\int [dU][d\psi][d\bar{\psi}]\bar{\psi}_y^{u,a}\psi_y^{d,a}\bar{\psi}_x^{d,b}\psi_x^{u,b}e^{-S} = \int [dU](M_{x,y}^{-1,u}[U])^{ab}(M_{y,x}^{-1,d}[U])^{ba}\mathrm{det}Me^{-S_g}.$$

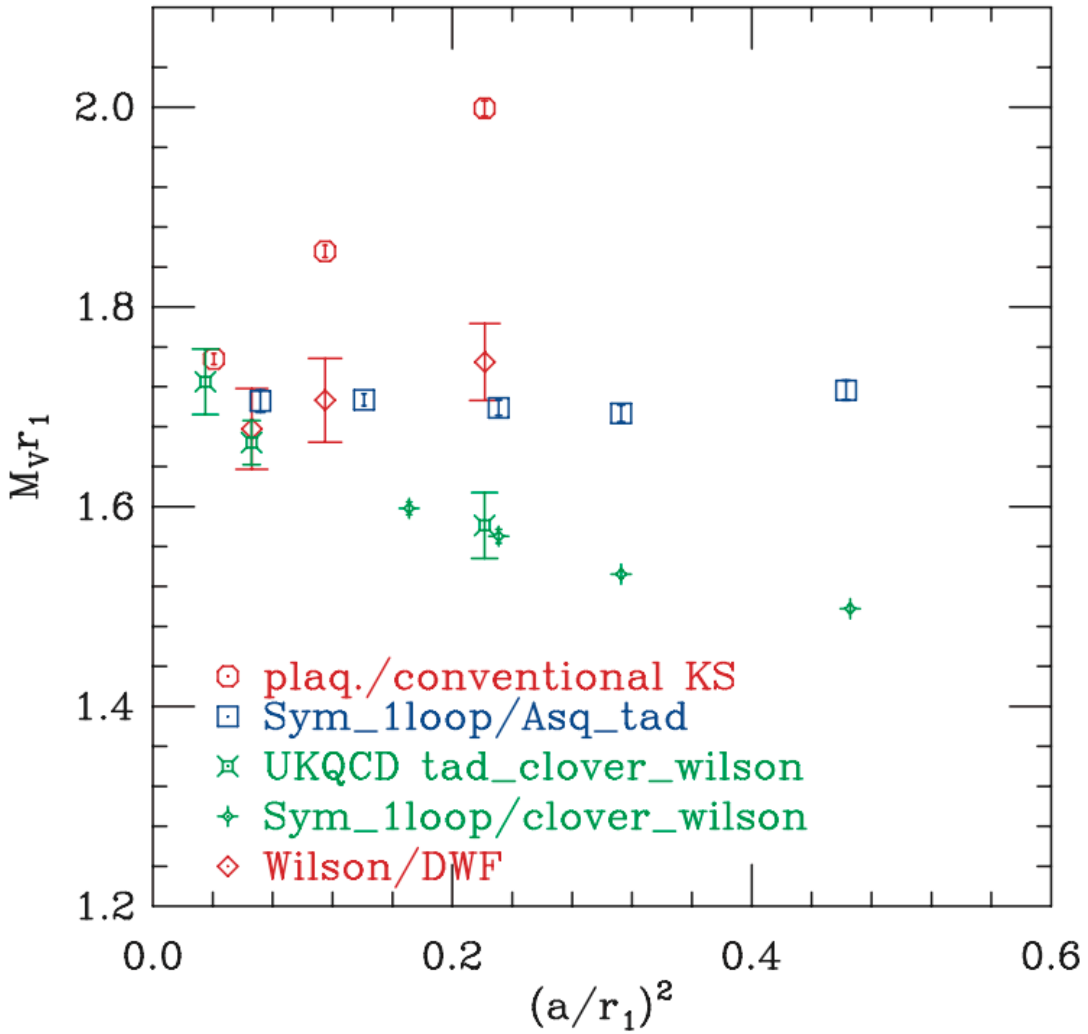
$$\langle 0|H^\dagger(T)H(0)|0\rangle = \sum_n \frac{A_{\mathrm{src},n}A_{\mathrm{snk},n}}{2E_n}e^{-E_nT}$$

Spectral decomposition
(generalizes asymptotic exponential decay)

Insert appropriate gamma matrices to create different quantum numbers

Example of results : Rho mass

from different fermion discretizations



r1= scale setting parameter
Circa 0.57GeV

Either for Glueballs and Mesons
the task is to identify the asymptotic exponent **m**

$$G(t)_{t\to\infty} \to e^{-\mathbf{m}t}$$

Operator Improvement -

Smoothing a rough landscape

CERN-PH-TH/2010-143

Properties and uses of the Wilson flow in lattice QCD

Martin Lüscher

CERN, Physics Department, 1211 Geneva 23, Switzerland

In a nutshell:

Evolves gauge fields towards minimum of the Action in fictitious time τ_F

Gaussian smearing over sphere with flow radius $\sqrt{8\tau_F}$

A basic proposal for illustration: smearing

Build a sequence of operators

$$\varphi^{(s)}(n) \to \varphi^{(s+1)}(n) = \varphi^{(s)}(n) + \epsilon \sum{}' \varphi^{(s)}(n')$$

