Lattice QCD and its phenomenological applications

Introduction to Lattice QCD - 2

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lattice virtual academy

Introduction to Lattice QCD — II

More on Euclidean formulation

Light quarks - chiral symmetry (and confinement)

Theta term, strong CP problem, topology, axion (brief recap)

Topology

Heavy quarks

Lattice QCD and axion cosmology

Imaginary time, temperature, dimensional reduction

Euclidean Field Theory —> Classical Statistical System

$$< \phi_a | e^{-iHt} | \phi_a >= \int d\pi \int_{\phi(x,0)=\phi_a(x)}^{\phi(x,t)=\phi_a(x)} d\phi e^{i \int_0^t dt \int d^3x (\pi(\vec{x},t) \frac{\partial \phi(\vec{x},t)}{\partial t} - H(\pi,\phi))}$$

$$\mathcal{Z} = Tr e^{-\beta(H-\mu\hat{N})} = \int d\phi_a < \phi_a | e^{-\beta(H-\mu N)} | \phi_a >$$

$$\boxed{\beta \equiv \frac{1}{T} \rightarrow it}$$

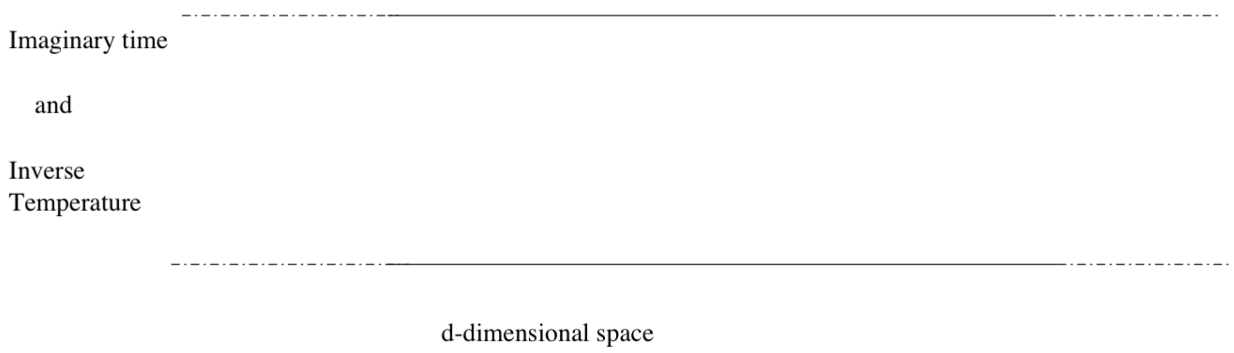


Table 1

The equivalences between a Euclidean field theory and Classical Statistical Mechanics.

Euclidean Field Theory	Classical Statistical Mechanics
Action	Hamiltonian
unit of action h	units of energy $\beta = 1/kT$
Feynman weight for amplitudes $e^{-\mathcal{S}/h} = e^{-\int \mathcal{L}dt/h}$	Boltzmann factor $e^{-\beta H}$
Vacuum to vacuum amplitude $\int \mathcal{D}\phi e^{-\mathcal{S}/h}$	Partition function $\sum_{conf.} e^{-\beta H}$
Vacuum energy	Free Energy
Vacuum expectation value $\langle 0 \vert \mathcal{O} \vert 0 \rangle$	Canonical ensemble average $\langle \mathcal{O} \rangle$
Time ordered products	Ordinary products
Green's functions $\langle 0 \vert T[\mathcal{O}_1 \ldots \mathcal{O}_n] \vert 0 \rangle$	Correlation functions $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$
Mass M	correlation length $\xi = 1/M$
Mass-gap	exponential decrease of correlation functions
Mass-less excitations	spin waves
Regularization: cutoff Λ	lattice spacing a
Renormalization: $\Lambda \rightarrow \infty$	continuum limit $a \rightarrow 0$
Changes in the vacuum	phase transitions

Table by Rajan Gupta

Boundary conditions for the fields

$$\mathcal{Z} = \int d\phi d\psi e^{-S(\phi,\psi)}$$

$$S(\phi,\psi) = \int_0^{1/T} dt \int d^d x \mathcal{L}(\phi,\psi)$$

Bosons : periodic

$$G_B(\vec{x},\vec{y};\tau,0) = Tr\{\hat{\rho} T_\tau[\hat{\phi}(\vec{x},\tau)\hat{\phi}(\vec{y},0)]\}/\mathcal{Z}$$

where T_τ is the imaginary time ordering operator:

$$T_\tau[\hat{\phi}(\tau_1)\hat{\phi}(\tau_2)] = \hat{\phi}(\tau_1)\hat{\phi}(\tau_2)\theta(\tau_1-\tau_2)+\hat{\phi}(\tau_2)\hat{\phi}(\tau_1)\theta(\tau_2-\tau_1)$$

Use now the commuting properties of the imaginary time ordering evolution and H:

$$[T_\tau, e^{-\beta H}] = 0$$

togheter with the Heisenberg time evolution

$$e^{\beta H}\phi(\vec{y},0)e^{-\beta H} = \phi(\vec{y},\beta)$$

to get:

$$G_B(\vec{x},\vec{y};\tau,0) = G_B(\vec{x},\vec{y};\tau,\beta)$$

Fermions

$$T_\tau[\hat{\psi}(\tau_1)\hat{\psi}(\tau_2)] = \hat{\psi}(\tau_1)\hat{\psi}(\tau_2)\theta(\tau_1 - \tau_2) - \hat{\psi}(\tau_2)\hat{\psi}(\tau_1)\theta(\tau_2 - \tau_1)$$

$$\hat{\psi}(\vec{x}, 0) = -\hat{\psi}(\vec{x}, \beta)$$

Antiperiodic

Fermions

$$T_\tau[\hat{\psi}(\tau_1)\hat{\psi}(\tau_2)] = \hat{\psi}(\tau_1)\hat{\psi}(\tau_2)\theta(\tau_1 - \tau_2) - \hat{\psi}(\tau_2)\hat{\psi}(\tau_1)\theta(\tau_2 - \tau_1)$$

$$\hat{\psi}(\vec{x}, 0) = -\hat{\psi}(\vec{x}, \beta)$$

Antiperiodic

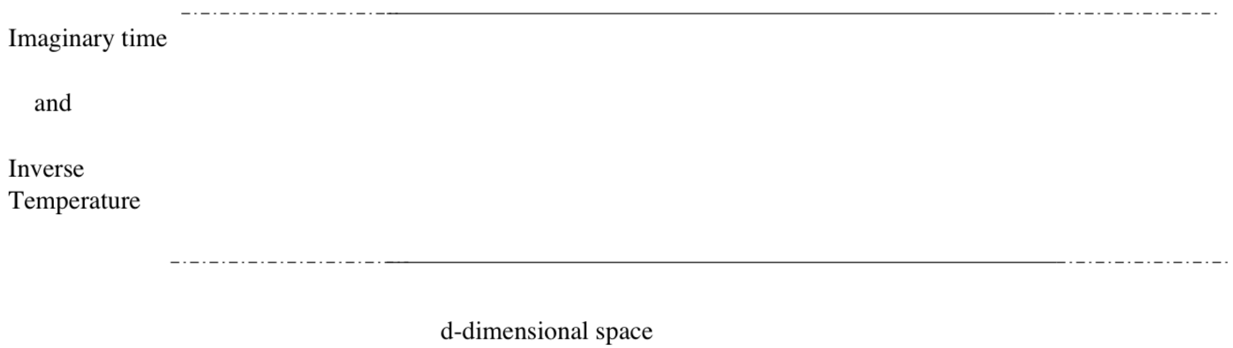
Bosons

$$T_\tau[\hat{\phi}(\tau_1)\hat{\phi}(\tau_2)] = \hat{\phi}(\tau_1)\hat{\phi}(\tau_2)\theta(\tau_1 - \tau_2) + \hat{\phi}(\tau_2)\hat{\phi}(\tau_1)\theta(\tau_2 - \tau_1)$$

$$G_B(\vec{x}, \vec{y}; \tau, 0) = G_B(\vec{x}, \vec{y}; \tau, \beta)$$

Periodic

Dimensional Reduction



+ coarse graining

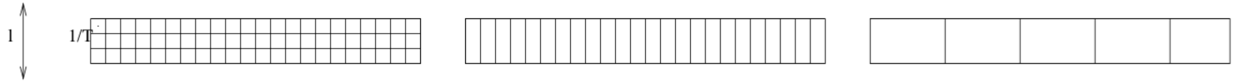


Figure 2: Sketchy view of the dimensional reduction from d+1 to d dimensions(from the leftmost to the middle picture) and subsequent coarse graining (from middle to rightmost)

Cases for Dimensional Reduction

1) T >>> any mass —> High T Electroweak transition

2) Diverging correlation length —> second order transition, basis for universality
High T QCD

Mode expansion and Decoupling

$$\phi(x,t) = \sum_{\omega_n = 2n\pi T} e^{i\omega_n t}\phi_n(x)$$ Bosons

$$\psi(x,t) = \sum_{\omega_n = (2n+1)\pi T} e^{i\omega_n t}\psi_n(x)$$ Fermions

In the expression for the Action

$$S(\phi,\psi) = \int_0^{1/T} dt \int d^d x \mathcal{L}(\phi,\psi)$$

the integral over time can then be traded with a sum over modes, and we reach the conclusion that a d+1 statistical field theory at $T > 0$ is equivalent to a d-dimensional theory with an <u>infinite</u> number of fields.

When dimensional reduction is possible, only one boson field survives

Finite temperature at a glance

- The partition function $\mathcal{Z}$ has the intepretation of the partition function of a statistical field theory in d+1 dimension, where the temperature has to be identified with the reciprocal of the (imaginary) time.

- The fields' boundary conditions follows from the Bose and Fermi s

$$\begin{aligned} \phi(t=0,\vec{x}) &= \phi(t=1/T,\vec{x}) \\ \psi(t=0,\vec{x}) &= -\psi(t=1/T,\vec{x}) \end{aligned}$$

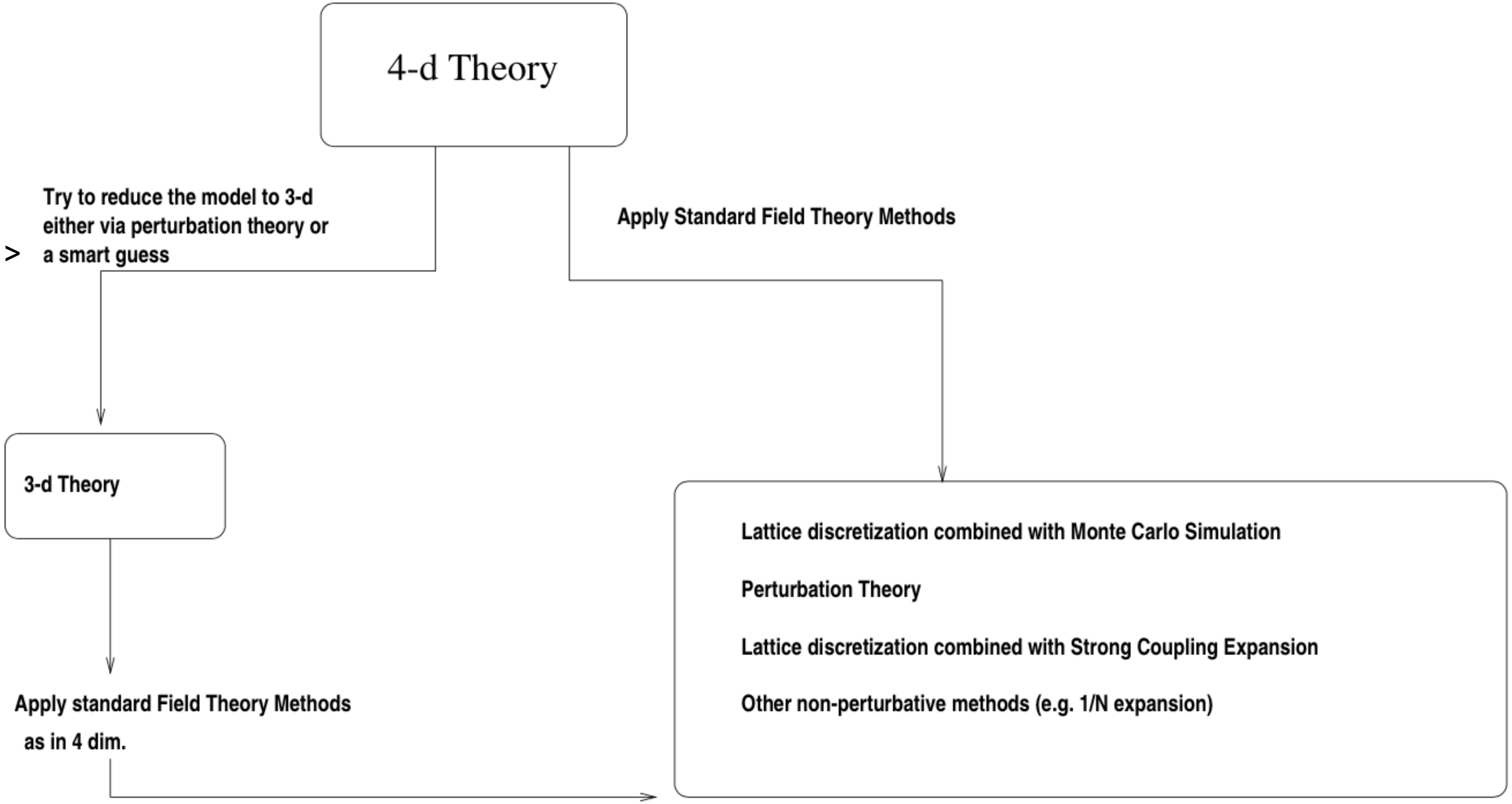
  i.e. fermionic and bosonic fields obey antiperiodic and periodic boundary conditions in time.

- "Dimensional reduction", when 'true' means that the system become effectvely 3-dimensional. In this case only the Fourier component of each Bose field with vanishing Matsubara frequency will contribute to the dynamics, while Fermions would decouple.

- The scenario above is very plausible and physically well founded, but it is by no means a theorem. Ab initio calculations can confirm or disprove it.

Computational schemes at a glance

Universality ->



Symmetry and pattern of breaking

Yang Mills

Massless QCD - light quarks

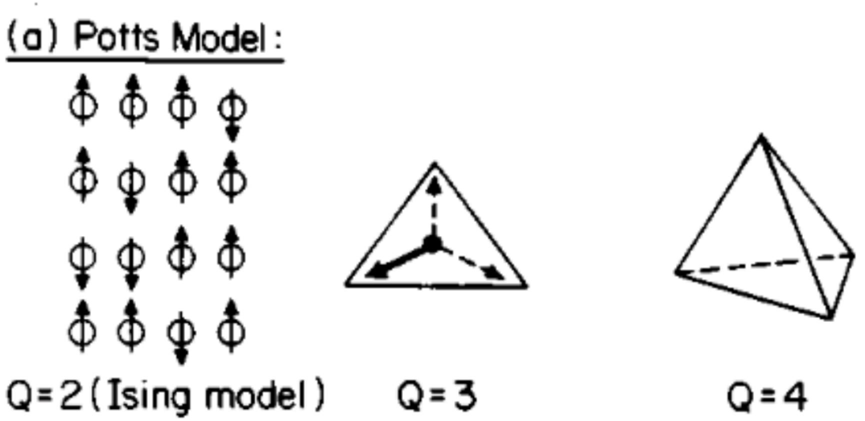
Interplay of chiral symmetry and confinement?

symmetry of Yang-Mills

3-dimensional system of Polyakov loops

SU(3) : Z3 center symmetry breaking at Tc —> 1st order 3 state Potts model

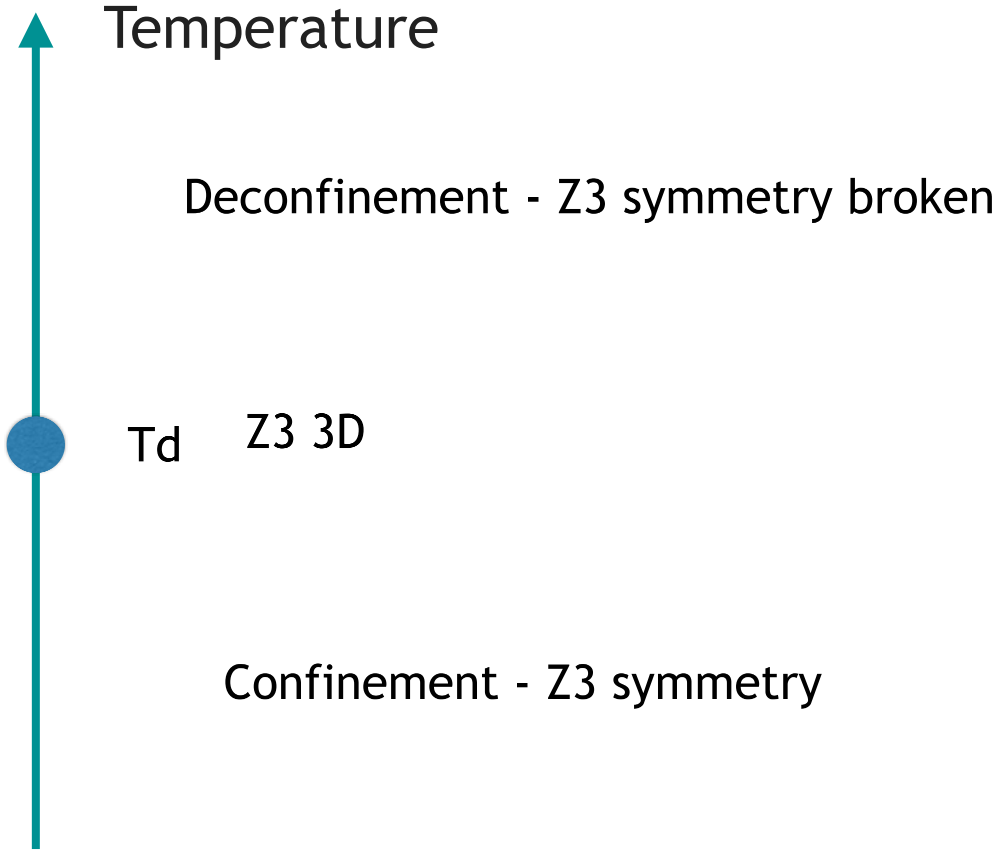
SU(2) : Z2 center symmetry breaking at Tc —> 2nd order 2 state Potts model (Ising)



Yang-Mills Theory Nc = 2 — Ising 3D universality class

Source		$SU(2)$	Ising
$< \|L\| >$	β/ν	0.525(8)	0.518(7)
$D < \|L\| >$	$(1-\beta)/\nu$	1.085(14)	1.072(7)
	$1/\nu$	1.610(16)	1.590(2)
	ν	0.621(6)	0.6289(8)
	β	0.326(8)	0.3258(44)
χ_ν	γ/ν	1.944(13)	1.970(11)
$D\chi_\nu$	$(1+\gamma)/\nu$	3.555(15)	3.560(11)
	$1/\nu$	1.611(20)	1.590(2)
	ν	0.621(8)	0.6289(8)
	γ	1.207(24)	1.239(7)
	$\gamma/\nu + 2\beta/\nu$	2.994(21)	3.006(18)
g_r	$-g_r^\infty$	1.403(16)	1.41
Dg_r	$1/\nu$	1.587(27)	1.590(2)
$(\omega = 1)$	ν	0.630(11)	0.6289(8)

Phases of Yang-Mills



Symmetries of QCD

$$\mathcal{L} = \sum_{a=1}^{n} \bar{q}_{La} \partial\!\!\!/ q_{La} + \bar{q}_{Ra} \partial\!\!\!/ q_{Ra} - m(\bar{q}_{La} q_{La} + \bar{q}_{Ra} q_{Ra}) + \theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a + \mathcal{L}_{gauge}$$

With $m = 0$, invariant under

$q_L \to V_L q_L q_R \to V_R q_R$, with $V \in U(n)$

Global symmetry:

$$U(n)_L \times U(n)_R \cong SU(n) \times \; SU(n) \times U(1)_V \times U(1)_A$$

.

baryon number

Spontaneously Broken

Explicitly broken

Experimental evidence

$(n^2 - 1)$ pseudoGB

Heavy η'

Explicit breaking of axial symmetry: solution of the UA(1) puzzle

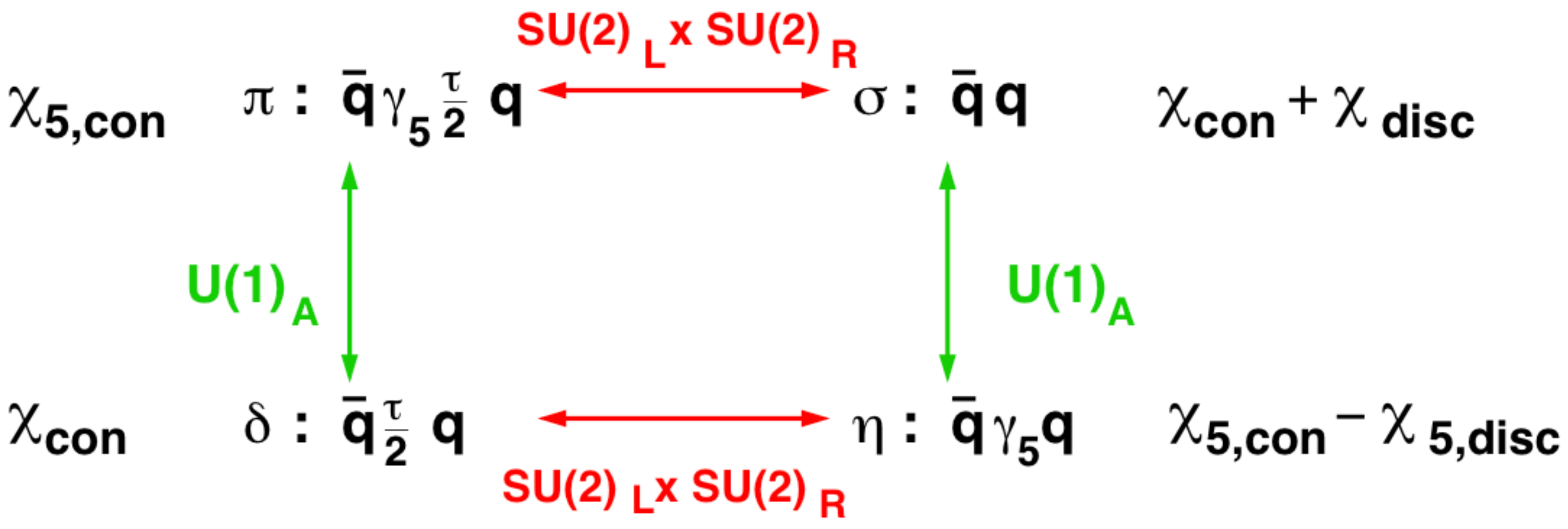
Pseudoscalar spectrum

Particle name	Particle symbol	Antiparticle symbol	Quark content	Rest mass (MeV/c^2)
Pion[6]	π^+	π^-	$u\bar{d}$	139.570 18 ±0.000 35
Pion[7]	π^0	Self	$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$ [a]	134.9766 ±0.0006
Eta meson[8]	η	Self	$\frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}}$ [a]	547.862 ±0.018
Eta prime meson[9]	$\eta'(958)$	Self	$\frac{u\bar{u}+d\bar{d}+s\bar{s}}{\sqrt{3}}$ [a]	957.78 ±0.06
Kaon[12]	K^+	K^-	$u\bar{s}$	493.677 ±0.016
Kaon[13]	K^0	$\bar{K}^0$	$d\bar{s}$	497.614 ±0.024

$U_A(1)$ puzzle

Pion, sigma => O(4) symmetry

Candidate universality class: O(4) in 3D



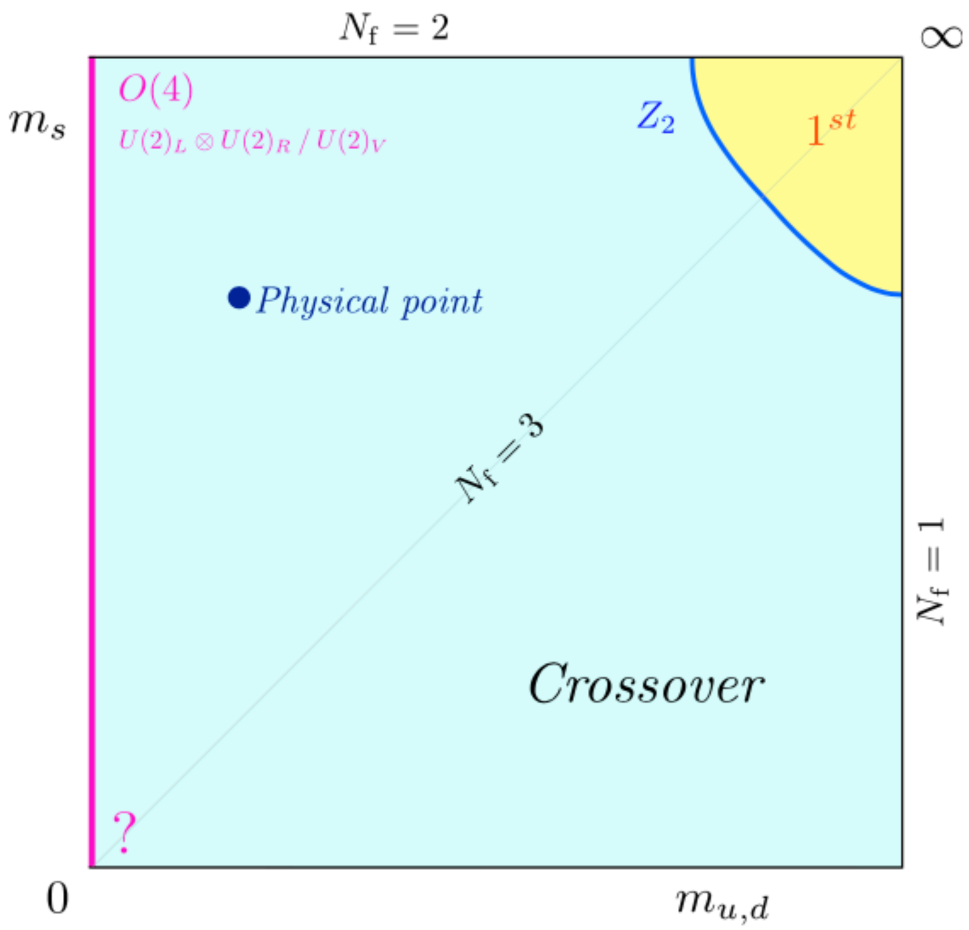
Universality class of the high T transition:

theoretical (*lack of*)guidance

Parisen Toldin, Pelissetto, Vicari 2003

N_f	$\mathrm{U}(1)_A$ broken	$\mathrm{U}(1)_A$ restored
1	crossover or 1^{st}ord	$\mathrm{O}(2) \to \mathbb{Z}_2$ or 1^{st}ord
2	$\mathrm{O}(4) \to \mathrm{O}(3)$ or 1^{st}ord Pisarski, Wilczek 1984	$\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R \to \mathrm{U}(2)_V$ or 1^{st}ord
≥ 3	1^{st}ord ?	1^{st}ord

Challenged by Cuteri, Philipsen, Sciarra 2020-2022


$N_f = 2$
∞
$O(4)$
$U(2)_L \otimes U(2)_R \, / \, U(2)_V$
m_s
Z_2
1^{st}
Physical point
$N_f = 3$
$N_f = 1$
Crossover
?
0
$m_{u,d}$

$SU(n) \times SU(n)$

$$M \quad \underset{m_\ell \to 0}{\sim} \quad \begin{cases} A\left(\frac{T_c^0 - T}{T_c^0}\right)^{\beta} & , \quad T < T_c^0 \\ 0 & , \quad T \geq T_c^0 \end{cases}$$

$$\chi_M \quad \underset{m_\ell \to 0}{\sim} \quad \begin{cases} \infty & , \quad T \leq T_c^0 \\ C\left(\frac{T - T_c^0}{T_c^0}\right)^{-\gamma} & , \quad T > T_c^0 \end{cases}$$

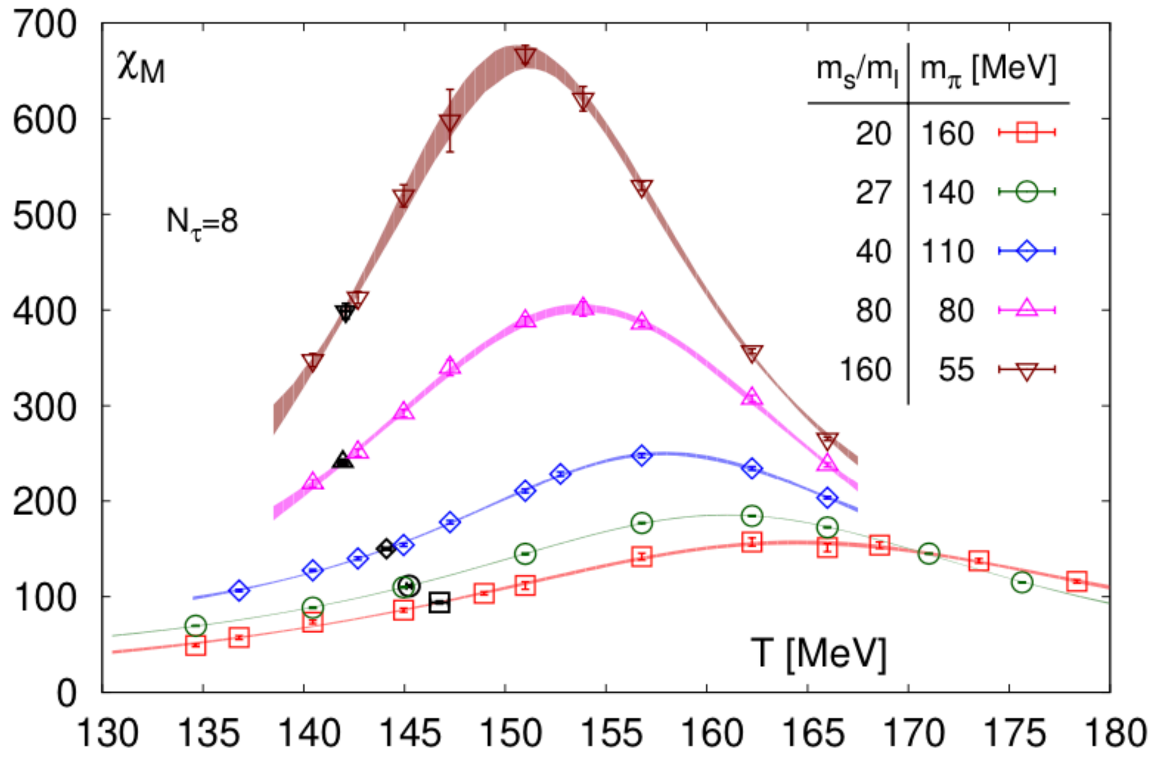
$SU(n) \times SU(n)$

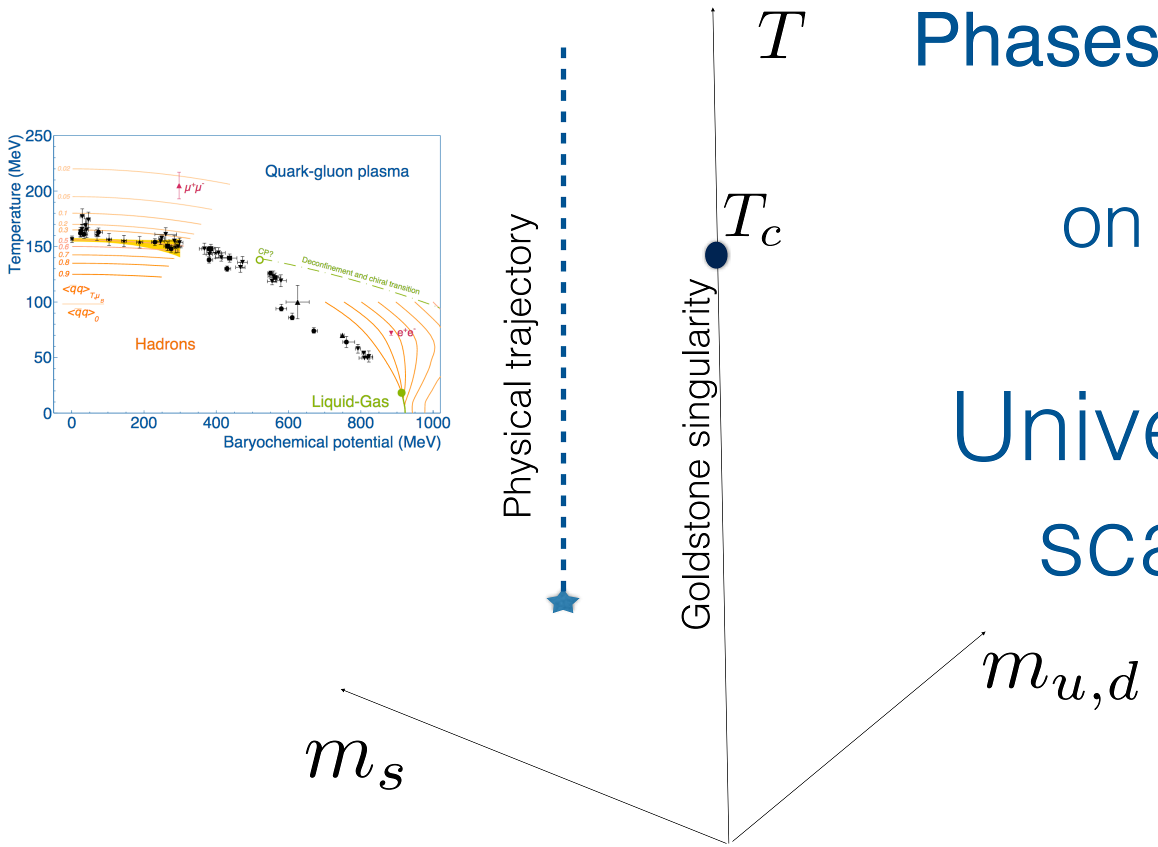
$T_c^0 = 132^{+3}_{-6}$ MeV

O(4) universality

$T_{pc} = 156.5(1.5)$ MeV

With physical quark masses





Phases of massless QCD:

imprinting
on observations?