Evolution in ‘smearing time $\tau = s\epsilon$

$$\partial\varphi/\partial\tau = \nabla^2 \varphi$$

In momentum space :

$$\exp(-Wk^2) \qquad \approx (W+\tau)^{-5/2}$$

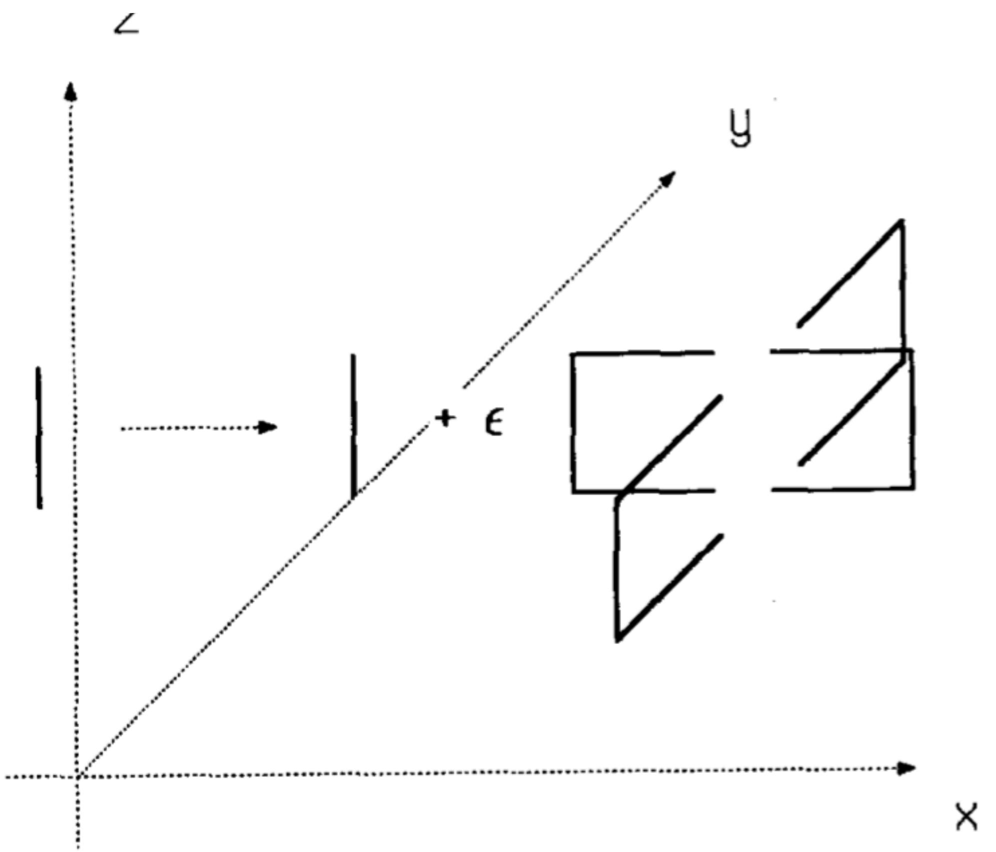


Fig. 1. The smearing procedure. We substitute a link with itself plus ϵ times the sum of the incomplete neighboring space like

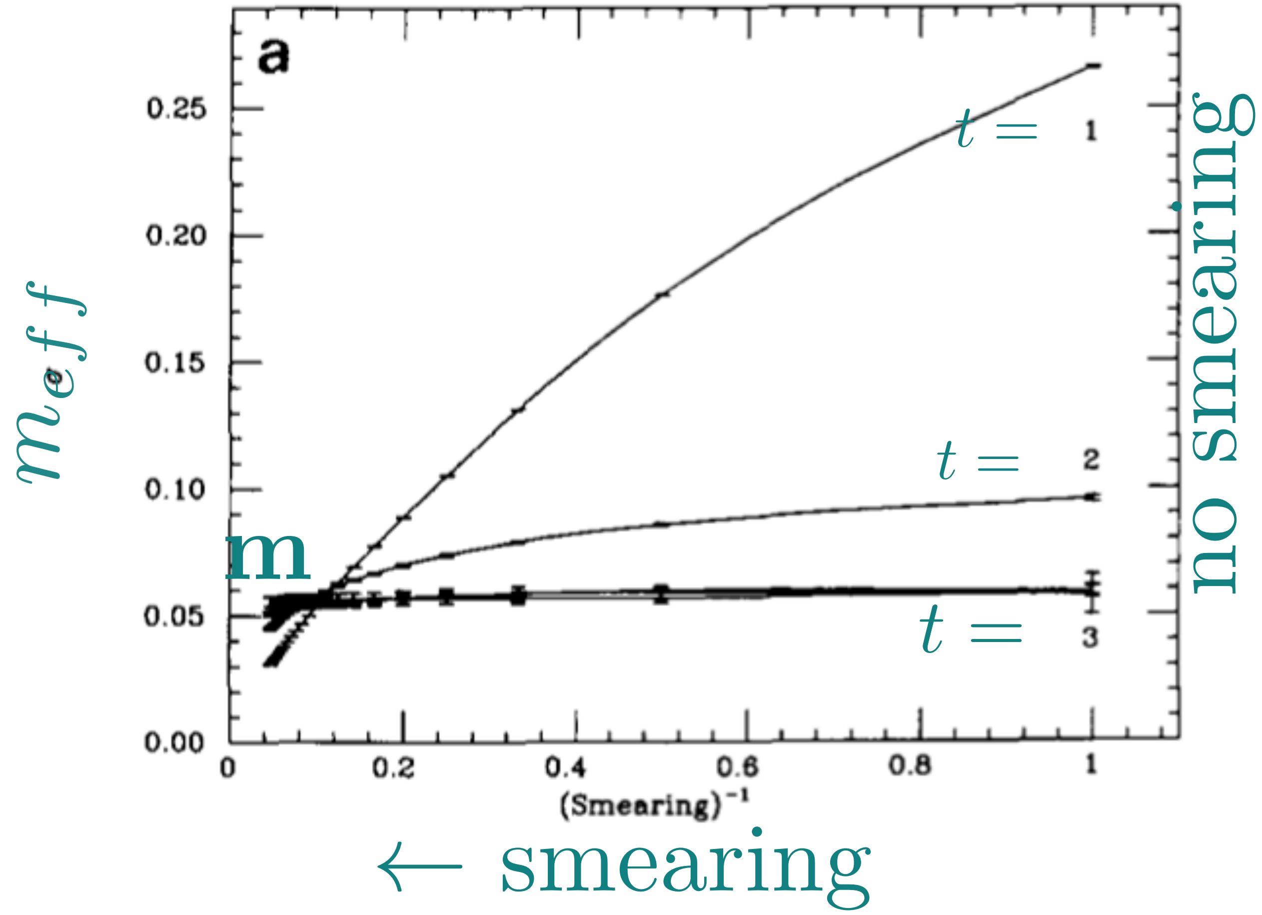
Smearing at work

$$G(t)_{t\to\infty} \to e^{-\mathbf{m}t}$$

Define

$$m_{eff}(t) \equiv -\frac{d\ln G(t)}{dt}$$

$$m_{eff}(t)_{t\to\infty} \to \mathbf{m}$$

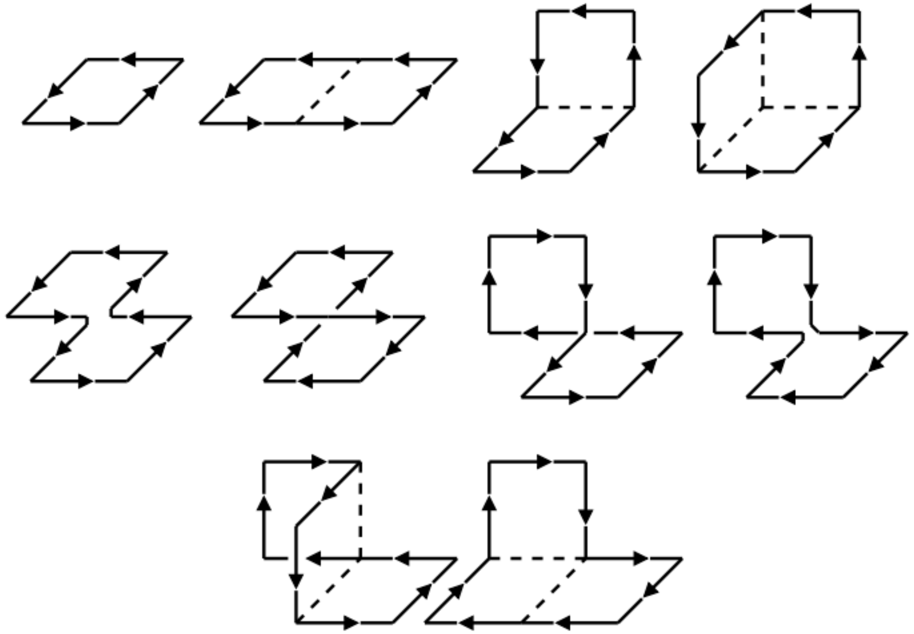


Smearing at work: Glueball masses

$$G(t) \equiv \langle O(t) \cdot O(0) \rangle - \langle O(t) \rangle \cdot \langle O(0) \rangle$$

$$\approx \exp(-mt) \, ,$$ Note: *t* Euclidean time!

$$G(t) = \int \delta(M-\omega) e^{-\omega t} \propto e^{-Mt}$$



Smearing

$$G_0^{(s)}(t) \equiv O^{(s)}(x,y) + O^{(s)}(y,z) + O^{(s)}(x,z)$$ 0++

$$G_2^{(s)}(t) \equiv -2O^{(s)}(x,y) + O^{(s)}(y,z) + O^{(s)}(x,z) \, .$$ 2++

CERN-PH-TH/2010-143

Many variations of this idea, substantial agreement

Properties and uses of the Wilson flow in lattice QCD

Martin Lüscher

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PROCEEDINGS OF SCIENCE

String tension from smearing and Wilson flow methods

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Recently, we proposed a new method to extract the string tension from 4-dimensionally smeared Wilson loops. In this talk, we first show that the results obtained using this smearing method are identical to those obtained by Wilson flow, once the time step is sufficiently small. We then demonstrate the practical advantage of our method by applying it to the calculation of string tension in SU(3) Yang-Mills theory.

Comparison of the gradient flow with cooling in $SU(3)$ pure gauge theory

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The gradient (Wilson) flow has been introduced recently in order to provide a solid theoretical framework for the smoothing of ultraviolet noise in lattice gauge configurations. It is interesting to ask how it compares with other, more heuristic and numerically cheaper smoothing techniques, such as standard cooling. In this study we perform such a comparison, focusing on observables related to topology. We show that, already for moderately small lattice spacings, standard cooling and the gradient flow lead to equivalent results, both for average quantities and configuration by configuration.