↓

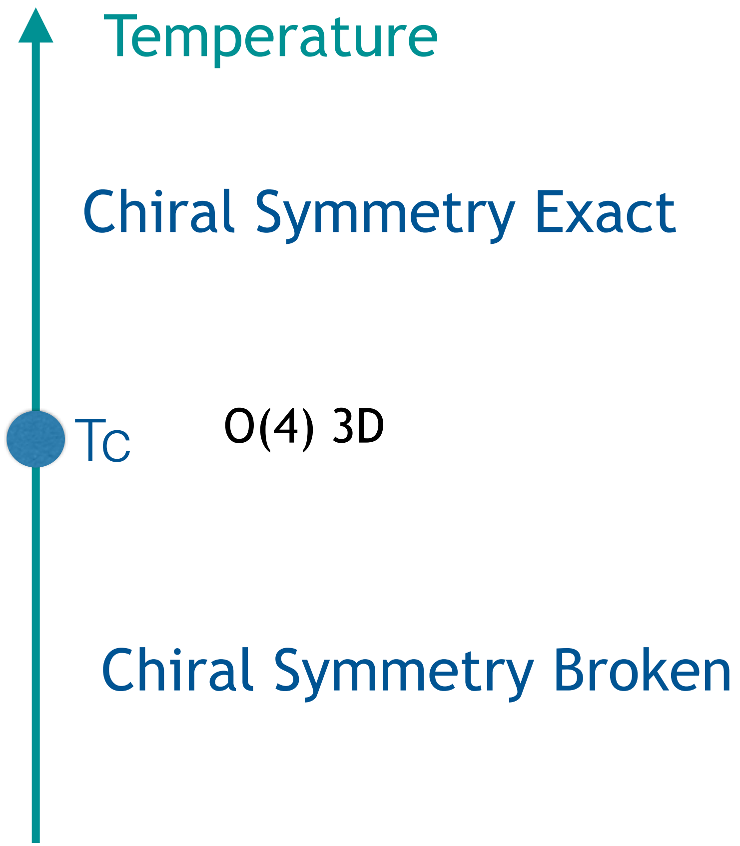
Universality class + scaling window

Fate of $U(1)_A$?

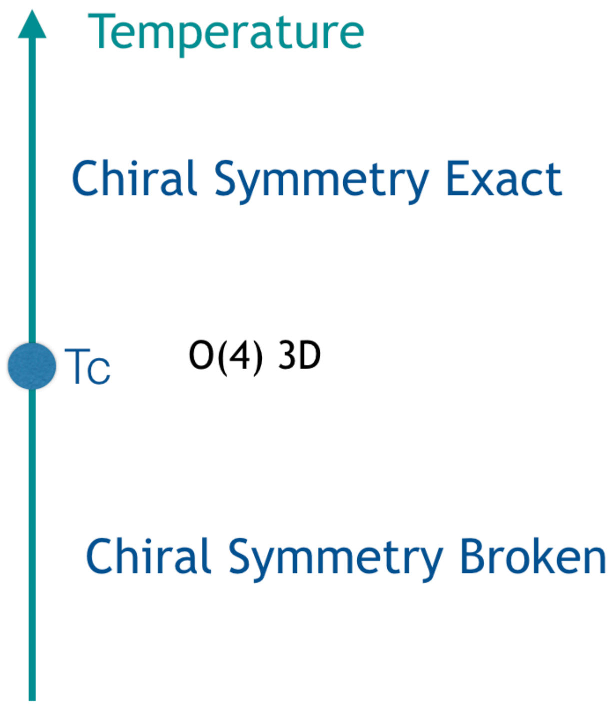
Probably still broken above Tc

The fate of the anomaly and its impact on the transition remains an important open problem

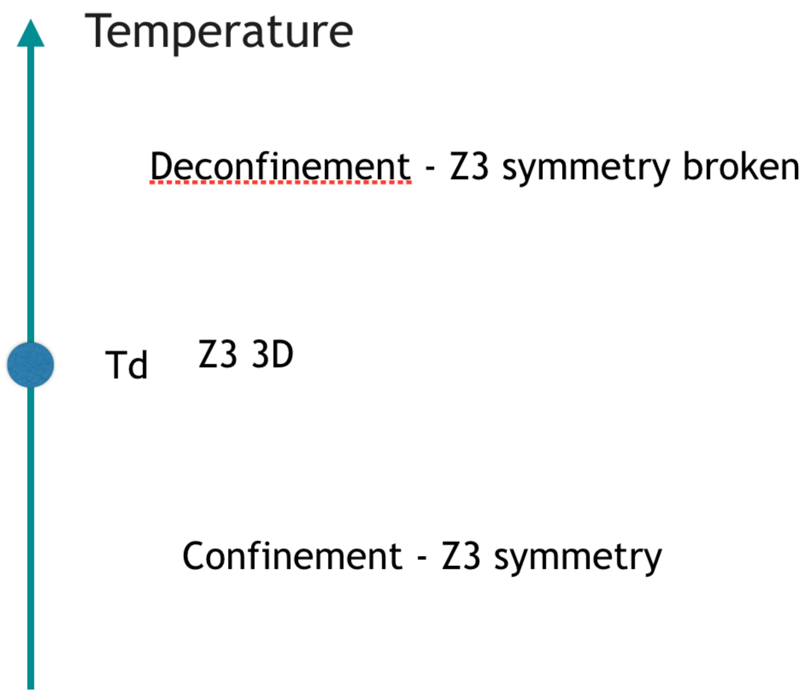
Phases of massless QCD



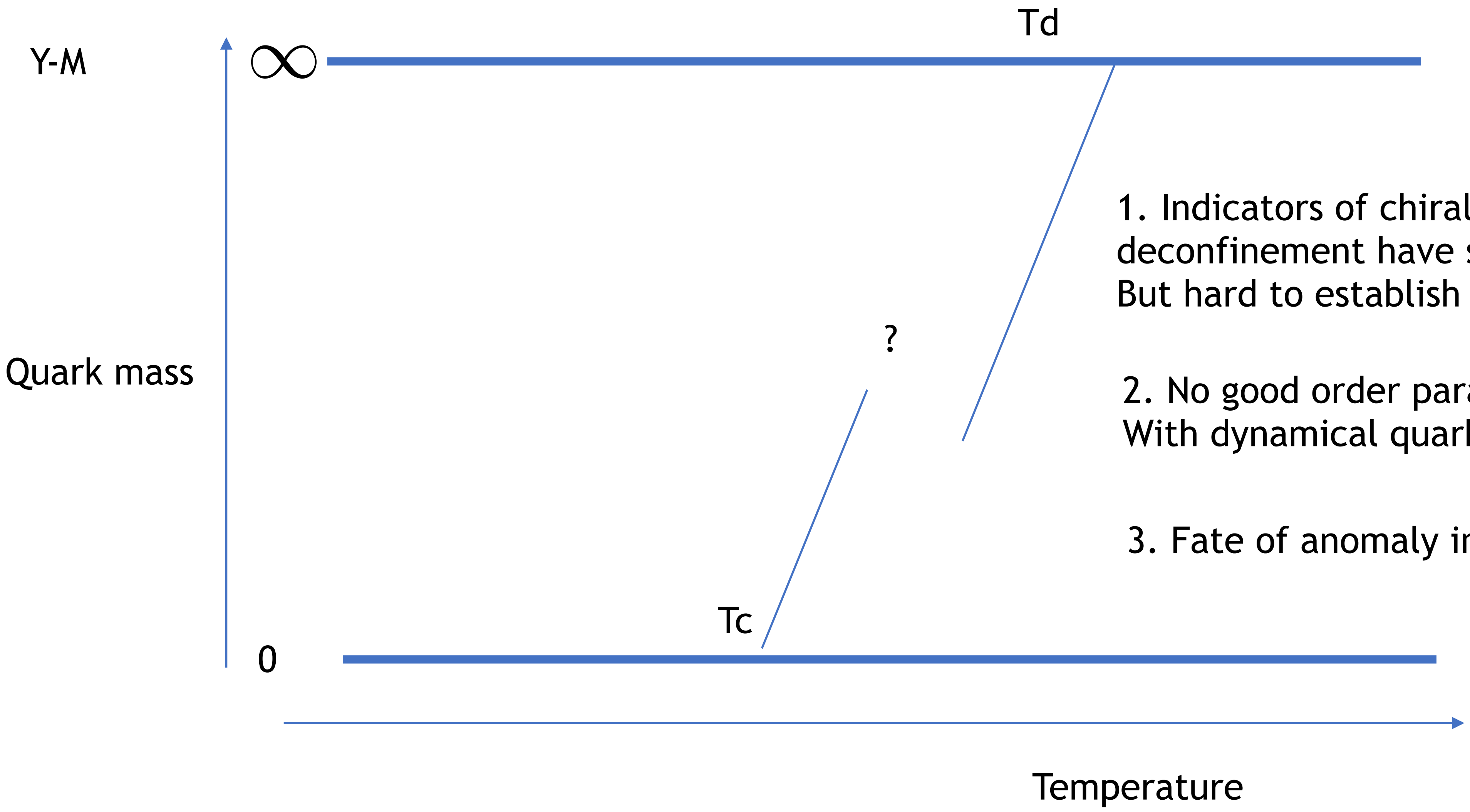
Phases of massless QCD



Phases of Yang Mills



Phases of QCD in the temperature, mass plane



1. Indicators of chiral symmetry & deconfinement have some correlation, But hard to establish it rigorously

2. No good order parameter for confinement With dynamical quarks

3. Fate of anomaly important and unclear

Topology and UA(1) problem

Focus on the θ term

$$\mathcal{L} = \mathcal{L}_{QCD} + \theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a$$

$$\frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a = q(x)$$

$$Q = \sum q(x)$$

Q — topological charge

CP-violating term

GCPF and θ

The GCPF of QCD is now a function of θ:

$$\mathcal{Z}(\theta, T) = \int \mathcal{D}[\Phi]\, e^{-T\sum_t \int d^3x \mathcal{L}(\theta)} = e^{-VF(\theta,T)}.$$

The energy density $F(\theta, T)$ is related with the probability of finding configurations with given topological charge $Q = \int d^4x\, q(x)$:

$$P_Q = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} e^{-VF(\theta)},$$

so the coefficients C_n of the Taylor expansion

$$F(\theta, T) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\theta^{2n}}{2n!} C_n$$

are given by the cumulants of the topological charge:

$$C_n = (-1)^{n+1} \frac{d^{2n}}{d\theta^{2n}} F(\theta, T)\Big|_{\theta=0} = \langle Q^{2n} \rangle_{conn}.$$

Topology, η' and solution of the $U_A(1)$ problem

It can be proven that

$$\frac{1}{32\pi^2}\int d^4x F\tilde{F} = Q$$ Gluonic definition

and

$$Q = n_+ - n_-$$ Fermionic definition

The η' mass may now be computed from the decay of the correlation

Q(x) Q(y)

$$\langle \partial_\mu j_5^\mu(x) \partial_\mu j_5^\mu(y) \rangle \propto \frac{1}{N^2} \langle F(x)\tilde{F}(x) F(y)\tilde{F}(y) \rangle$$

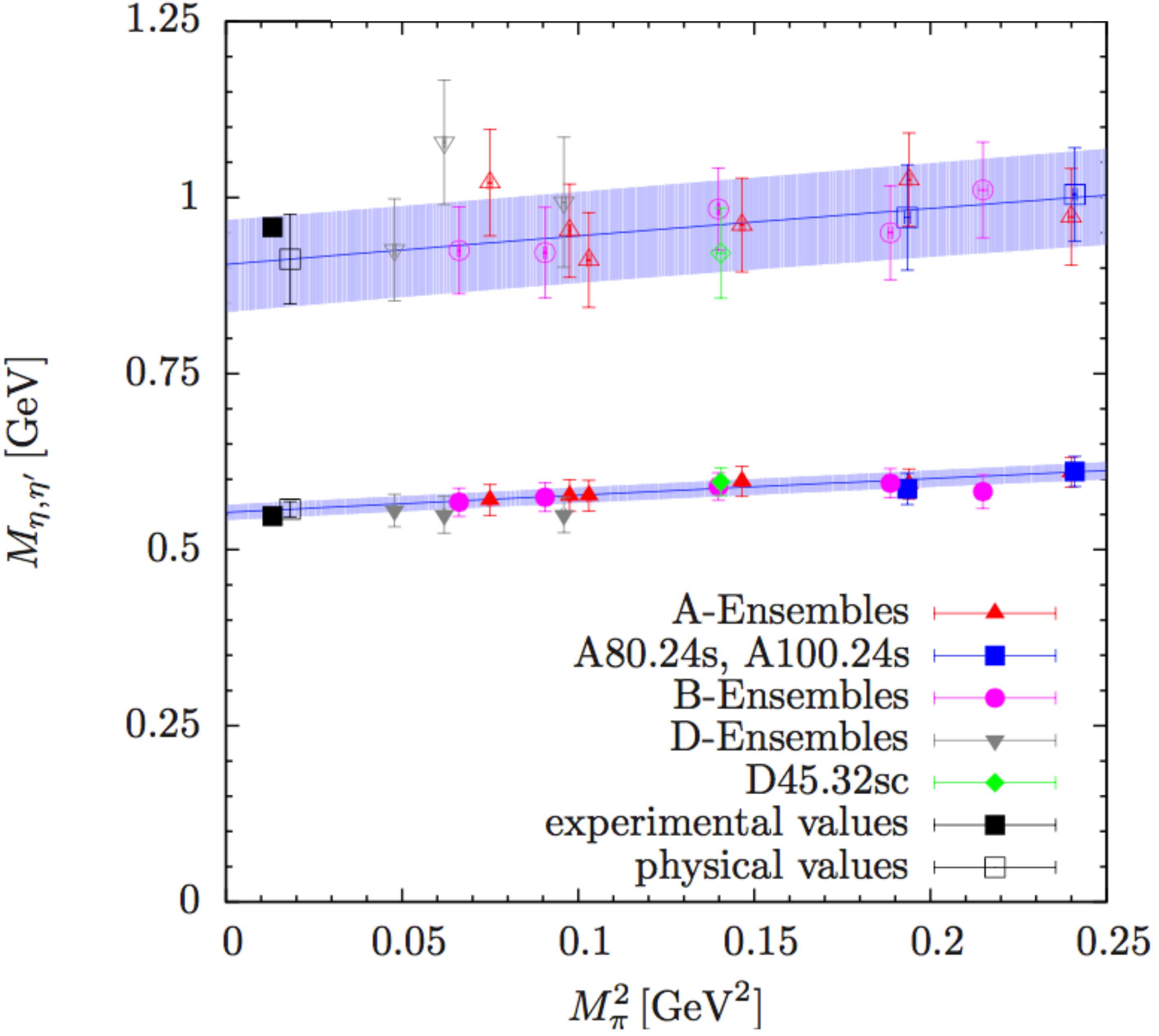
which at leading order gives the Witten-Veneziano formula

$$m^2_{\eta'} = \frac{2N_f}{F^2_\pi}\chi_t^{\mathrm{qu}}$$

Successful at T=0

Topology observable effects:

EVIDENCES OF THE EXPLICIT $U(1)_A$ BREAKING.

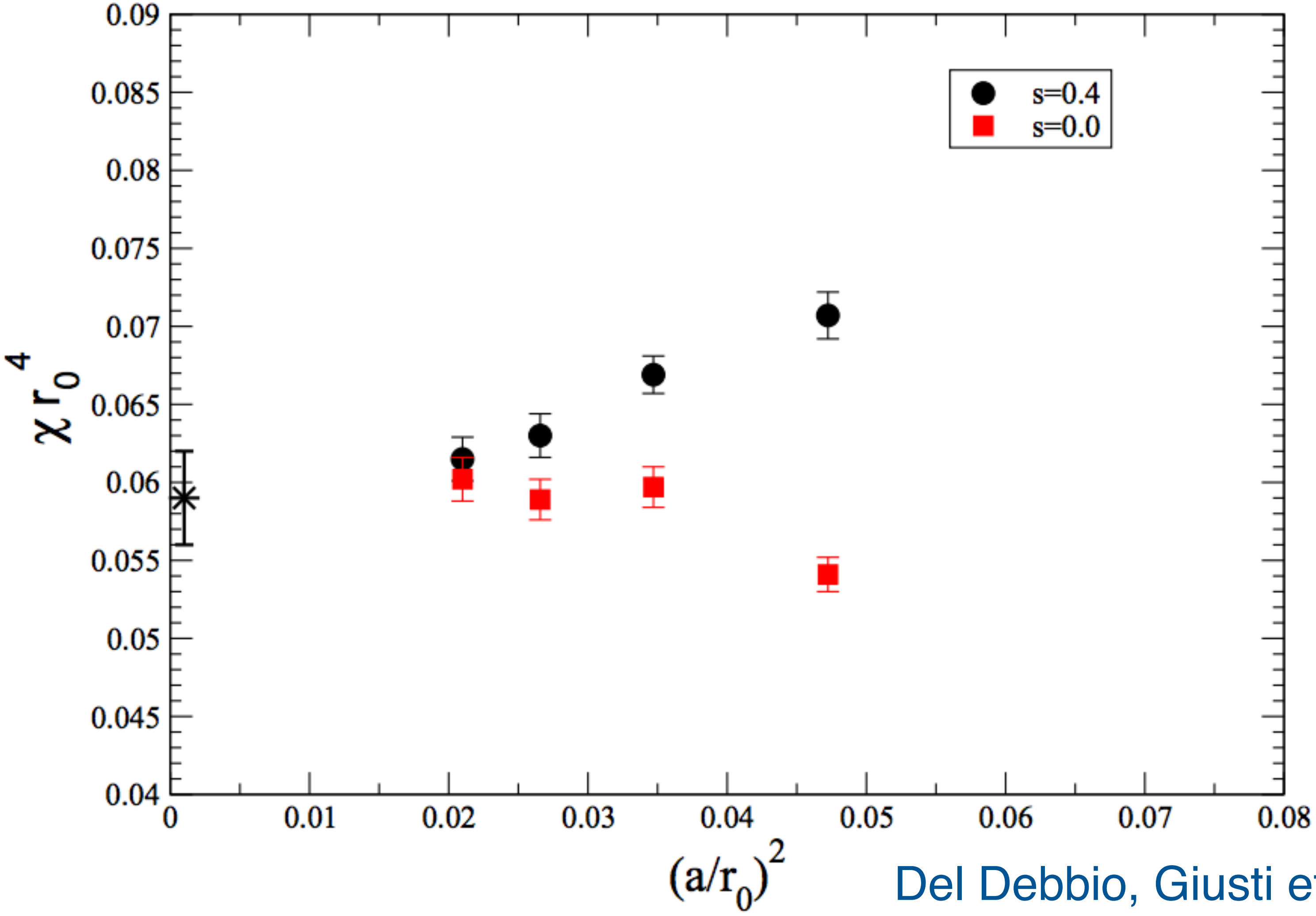


ETMC 2017

Topology observable effects:

EVIDENCES OF THE EXPLICIT $U(1)_A$ BREAKING.