The topological susceptibility of the pure SU(3) Yang–Mills vacuum on the lattice ☆

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H. Panagopoulos and R. Tripiccione

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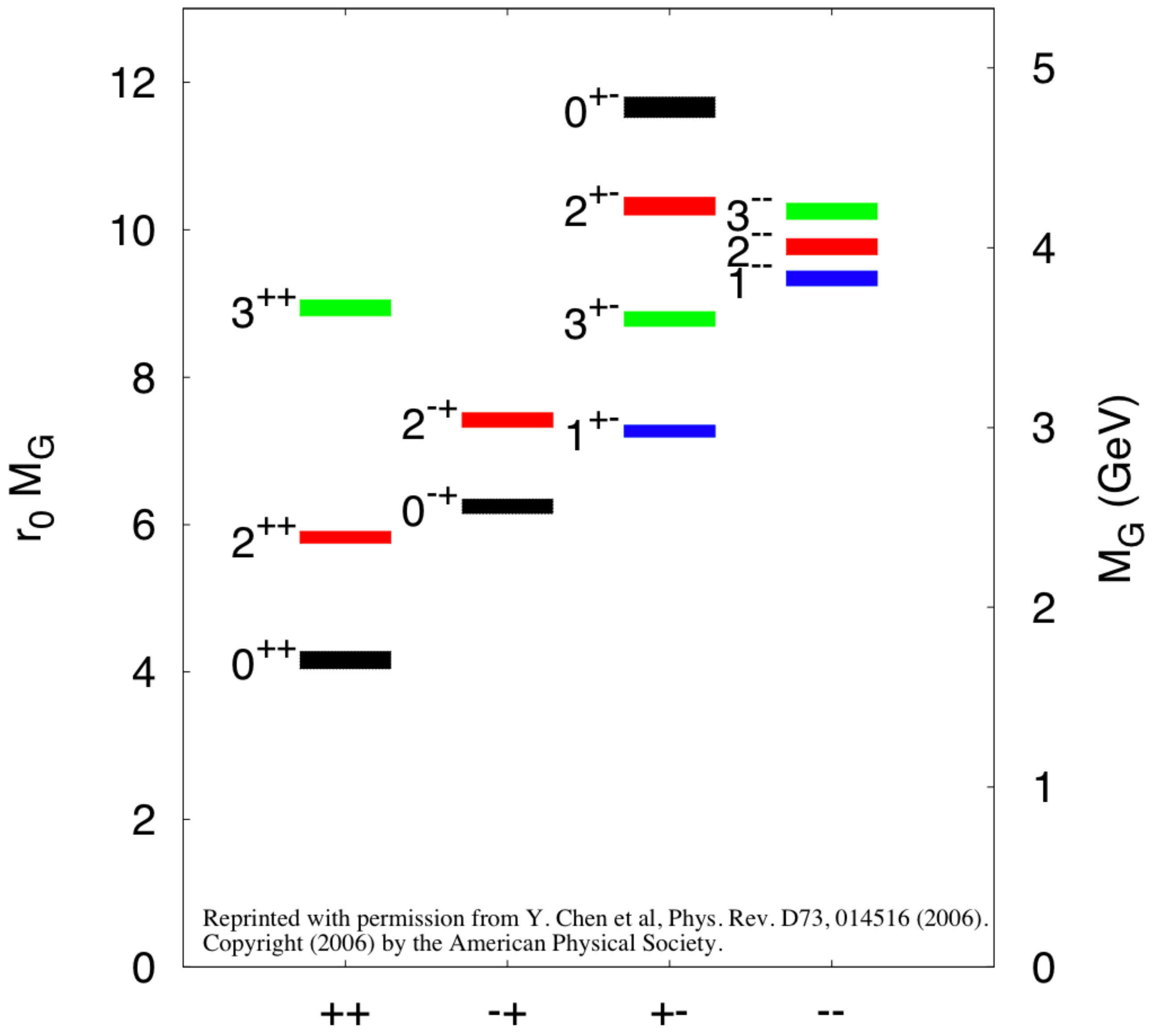
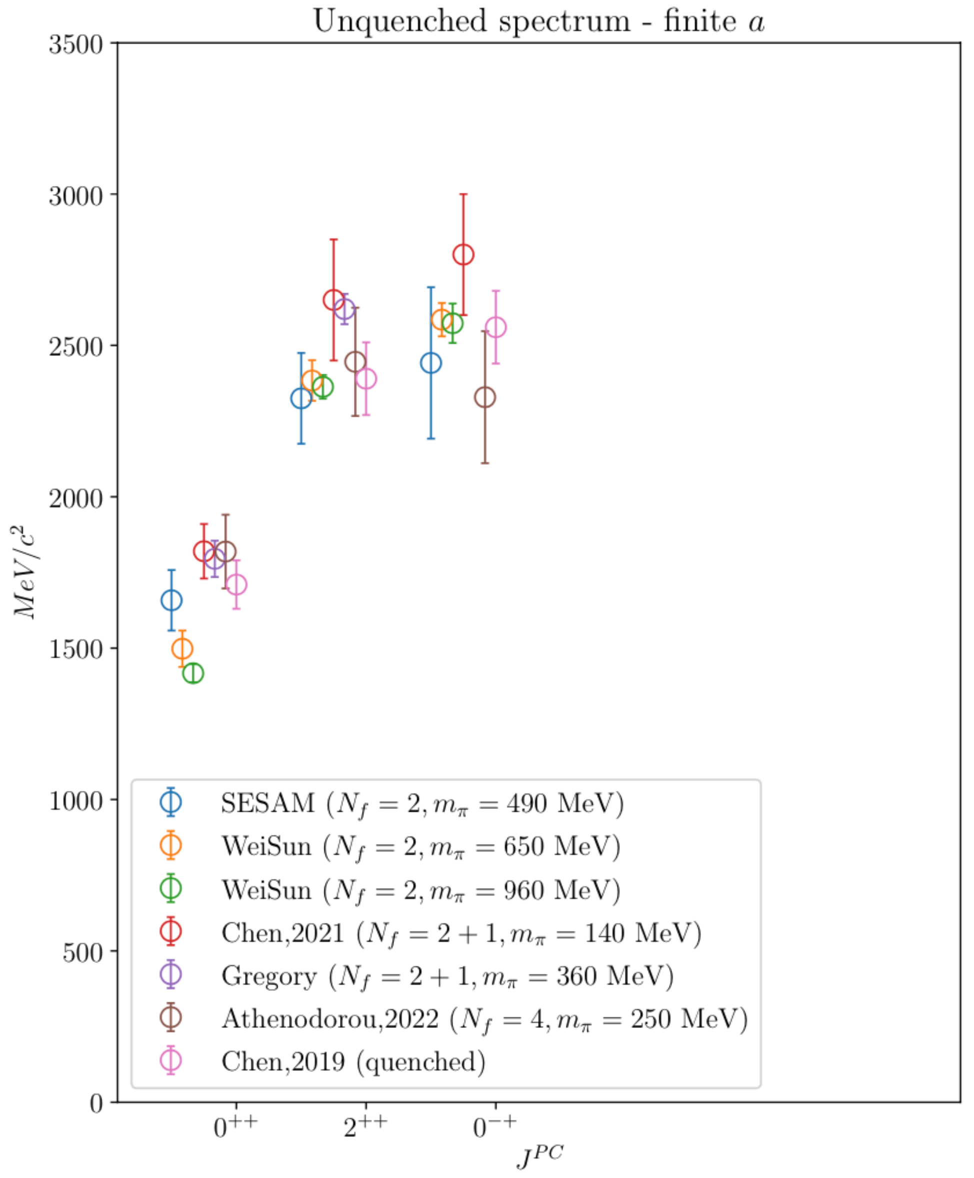
Received 2 August 1990