$\chi = (191 \pm 5\,\mathrm{MeV})^4\,,$ Yang-Mills Topological Susceptibility



Del Debbio, Giusti et al. (2015)

Strong CP problem and the QCD axion

Lattice QCD and θ term

Lattice simulations rely on a sampling of the phase space weighted with the e^{-S}, where S the Euclidean action.

$$Z = \int \mathcal{D}[U]\, e^{-S_{lat}[U]}$$

- Sign problem: In Euclidean space-time the Minkowskian Lagrangian becomes *complex* for real values of θ.
- Way out: Taylor expansion around $\theta = 0$ or an analytic continuation from imaginary values of θ **Bonati, D'Elia et al. 2019**.

In the following, topological susceptibility and cumulants will be measured at $\theta = 0$:

$$\chi_{top}(T) \equiv \left. \frac{\partial^2 F(\theta, T)}{\partial \theta^2} \right|_{\theta=0}$$

Topology and the Strong CP problem

How 'large' is θ ?

The QCD Lagrangian admits a CP violating term

$$\mathcal{L} = \mathcal{L}_{QCD} + \theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a,$$

$\frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a$ is the topological charge density $q(x)$,

$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, and $\theta\, q(x)$ is known as the θ-term.

Without the θ-term strong interactions conserve CP. With the θ-term the neutron acquires an electric dipole moment d_n:

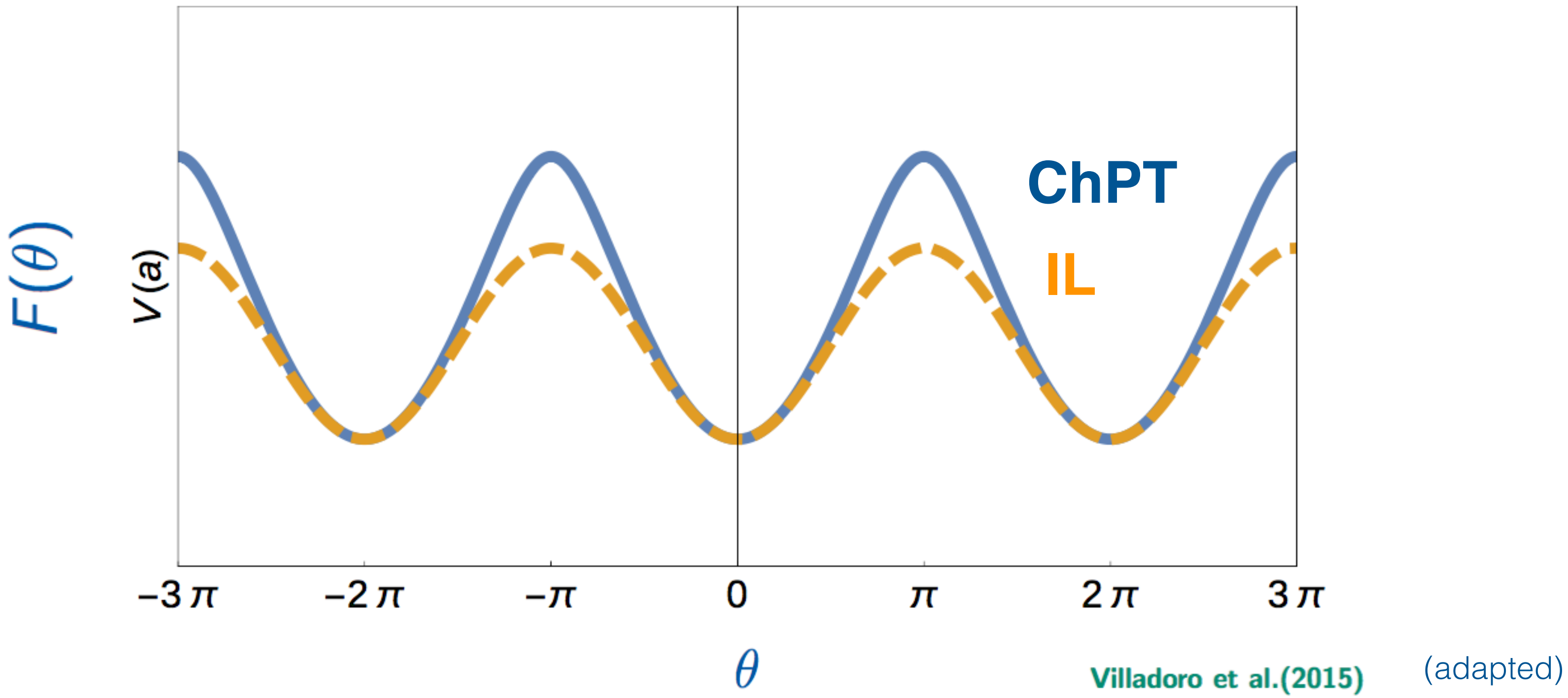
with QCD sum rules: $d_n = 2.4 \times 10^{-16}\theta$ e cm

chiral perturbation theory: $d_n = 3.3 \times 10^{-16}\theta$ e cm

Experiments: $|d_n| < 1.8 \times 10^{-26}$ e cm at a 90% C.L,

$$\theta < 0.5 \times 10^{-10}.$$

The free energy $F(\theta, T)$ has a minimum at $\theta = 0$
BUT this does not solve the strong CP problem



Solution of the strong CP problem: the axion

Suppose θ were a dynamical parameter: in such a case, dynamics would force its value to zero, thus solving the strong CP problem. Postulate Axion! a pseudo-Goldstone boson of a spontaneously broken symmetry known as the Peccei-Quinn (PQ) symmetry, which couples to the QCD topological charge, with a coupling suppressed by a scale f_A.
Axion field $a(x) = f_A \theta(x)$ is now a space-time dependent θ parameter.
The axion–QCD Lagrangian:

$$\mathcal{L} = \mathcal{L}_{QCD} + \theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a + \partial^2_\mu a^2 + \frac{a}{f_A} \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a .$$

Assume a shift symmetry: $a \to a + \alpha$. The θ dependence has been traded with a dependence on the axion field, whose minimum is at zero:

This solves the strong CP problem. !

The axion mass

At leading order in $1/f_A$ – well justified as $f_A \gtrsim 4 \times 10^8$ GeV – the axion can be treated as an external source, and its mass is given by

$$m_A^2(T) f_A^2 = \left. \frac{\partial^2 F(\theta, T)}{\partial \theta^2} \right|_{\theta=0} \equiv \chi_{top}(T) \ .$$

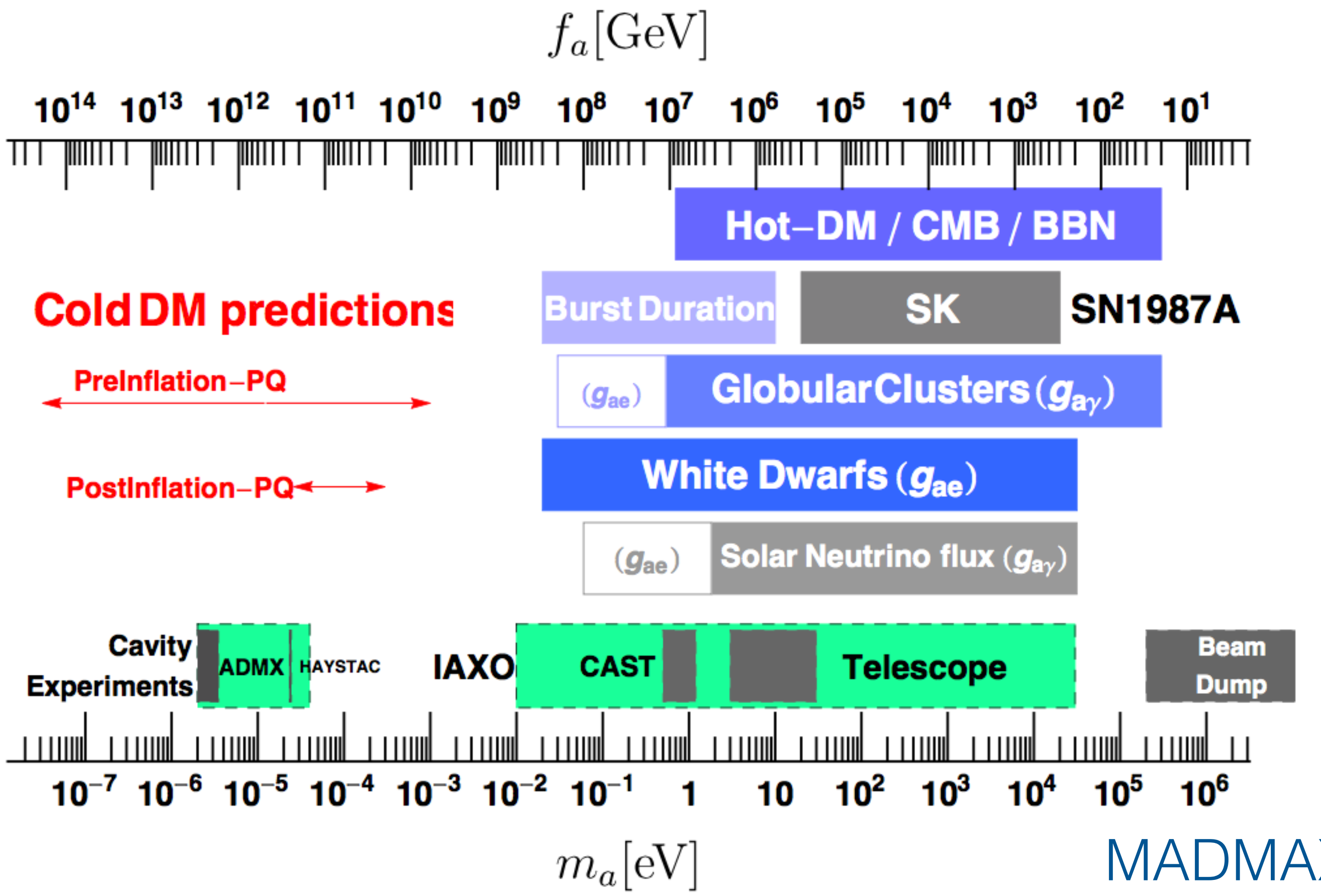
At zero and low temperature, chiral perturbation theory gives:

$$m_A^2 = \frac{m_u m_d}{(m_u + m_d)^2} \frac{m_\pi^2 f_\pi^2}{f_A^2},$$

In very brief summary, the essence of this discussion is the close relation between axion mass and topological susceptibility:

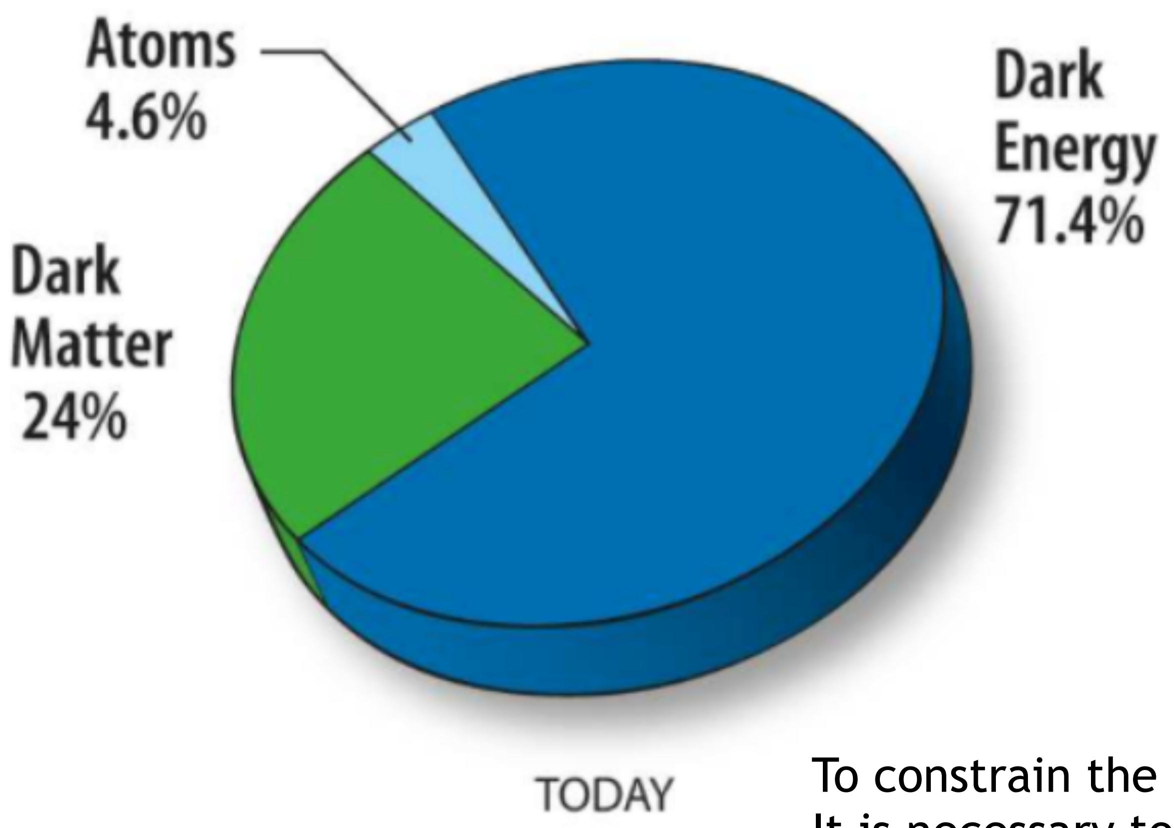
$$m_A^2 f_A^2 = \chi_{top},$$

QCD axion mass landscape



MADMAX (2019)

Candidate to Dark Matter:
QCD axion



To constrain the axion mass values,
It is necessary to know the
Topological susceptibility at high temperatures

Topology from Lattice QCD around and above the critical temperature

Analytic knowledge

At zero and low (below or around T_c) temperatures $F(\theta, T)$ can be computed by considering low energy effective Lagrangians

Finite temperature corrections have been computed, and their validity stretches till $T \simeq T_c$.

In the Quark Gluon Plasma phase the basic expression for $F(\theta, T)$ comes from dilute gas approximation (DIGA) and high temperature perturbation theory at leading order:

$$F(\theta, T) - F(0, T) \simeq T^{4-\beta_0} \left(\frac{m_l}{T}\right)^{N_{f,l}} (1 - \cos\theta),$$

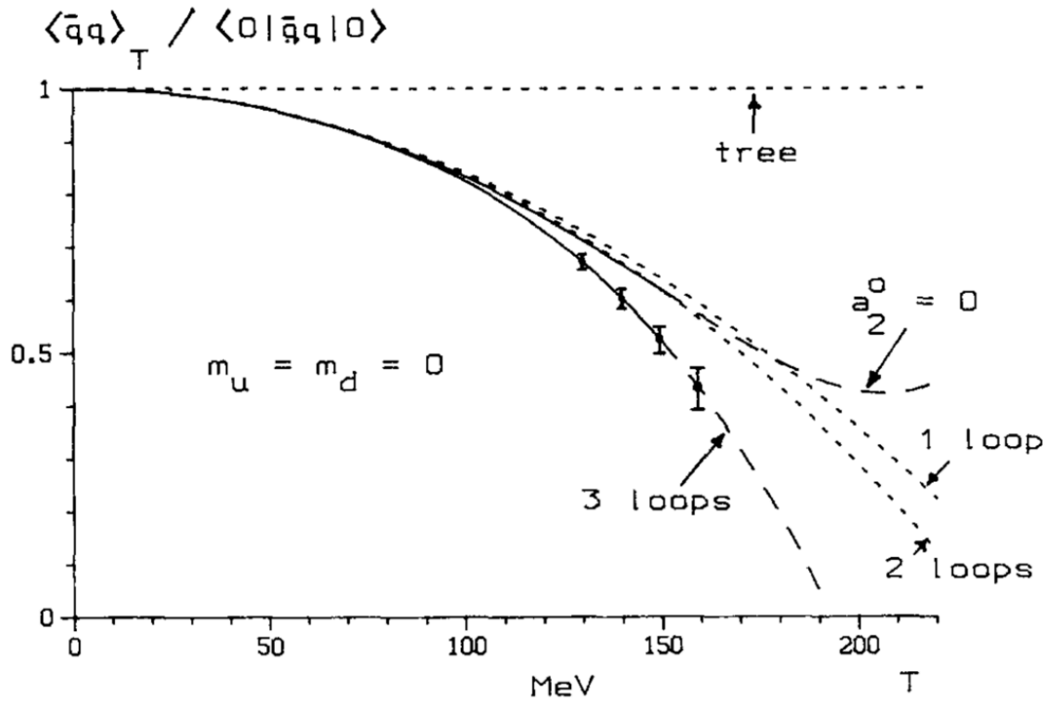
where $\beta_0 = 11N_c/3 - 2N_f/3$ and $N_{f,l}$ is the number of light flavors.