Using a "field theoretic" approach, we compute the topological susceptibility χ of the pure gauge SU(3) theory on the lattice. We also apply an algorithm of gradual cooling, and use these two approaches as a cross-check on each other. The final value we find for χ confirms results found earlier using an abrupt-cooling algorithm.

2006 data, From PDG 2022

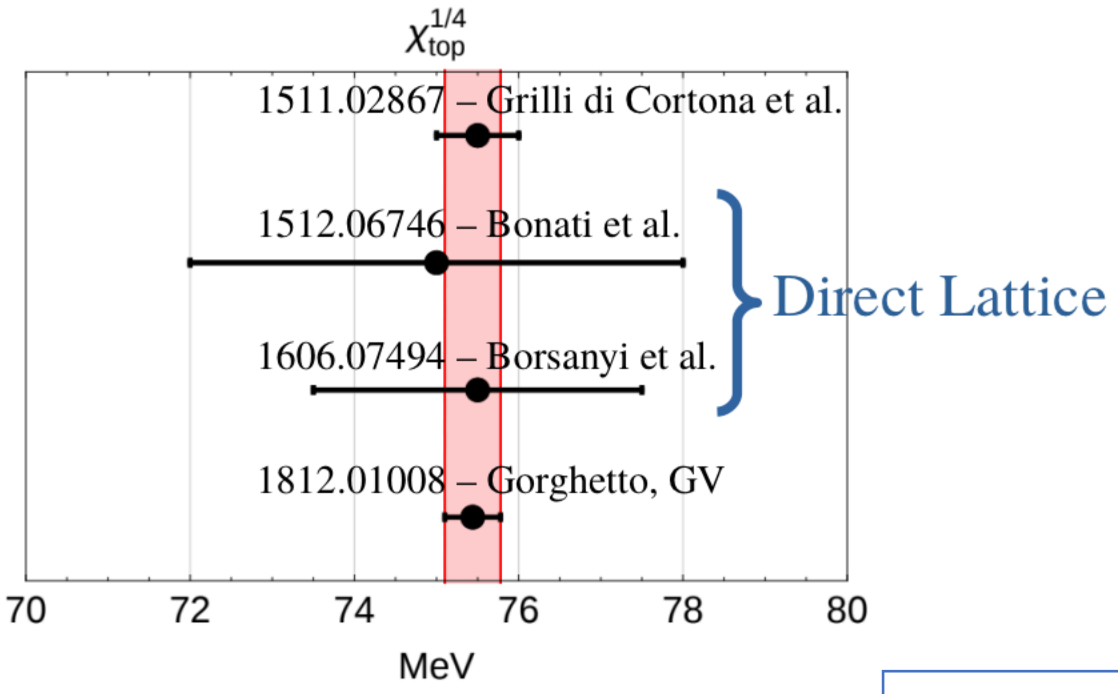


2022 compilation, Vadacchino@lat22

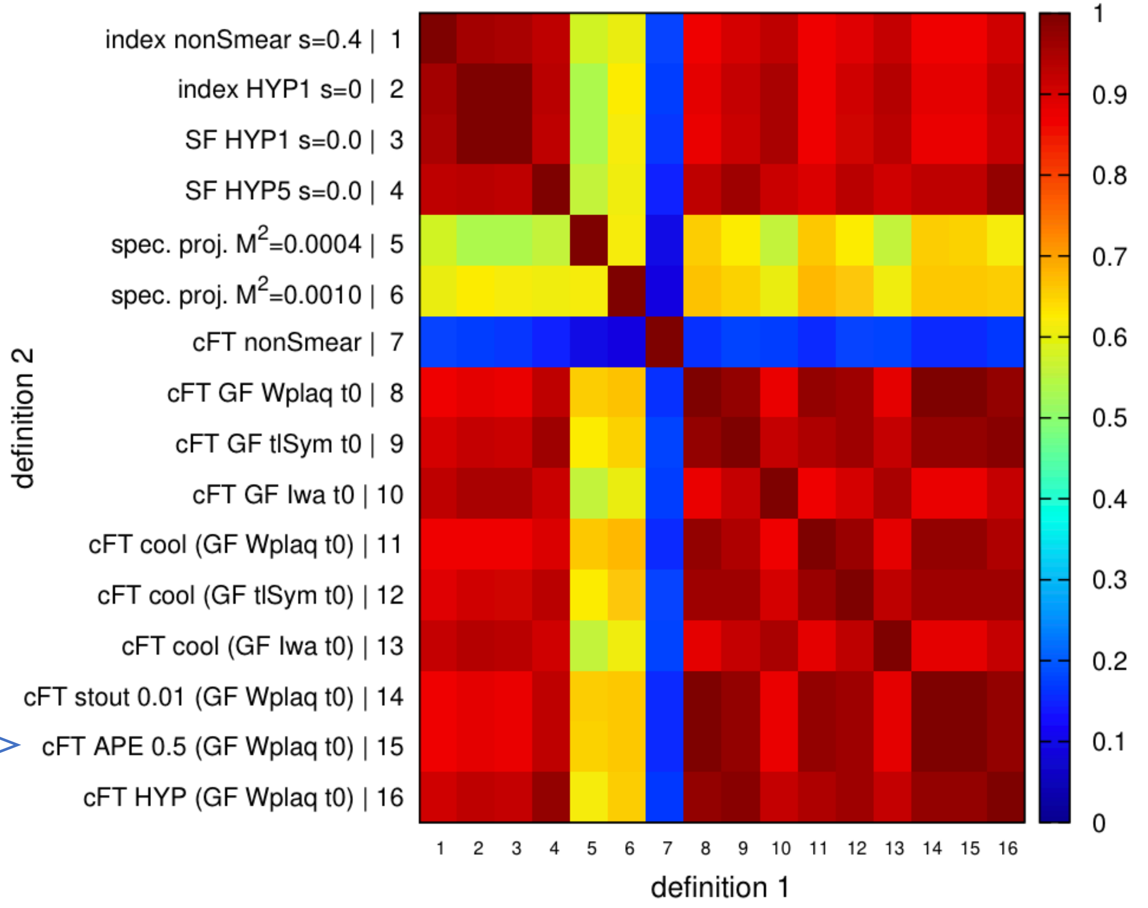


Topology - susceptibility