Intermediate temperatures, around T_c:
Lattice simulations

From chiral perturbation theory

$$\frac{\chi_{top}(T)}{\chi_{top}} \overset{\text{NLO}}{=} \frac{m_\pi^2(T) f_\pi^2(T)}{m_\pi^2 f_\pi^2} = \frac{\langle \bar{q}q \rangle_T}{\langle \bar{q}q \rangle}$$

$$\langle \bar{q}q \rangle \overset{m \to 0}{=} \langle 0|\bar{q}q|0 \rangle \left(1 - \frac{T^2}{8F^2} - \frac{T^4}{384F^4} - \frac{T^6}{288F^6} \ln \frac{\Lambda_q}{T} + O(T^8) \right)$$



Gasser-Leutwyler 1987-1989

For axion applications we need T approx. 500-600 MeV

1) Comparison with DIGA: Only instanton-anti-instanton pair contribute

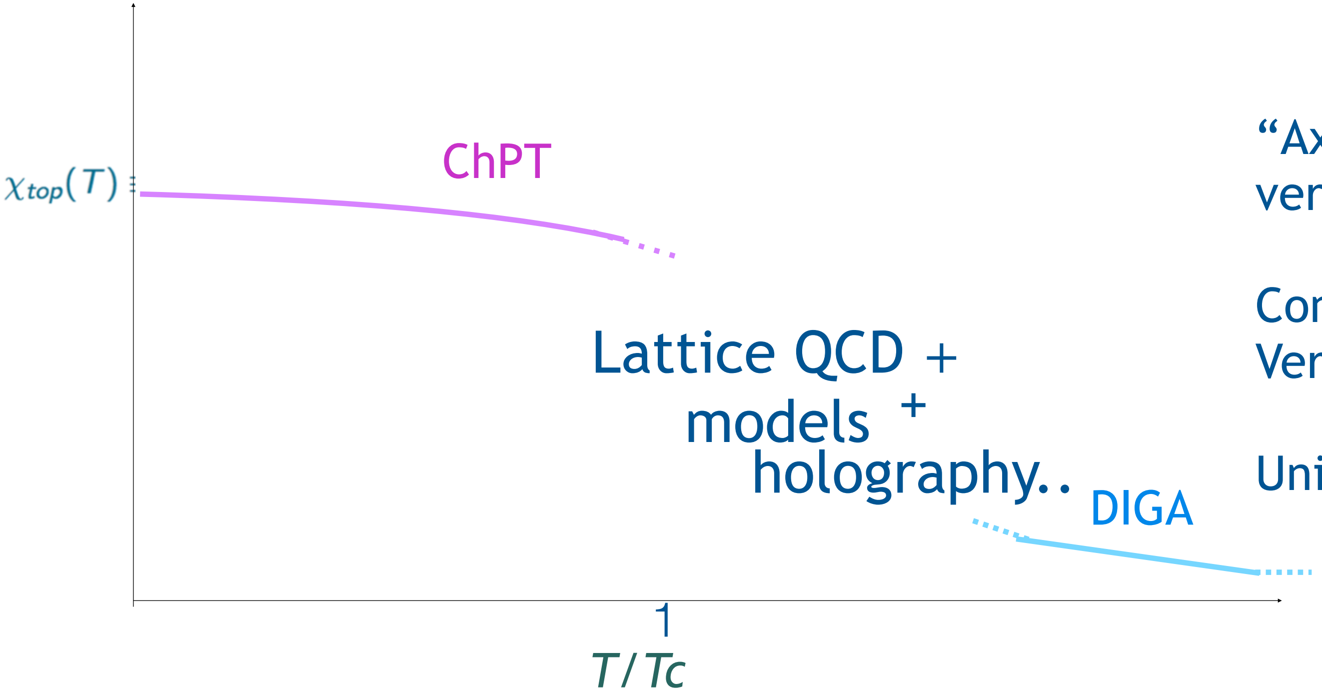
$$F(\theta,T) - F(0,T) \simeq T^{4-\beta_0}\left(\frac{m_l}{T}\right)^{N_{f,l}}(1-\cos\theta),$$

where $\beta_0 = 11N_c/3 - 2N_f/3$ and $N_{f,l}$ is the number of light flavors.

$$\chi(T) \sim T^4\left(\frac{m}{T}\right)^{N_f} e^{-8\pi^2/g^2(T)} \sim m^{N_f} T^{4-\frac{11}{3}N_c-\frac{1}{3}N_f}$$

2) Fate of the eta’

What do we know about $\chi_{top}(T) \equiv \left.\frac{\partial^2 F(\theta, T)}{\partial \theta^2}\right|_{\theta=0}$



"Axion mass
very small

Compton length
Very large

Universe very small.."

$$\chi(T) \sim T^4 \left(\frac{m}{T}\right)^{N_f} e^{-8\pi^2/g^2(T)} \sim m^{N_f} T^{4-\frac{11}{3}N_c-\frac{1}{3}N_f}$$

Lattice topology Michael Mueller-Preussker(2015)

- Gluonic: **Luscher(2010), Bonati,d'Elia e al (2014),Alexandrou et al . (2015)**

$$Q = \frac{a^4}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \sum_n \mathrm{Tr}[F_{lat}^{\mu\nu}(n) F_{lat}^{\rho\sigma}(n)],$$

Need smooth configurations,using smearing,cooling, gradient flow..

$$\dot{V}_\mu(n,\tau) = -g^2[\partial_{n,\mu} S_G(V(\tau))] V_\mu(n,\tau), \qquad V_\mu(n,0) = U_\mu(n),$$

Pros: Easy
Cons: suffers very much from lattice artifacs

- Fermionic: **Atiyah Singer(1971,1984)**

$$Q = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \int \mathrm{Tr}[F^{\mu\nu}(x) F^{\rho\sigma}(x)]\, d^4x = n_+ - n_-$$

Pros: not affected but UV fluctuations
Cons: very high computational cost

- Fermionic - simple but approximate: **Kogut et al.(1996),Petreczky, Sharma(2016)**

$$\chi_{top} = \frac{\langle Q^2 \rangle}{V} = m_l^2\, \chi_{5,disc}$$

$$\chi_{top}(T \gtrsim T_c) = m_l^2\, \chi_{disc} = m_l^2 \frac{V}{T} \left(\langle (\bar\psi\psi)^2 \rangle_l - \langle \bar\psi\psi \rangle_l^2 \right).$$

- Fermionic - simple but approximate: **Kogut et al.(1996),Petreczky, Sharma(2016)**

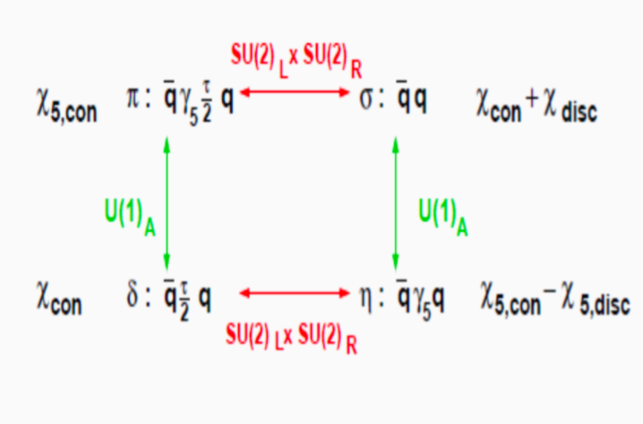
$$\chi_{top} = \frac{\langle Q^2 \rangle}{V} = m_l^2 \, \chi_{5,disc}$$

$$\chi_{top}(T \gtrsim T_c) = m_l^2 \, \chi_{disc} = m_l^2 \, \frac{V}{T} \left(\langle (\bar{\psi}\psi)^2 \rangle_l - \langle \bar{\psi}\psi \rangle_l^2 \right) .$$ Above Tc

□

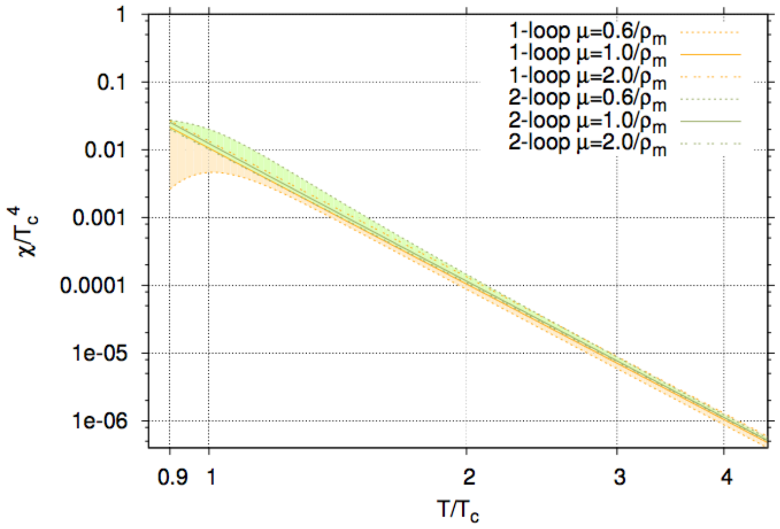
Recall:

- Degeneracy pattern for two flavors: MILC plot



Yang-Mills

Two loop RG DIGA

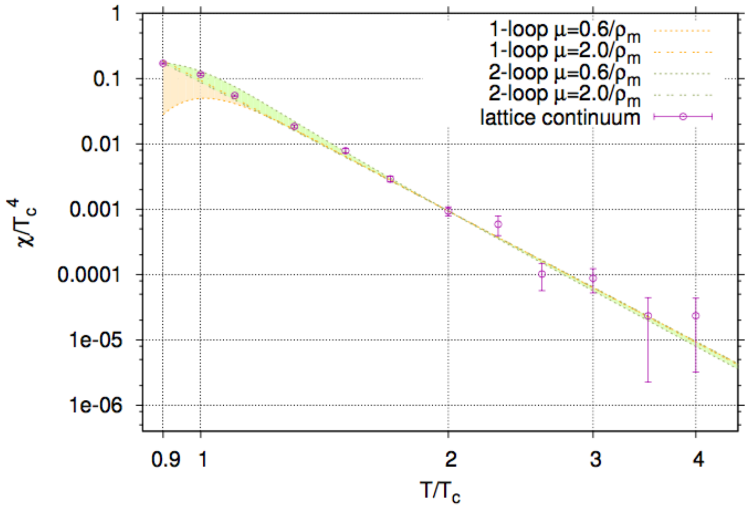


T/T_c	1.5	2	3	4	5
$b\ (\kappa = 0.6)$	-6.04	-6.26	-6.43	-6.50	-6.55
$b\ (\kappa = 1)$	-6.37	-6.46	-6.55	-6.59	-6.62
$b\ (\kappa = 2)$	-6.55	-6.59	-6.64	-6.67	-6.69

Table 2: Temperature slopes of the topological susceptibility predicted in the two-loop RGI DIGA, for a range of renormalization scales ac-

Borsanyi et al. 2015

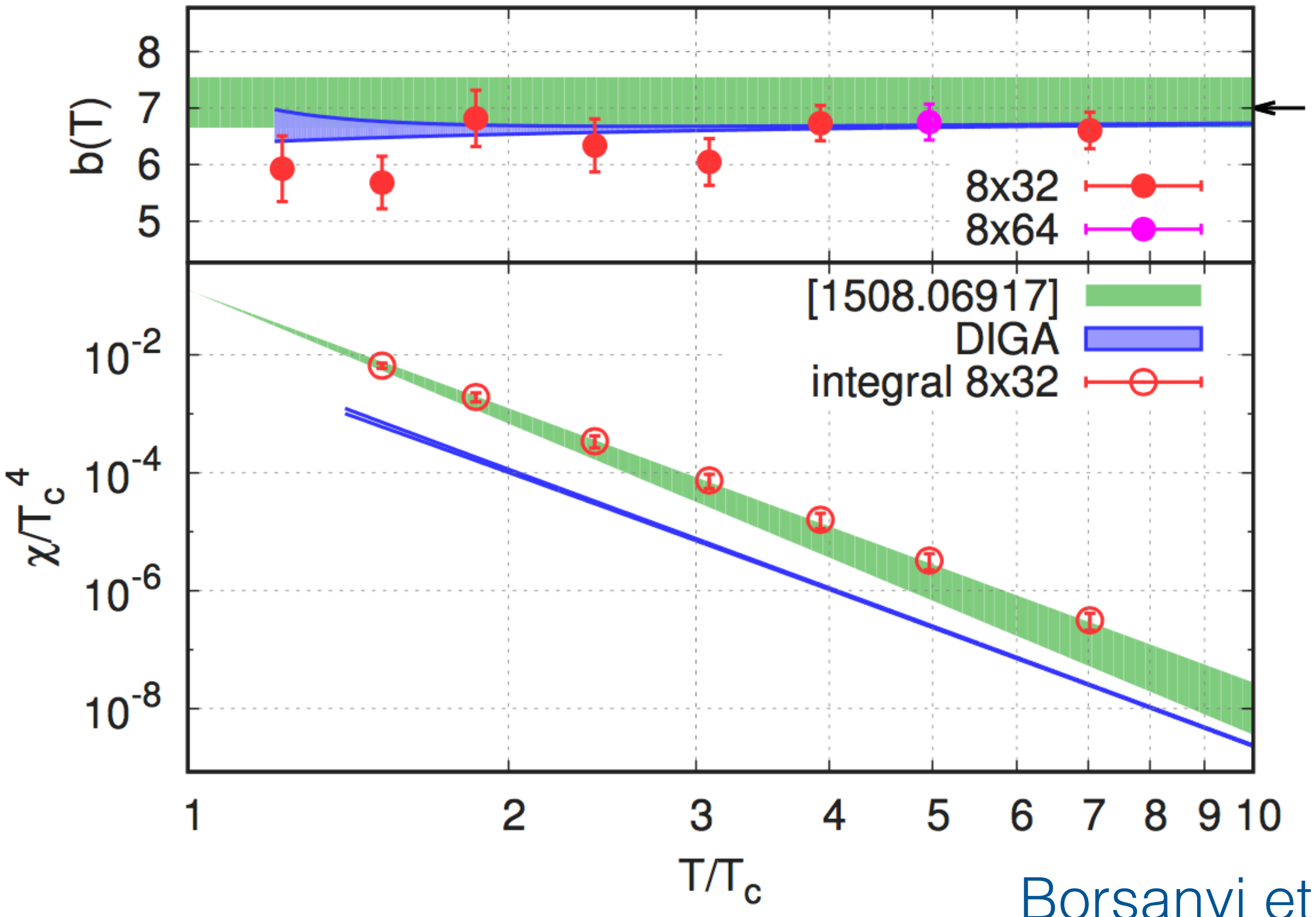
Lattice results



Exponent OK

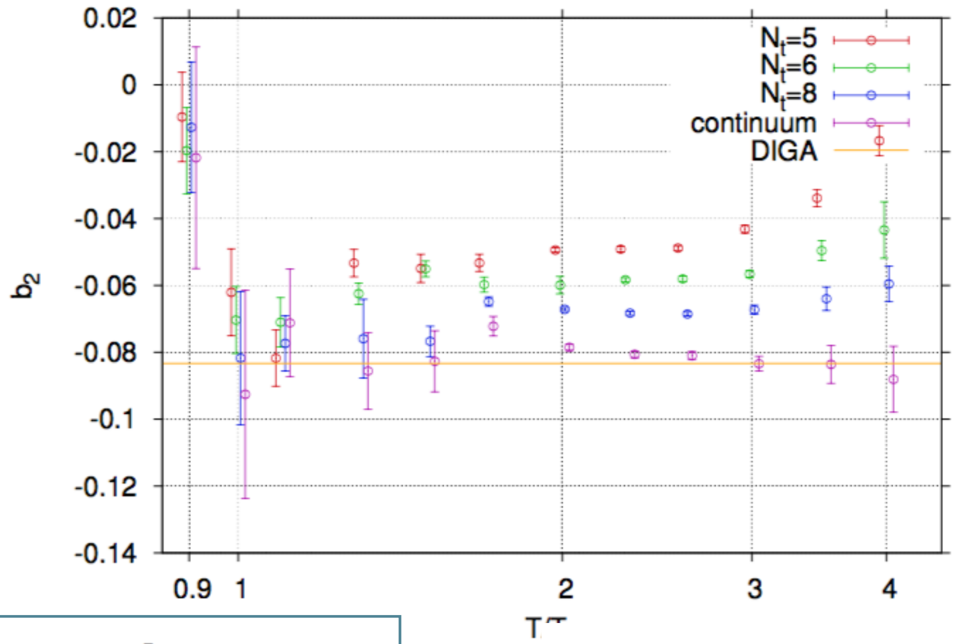
One order of magnitude discrepancy in the absolute value

T/T_c	1	2	3	4	5
$\alpha_{\overline{\mathrm{MS}}}(1/\rho_m)$	0.36	0.23	0.19	0.17	0.16



Borsanyi et al. 2016

Beyond topological susceptibility



$$V(A,T) = \frac{1}{2}\chi(T)\theta^2\left[1 + b_2(T)\theta^2 + \ldots\right]|_{\theta=A/f_A},$$

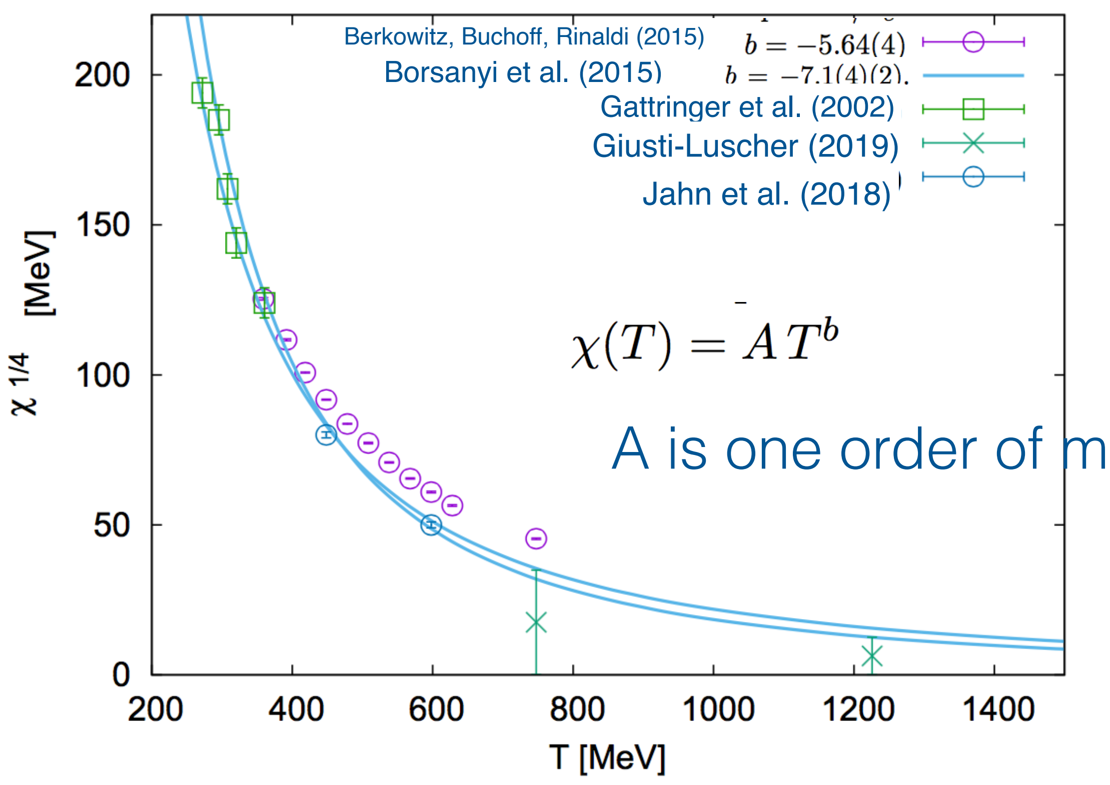
$$b_2(T) = -\frac{\langle Q^4\rangle_T - 3\langle Q^2\rangle_T^2}{12\langle Q^2\rangle_T}\,|_{\theta=0}\,.$$

Bonati,D'Elia,Panagopoulos,Vicari 2013

$$C_n = (-1)^{n+1}\frac{d^{2n}}{d\theta^{2n}}F(\theta,T)\Big|_{\theta=0} = \langle Q^{2n}\rangle_{conn}.$$

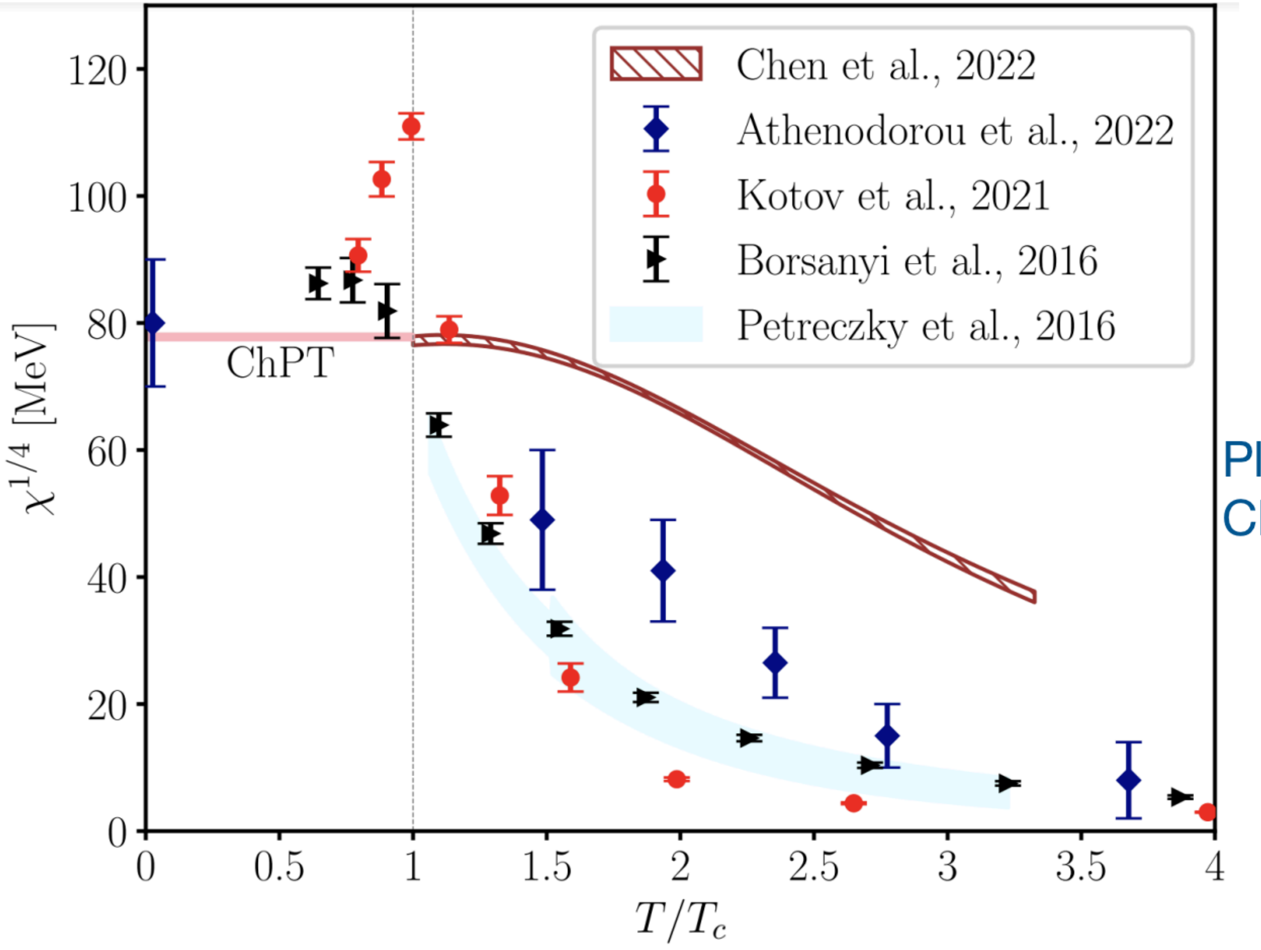
Borsanyi et al. 2016

Yang-Mills - summary



A is one order of magnitude off!

QCD - summary



Plot by
Claudio Bonanno

Na6-Strong2020, MpL et al, 2023

Heavy Quarks

Quarkonia
Potential models
Temperature effects
NRQCD

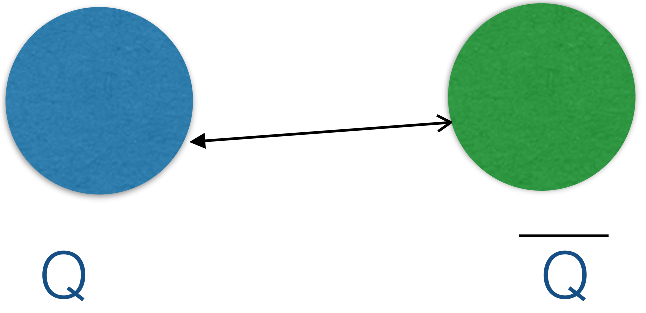
Quarkonia: heavy quarks ⇒ non-relativistic potential theory

Jacobs et al. 1986

Schrödinger equation $\left\{2m_c - \frac{1}{m_c}\nabla^2 + V(r)\right\}\Phi_i(r) = M_i\Phi_i(r)$

with confining ("Cornell") potential $V(r) = \sigma\, r - \frac{\alpha}{r}$

$$E(r) = 2m + \frac{1}{2mr^2} + V(r),$$



Minimizing the energy:

$$\frac{1}{mr^3_{J/\psi}} - \frac{\alpha_{\text{eff}}}{r^2_{J/\psi}} - \sigma = 0$$

Relation between the parameters of the potential, the mass and the radius

Quarkonia: heavy quarks ⇒ non-relativistic potential theory

Jacobs et al. 1986

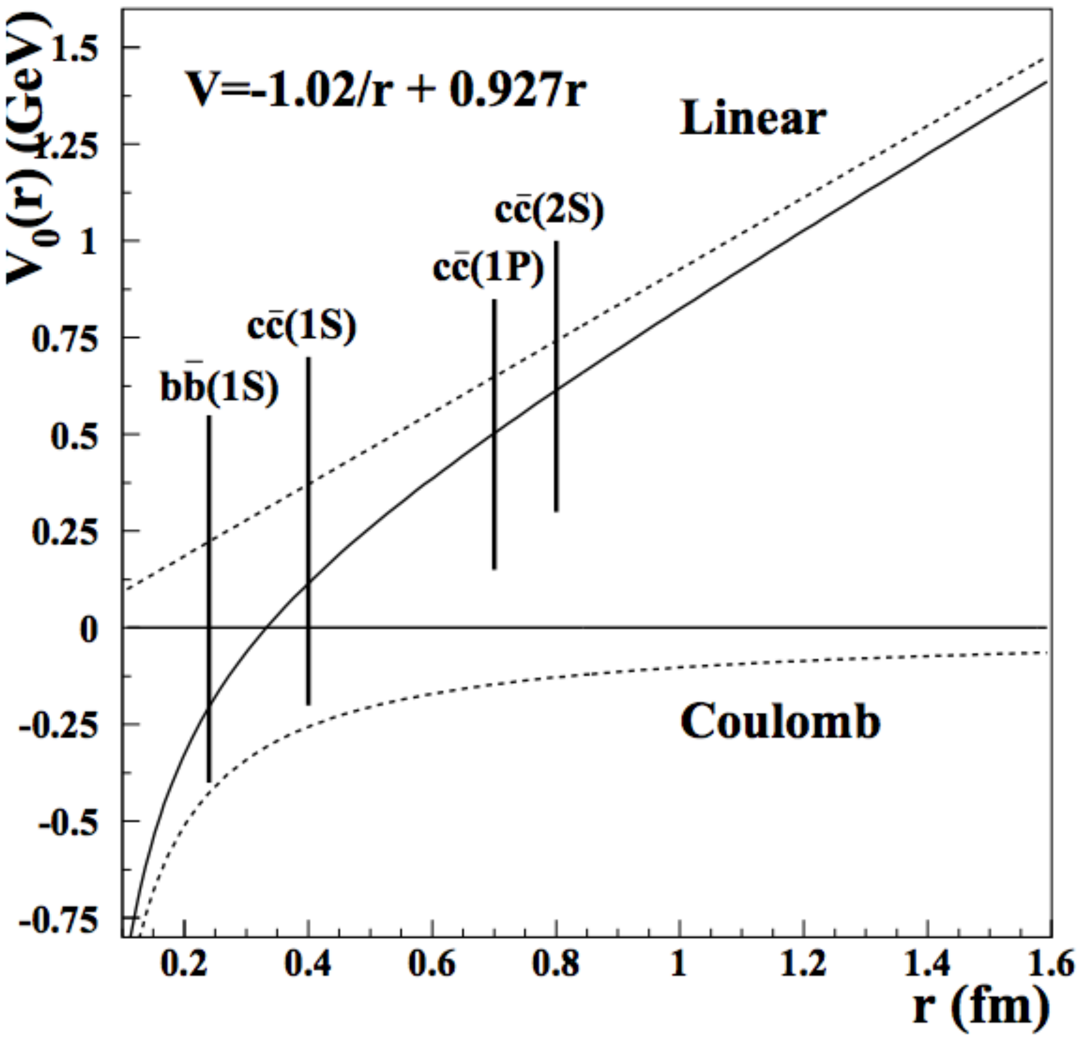
Schrödinger equation $\left\{2m_c - \frac{1}{m_c}\nabla^2 + V(r)\right\}\Phi_i(r) = M_i\Phi_i(r)$

with confining ("Cornell") potential $V(r) = \sigma\, r - \frac{\alpha}{r}$

state	J/ψ	χ_c	ψ'	Υ	χ_b	Υ'	χ_b'	Υ''
mass [GeV]	3.10	3.53	3.68	9.46	9.99	10.02	10.26	10.36
radius [fm]	0.25	0.36	0.45	0.14	0.22	0.28	0.34	0.39

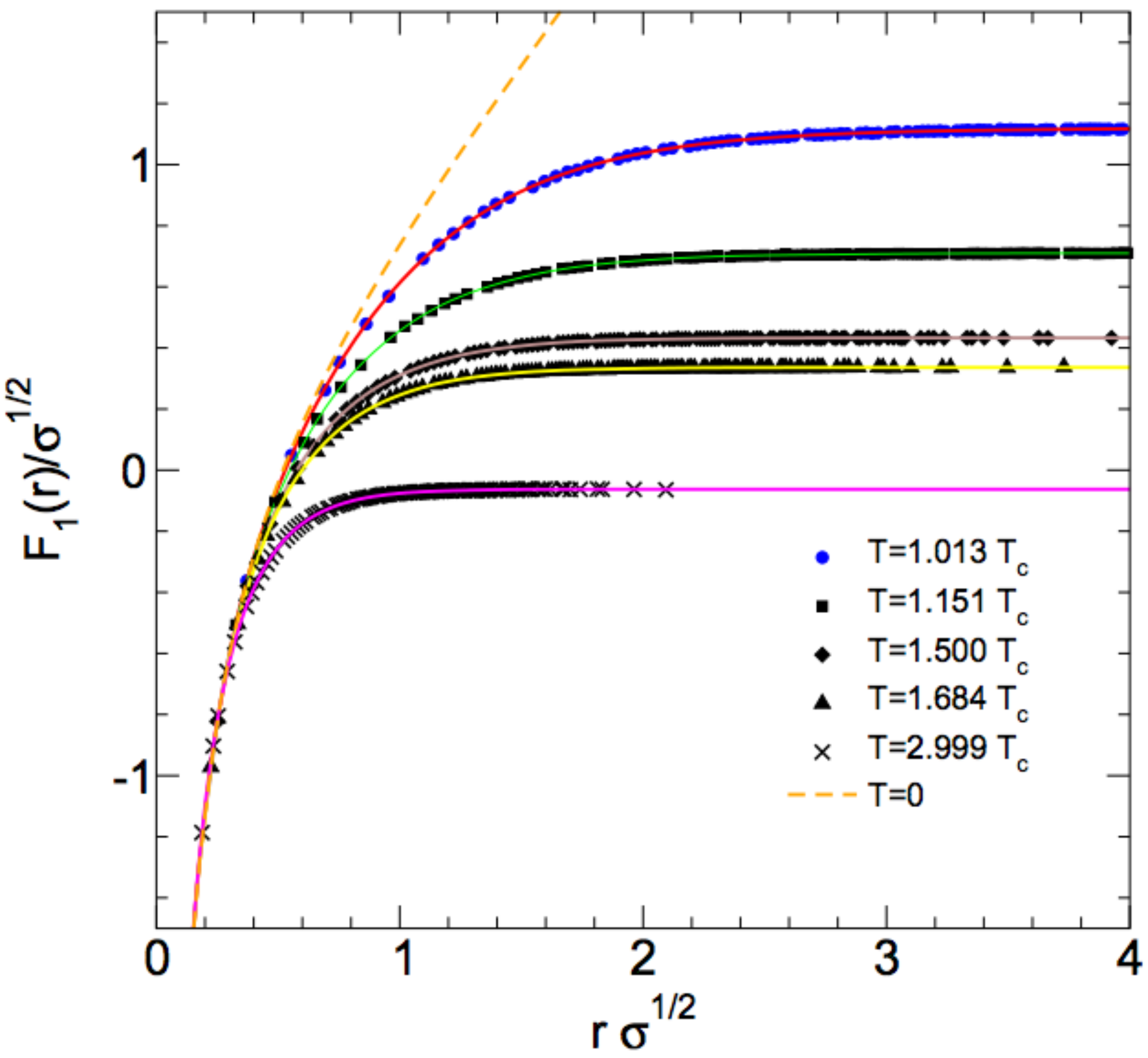
Bound by Coulombic part vs confining binding

Potential probes gauge dynamics



Seth 2009

Temperature effects

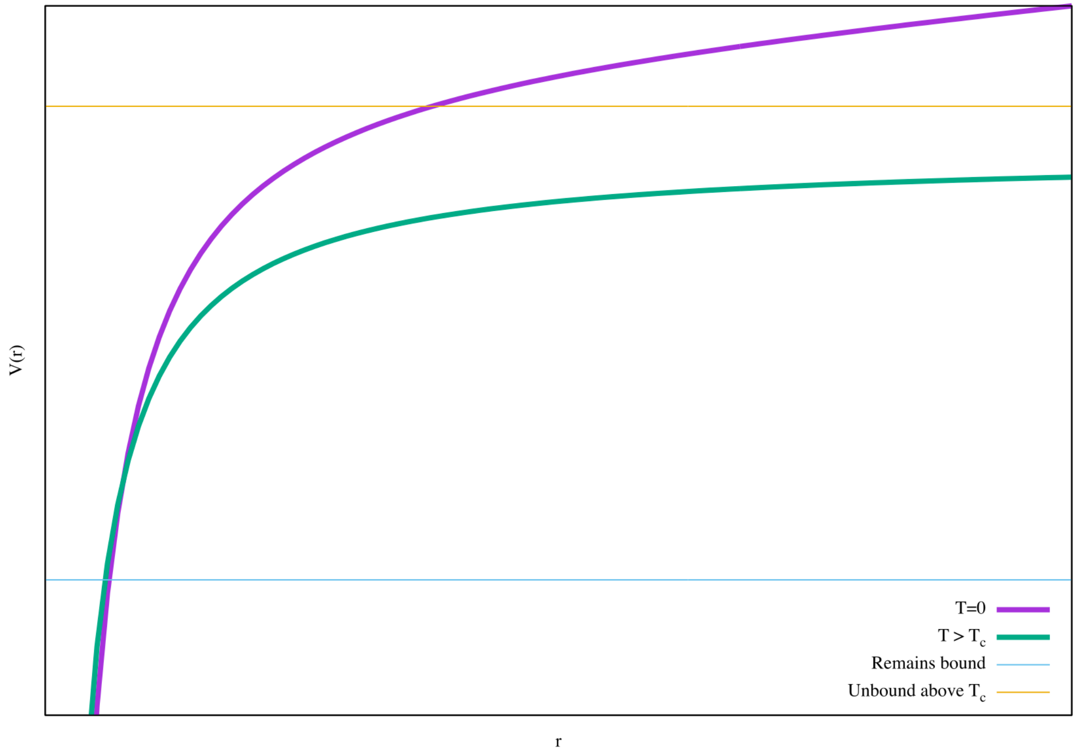


Heavy quark potential from: models + high T p.t.