Many varieties of smearing/smoothing/flowing/cooling.. good agreement at T=0



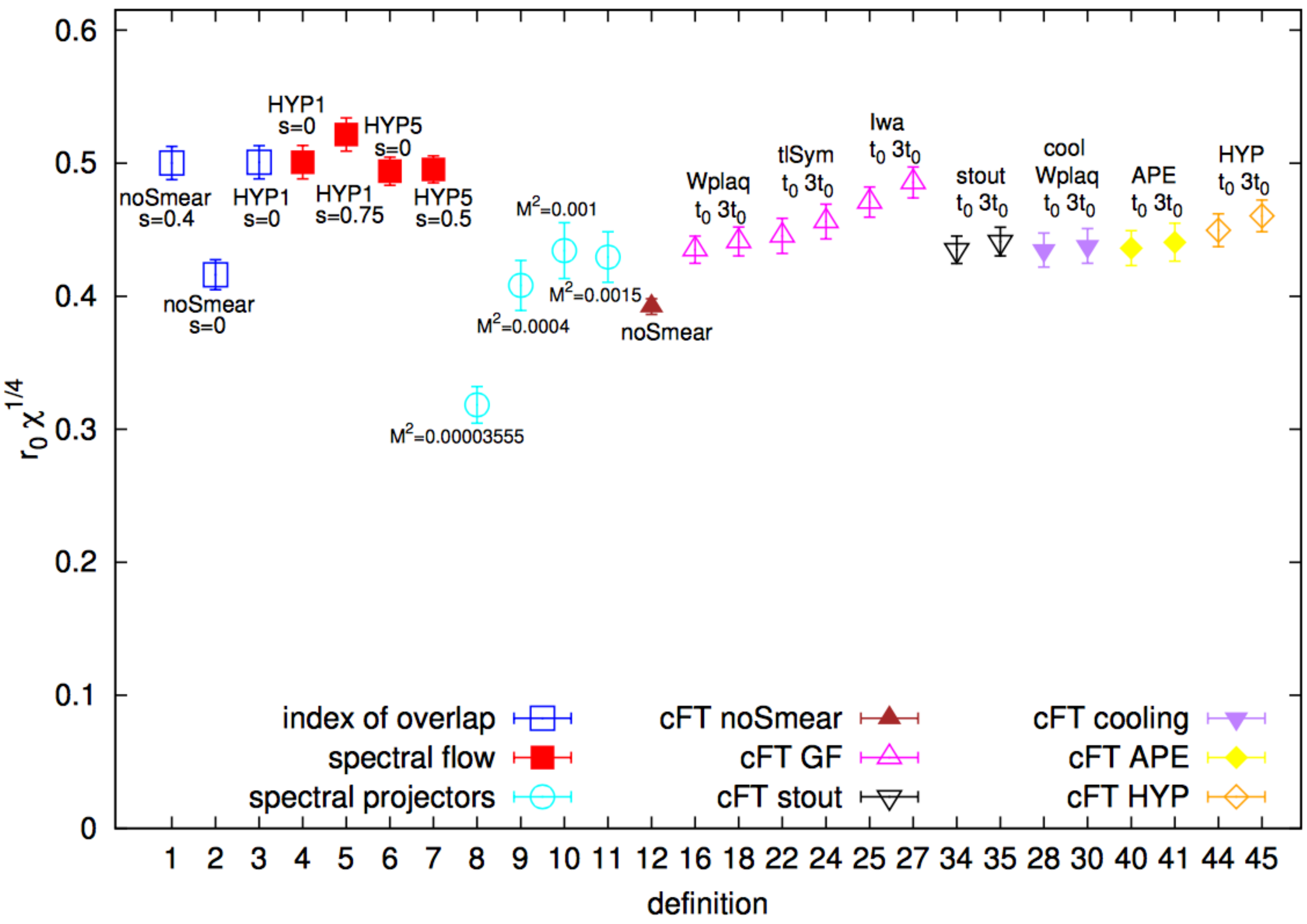
Plot by G. Villadoro



ETMC collaboration, 2017

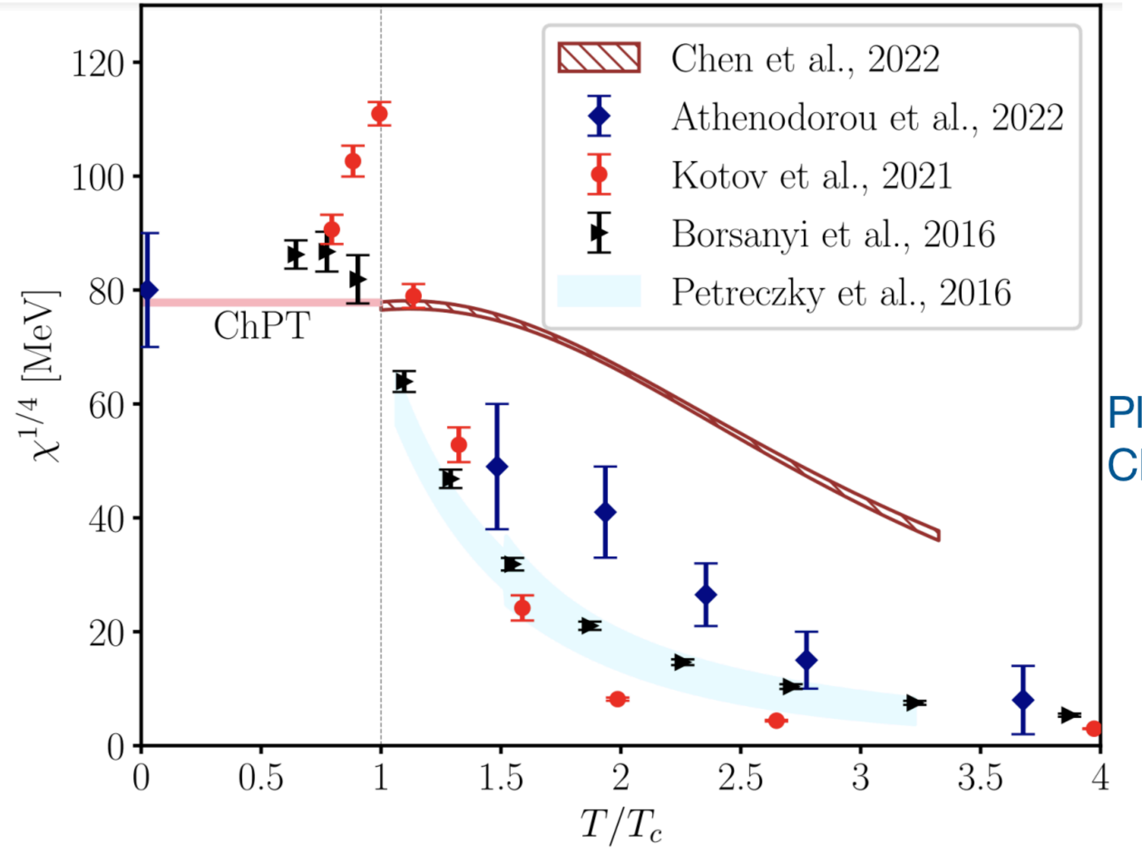
Comparison of topological charge definitions