High temperature Debye Screening

$$V(r) = \sigma\, r - \frac{\alpha}{r} \quad \xrightarrow{T} \quad V(r) \sim \frac{e^{-m_D^0 r}}{4\pi r},$$

$$m_D^0 = \left(\frac{N}{3} + \frac{N_f}{6}\right)^{1/2} gT$$



Caveat: recent works challenge this picture

Potential model recipe:

1) Find Potential as a function of temperature

2) Solve Schroedinger eq. as a function of T

3) Check who remains bound (binding energy)

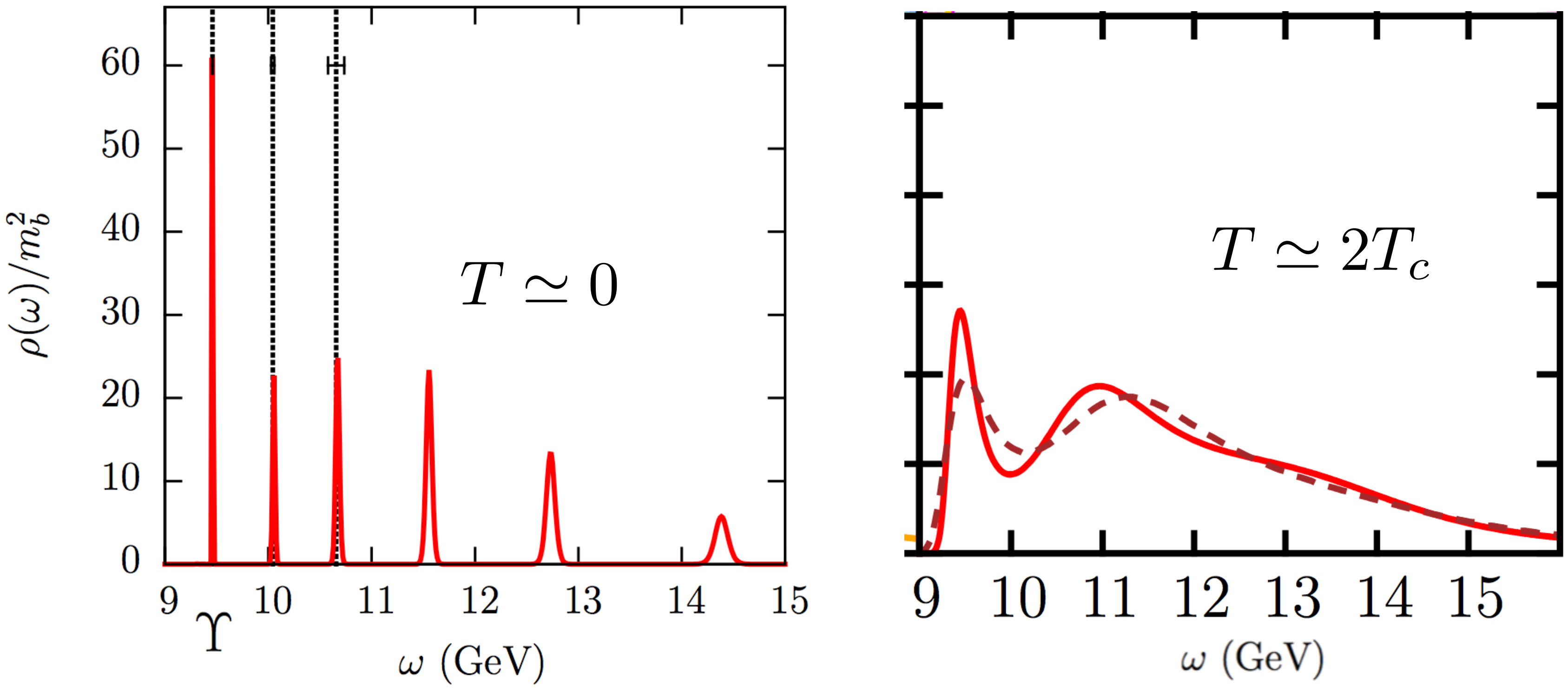
Various uncertainties ..

Compromise: Bottomonium via NRQCD

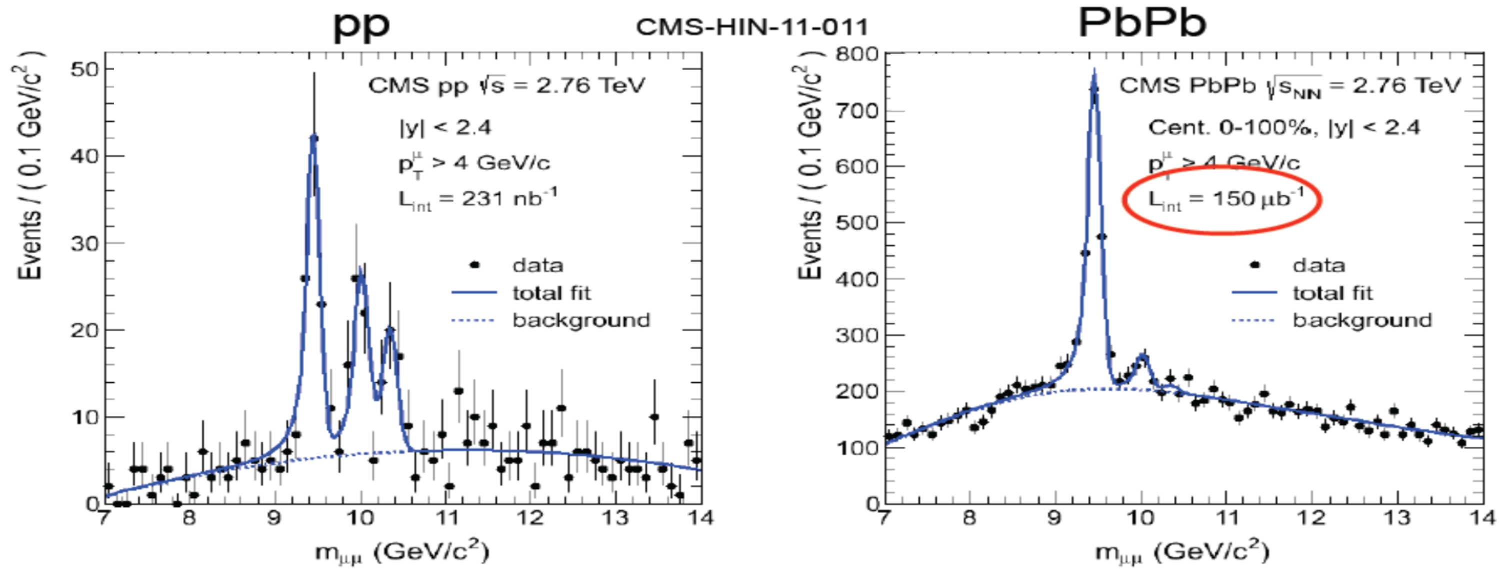
Zero temperature NRQCD works beautifully for the spectrum

$n^{S+1}L_J$	State	$a_\tau M$	$E_0 + M$ (MeV)	$M_{\rm expt}$ (MeV)
1^1S_0	η_b	0.20549(4)	9409(12)	9398.0(3.2)
2^1S_0	η_b'	0.311(3)	10004(21)	9999(4)
1^3S_1	Υ	0.21460(5)	9460*	9460.30(26)
2^3S_1	Υ'	0.318(3)	10043(22)	10023.26(31)
1^1P_1	h_b	0.2963(4)	9920(15)	9899.3(1.0)
1^3P_0	χ_{b0}	0.2921(4)	9896(15)	9859.44(52)
1^3P_1	χ_{b1}	0.2964(4)	9921(15)	9892.78(40)
1^3P_2	χ_{b2}	0.2978(4)	9928(15)	9912.21(40)

Bottomonium spectral functions from the lattice



Bottomonium as a probe of QGP

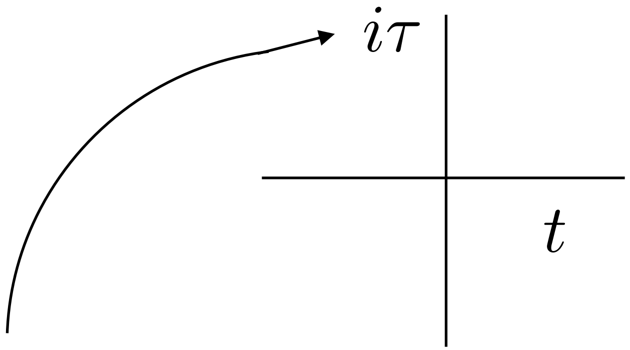


$$N_{\Upsilon(2S)}/N_{\Upsilon(1S)}|_{pp} = 0.56 \pm 0.13 \pm 0.01$$
$$N_{\Upsilon(3S)}/N_{\Upsilon(1S)}|_{pp} = 0.21 \pm 0.11 \pm 0.02$$

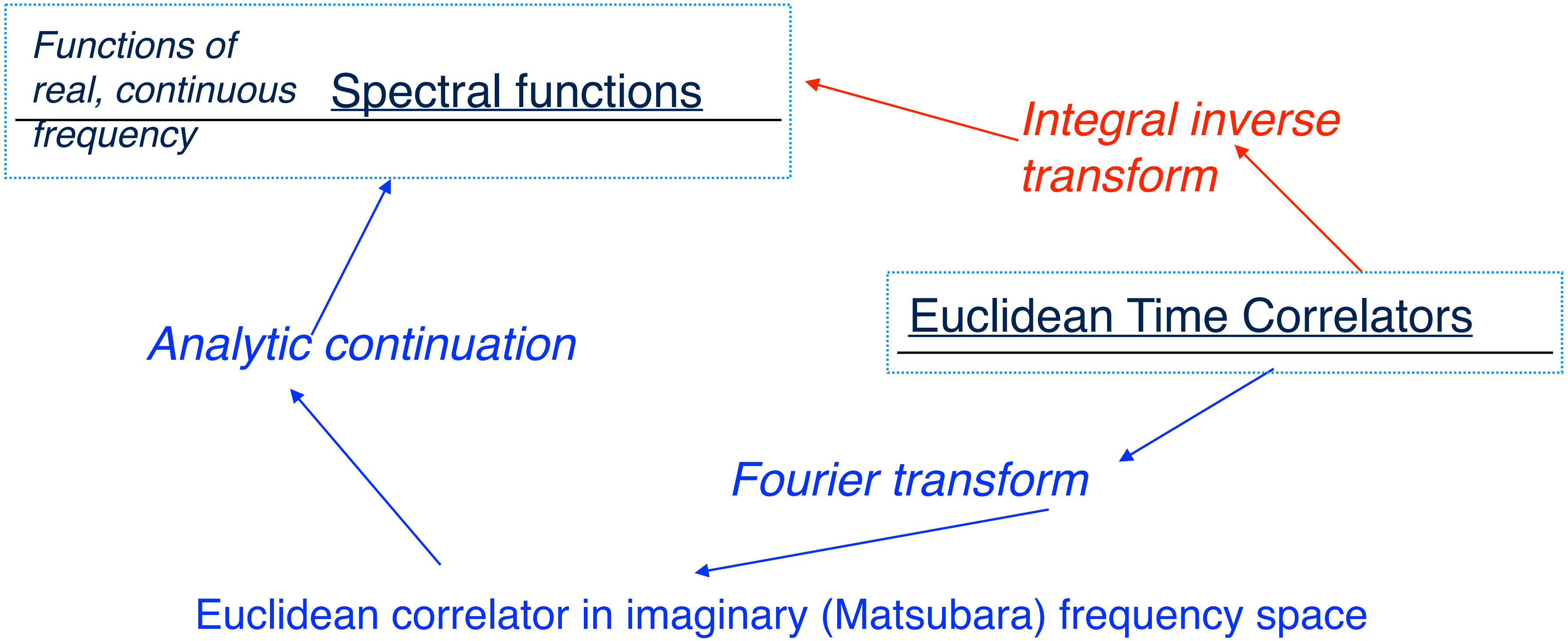
$$N_{\Upsilon(2S)}/N_{\Upsilon(1S)}|_{PbPb} = 0.12 \pm 0.03 \pm 0.01$$
$$N_{\Upsilon(3S)}/N_{\Upsilon(1S)}|_{PbPb} < 0.07$$

CMS
Eur.Phys.J. C76 (2016) no.3, 107

Objects of interest: Spectral functions



Computed on the lattice: Euclidean (imaginary) Time Correlators



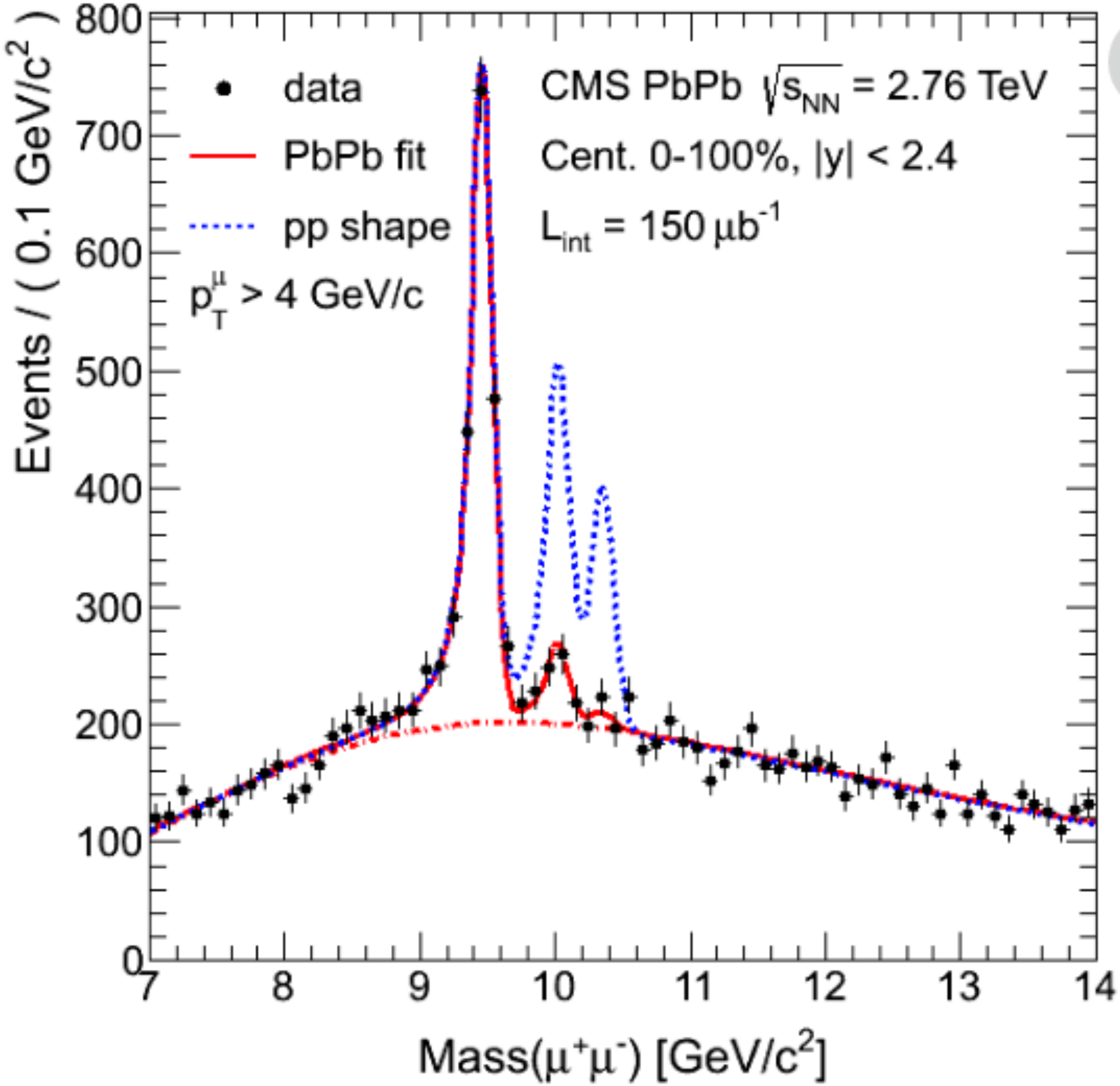
Relativistic

$$D(\tau) = \int_0^\infty \frac{e^{-\tau\omega} + e^{-(\beta-\tau)\omega}}{1 - e^{-\beta\omega}} S(\omega) d\omega$$