nr	full name	smearing type	short name	type
1	index of overlap Dirac operator $s = 0.4$	–	index nonSmear $s = 0.4$	F
2	index of overlap Dirac operator $s = 0.0$	–	index nonSmear $s = 0$	F
3	index of overlap Dirac operator $s = 0.0$	HYP1	index HYP1 $s = 0$	F
4	Wilson-Dirac op. spectral flow $s = 0.0$	HYP1	SF HYP1 $s = 0.0$	F
5	Wilson-Dirac op. spectral flow $s = 0.75$	HYP1	SF HYP1 $s = 0.75$	F
6	Wilson-Dirac op. spectral flow $s = 0.0$	HYP5	SF HYP5 $s = 0.0$	F
7	Wilson-Dirac op. spectral flow $s = 0.5$	HYP5	SF HYP5 $s = 0.5$	F
8	spectral projectors $M^2 = 0.00003555$	–	spec. proj. $M^2 = 0.0000355$	F
9	spectral projectors $M^2 = 0.0004$	–	spec. proj. $M^2 = 0.0004$	F
10	spectral projectors $M^2 = 0.0010$	–	spec. proj. $M^2 = 0.0010$	F
11	spectral projectors $M^2 = 0.0015$	–	spec. proj. $M^2 = 0.0015$	F
12	field theoretic (clover)	–	cFT nonSmear	G
13	field theoretic (plaquette)	GF (Wplaq,t_0)	pFT GF Wplaq t_0	G
14	field theoretic (plaquette)	GF (Wplaq,$2t_0$)	pFT GF Wplaq $2t_0$	G
15	field theoretic (plaquette)	GF (Wplaq,$3t_0$)	pFT GF Wplaq $3t_0$	G
16	field theoretic (clover)	GF (Wplaq,t_0)	cFT GF Wplaq t_0	G
17	field theoretic (clover)	GF (Wplaq,$2t_0$)	cFT GF Wplaq $2t_0$	G
18	field theoretic (clover)	GF (Wplaq,$3t_0$)	cFT GF Wplaq $3t_0$	G
19	field theoretic (improved)	GF (Wplaq,t_0)	iFT GF Wplaq t_0	G
20	field theoretic (improved)	GF (Wplaq,$2t_0$)	iFT GF Wplaq $2t_0$	G
21	field theoretic (improved)	GF (Wplaq,$3t_0$)	iFT GF Wplaq $3t_0$	G
22	field theoretic (clover)	GF (tlSym,t_0)	cFT GF tlSym t_0	G
23	field theoretic (clover)	GF (tlSym,$2t_0$)	cFT GF tlSym $2t_0$	G
24	field theoretic (clover)	GF (tlSym,$3t_0$)	cFT GF tlSym $3t_0$	G
25	field theoretic (clover)	GF (Iwa,t_0)	cFT GF Iwa t_0	G
26	field theoretic (clover)	GF (Iwa,$2t_0$)	cFT GF Iwa $2t_0$	G
27	field theoretic (clover)	GF (Iwa,$3t_0$)	cFT GF Iwa $3t_0$	G
28	field theoretic (clover)	cool (Wplaq,t_0)	cFT cool (GF Wplaq t_0)	G
29	field theoretic (clover)	cool (Wplaq,$3t_0$)	cFT cool (GF Wplaq $3t_0$)	G
30	field theoretic (clover)	cool (tlSym,t_0)	cFT cool (GF tlSym t_0)	G
31	field theoretic (clover)	cool (tlSym,$3t_0$)	cFT cool (GF tlSym $3t_0$)	G
32	field theoretic (clover)	cool (Iwa,t_0)	cFT cool (GF Iwa t_0)	G
33	field theoretic (clover)	cool (Iwa,$3t_0$)	cFT cool (GF Iwa $3t_0$)	G
34	field theoretic (clover)	stout (0.01,t_0)	cFT stout 0.01 (GF Wplaq t_0)	G
35	field theoretic (clover)	stout (0.01,$3t_0$)	cFT stout 0.01 (GF Wplaq $3t_0$)	G
36	field theoretic (clover)	stout (0.1,t_0)	cFT stout 0.1 (GF Wplaq t_0)	G
37	field theoretic (clover)	stout (0.1,$3t_0$)	cFT stout 0.1 (GF Wplaq $3t_0$)	G
38	field theoretic (clover)	APE (0.4,t_0)	cFT APE 0.4 (GF Wplaq t_0)	G
39	field theoretic (clover)	APE (0.4,$3t_0$)	cFT APE 0.4 (GF Wplaq $3t_0$)	G
40	field theoretic (clover)	APE (0.5,t_0)	cFT APE 0.5 (GF Wplaq t_0)	G
41	field theoretic (clover)	APE (0.5,$3t_0$)	cFT APE 0.5 (GF Wplaq $3t_0$)	G
42	field theoretic (clover)	APE (0.6,t_0)	cFT APE 0.6 (GF Wplaq t_0)	G
43	field theoretic (clover)	APE (0.6,$3t_0$)	cFT APE 0.6 (GF Wplaq $3t_0$)	G
44	field theoretic (clover)	HYP (t_0)	cFT HYP (GF Wplaq t_0)	G
45	field theoretic (clover)	HYP ($3t_0$)	cFT HYP (GF Wplaq $3t_0$)	G



noSmear s=0.4
noSmear s=0
HYP1 s=0
HYP1 s=0
HYP5 s=0
HYP1 s=0.75
HYP5 s=0.5
M²=0.00003555
M²=0.0004
M²=0.001
M²=0.0015
noSmear
Wplaq t0 3t0
tlSym t0 3t0
Iwa t0 3t0
stout t0 3t0
cool Wplaq t0 3t0
APE t0 3t0
HYP t0 3t0
index of overlap
spectral flow
spectral projectors
cFT noSmear
cFT GF
cFT stout
cFT cooling
cFT APE
cFT HYP
r0 χ^1/4
definition
0
0.1
0.2
0.3
0.4
0.5
0.6
1 2 3 4 5 6 7 8 9 10 11 12 16 18 22 24 25 27 34 35 28 30 40 41 44 45

But at finite temperature topology poses specific challenges…



Plot by
Claudio Bonanno

Na6-Strong2020, MpL et al, 2023

A good playground for collaborations.