Non-relativistic

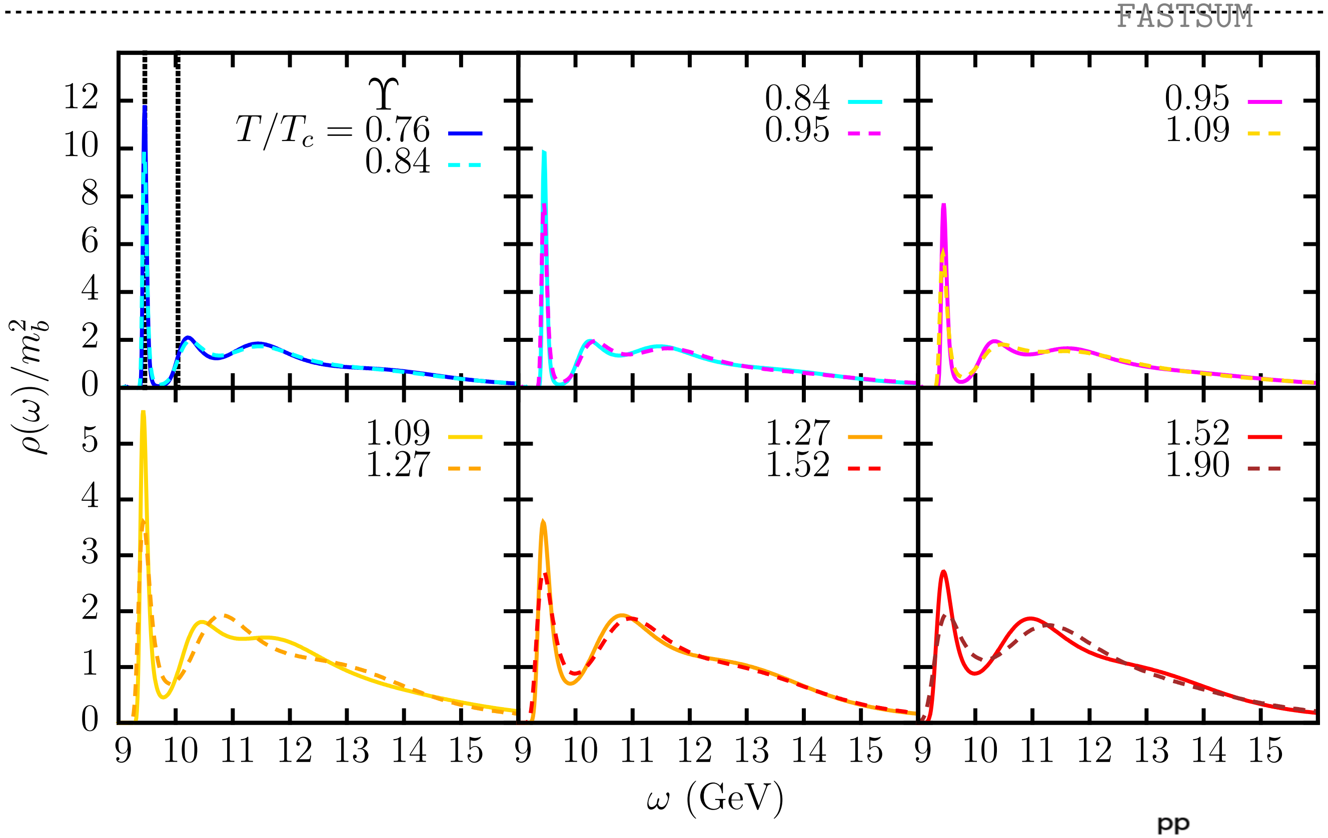
Inverse Laplace:
makes life easier..

$$D(\tau) = \int_{-M_0}^\infty e^{-\tau\omega} S(\omega) d\omega$$



data
PbPb fit
pp shape
$p_T^\mu > 4$ GeV/c
CMS PbPb $\sqrt{s_{NN}}$ = 2.76 TeV
Cent. 0-100%, |y| < 2.4
L_{int} = 150 μb^{-1}
Events / (0.1 GeV/c^2)
Mass($\mu^+\mu^-$) [GeV/c^2]
0
100
200
300
400
500
600
700
800
7
8
9
10
11
12
13
14

Upsilon's spectral functions from MEM (NRQCD)

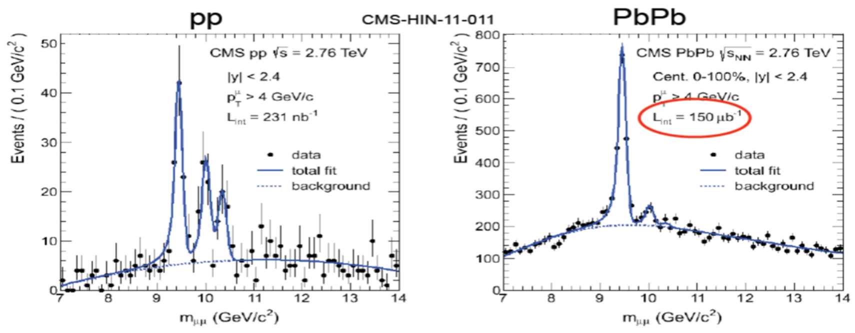


Persistence of the ground state at all temperatures

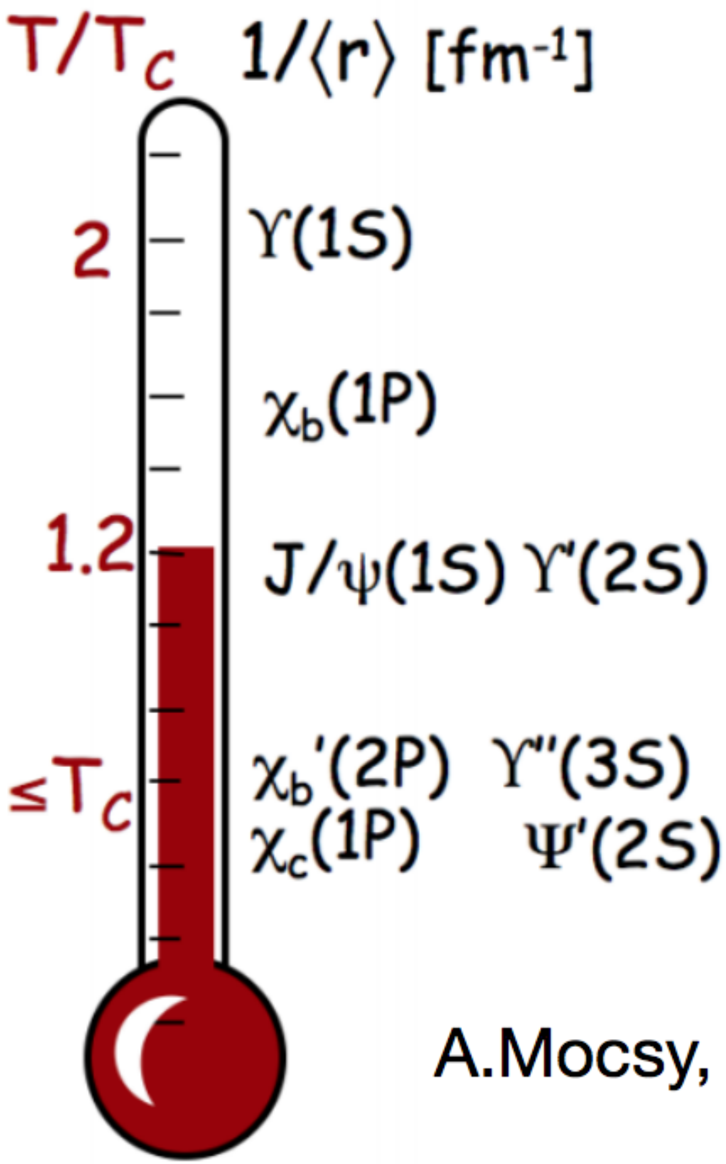
Melting of excited states

Modifications of the ground state

Pattern reminiscent of experimental observations



The picture of sequential melting is qualitatively correct, but quantitative uncertainties remain — cross checks from different methods important



A.Mocsy, arXiv:0811.0337

Review by C.Allton:

https://www.ggi.infn.it/talkfiles/slides/slides5843.pdf

Study of Numerical Methods

1. Exponential (Conventional δ f'ns) 2. Gaussian Ground State (+ δ f'n excited)	Maximum Likelihood (Minimise χ^2)
3. Moments of Correlation F'ns	Direct Method - "no" fit
4. BR Method 5. Maximum Entropy Method	Bayesian Approaches
6. Kernel Ridge Regression	Machine Learning
7. Backus Gilbert	from Geophysics

Axion cosmology and lattice

T_c Peccei Quinn $\gtrsim 10^7$ -10^8 GeV

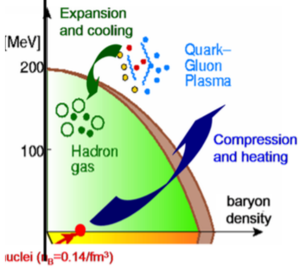
T_c Electroweak $\simeq 160$ GeV (SM)

Time from Big Bang

Temperature

$m_a(T) = \sqrt{\chi(T)}/f_a$

Quark Gluon Plasma: Topology



Peccei-Quinn transition: spontaneous breaking of a U(1) symmetry

[aside: possible gravitational waves..]

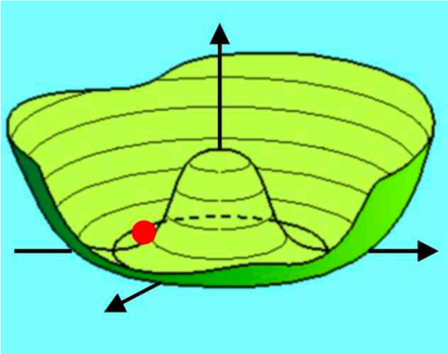
Low energy effective Lagrangian for the axion:

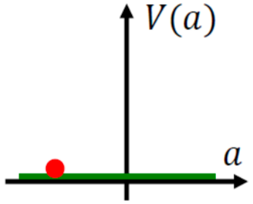
$$\mathcal{L} = \mathcal{L}_{QCD} + \theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a + \partial^2_\mu a^2 + \frac{a}{f_A} \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a .$$

Axion: Goldston boson of U(1) broken symmetry

T ~ f_a (very early universe)

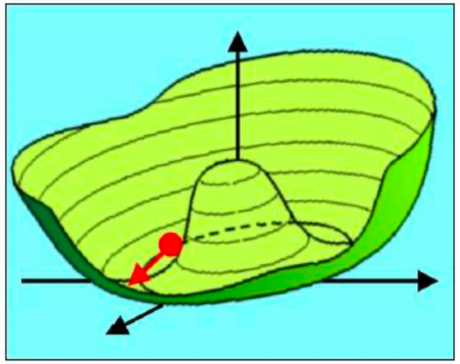
- U_{PQ} (1) spontaneously broken
- Axion fields settles in the "Mexican hat"
- Axion field frozen at initial value $a(t_i) = \theta_i f_a$

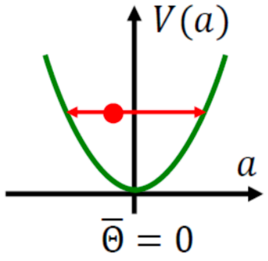




Graphic by A.Mirizzi

Axion mass: U(1) symmetry explicitly broken by topological fluctuations





$$m_A^2(T) f_A^2 = \frac{\partial^2 F(\theta, T)}{\partial \theta^2}\bigg|_{\theta=0} \equiv \chi_{top}(T) \ .$$

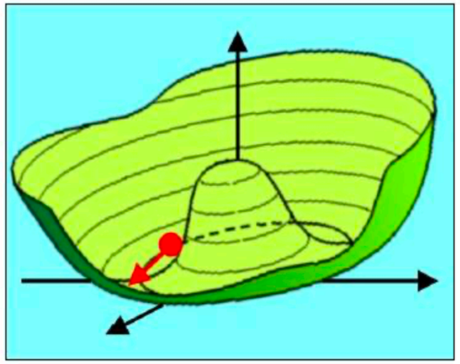
Axion mass: U(1) symmetry explicitly broken by topological fluctuations

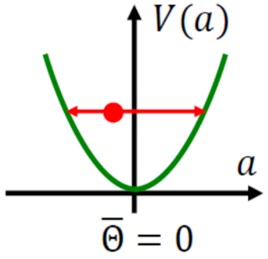
When does the explicit breaking become effective?

T ~ 1 GeV (H ~ 10^{-9} eV)

- Axion mass turns on quickly
- Field start oscillating when $m_a \geq 3H$
- Classical field oscillations (axion at rest)

Vacuum realignment





$$m_A^2(T) f_A^2 = \left.\frac{\partial^2 F(\theta, T)}{\partial \theta^2}\right|_{\theta=0} \equiv \chi_{top}(T) \ .$$

Axion cosmology (…just a glimpse..)

1) $$H^2 = \frac{1}{3M_p^2}\left(\frac{\pi^2}{30}g_{*,R}T^4 + f_a^2\left(\frac{1}{2}\dot{\theta}_a^{\,2} + m_a^2(T)(1-\cos\theta_a)\right)\right)$$

$$\ddot{\theta} + 3H\dot{\theta}_a + m_a^2(T)\sin\theta_a = 0$$

2) $$\frac{\dot{\rho}_a}{f_a^2} = 2m_a\dot{m}_a(1-\cos\theta_a) - 3H\dot{\theta}_a^{\,2},$$

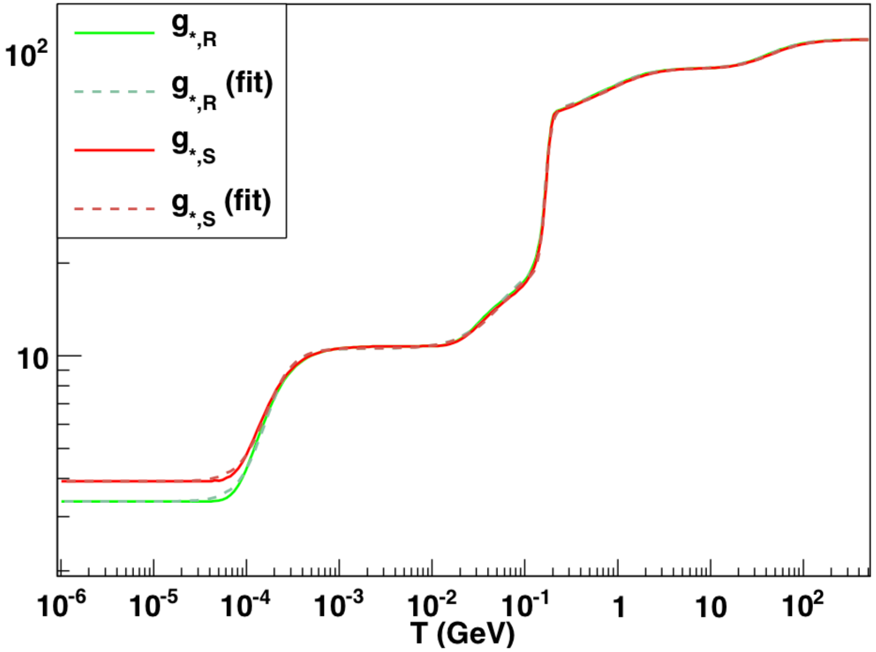
Freezout at: 3H(T) = m_a(T)

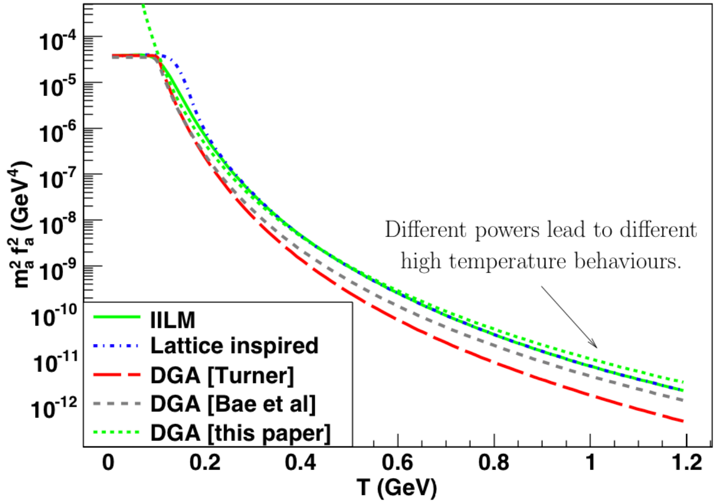
3) After freezout:

Axions oscillate in their potential
Bose condensed gas with

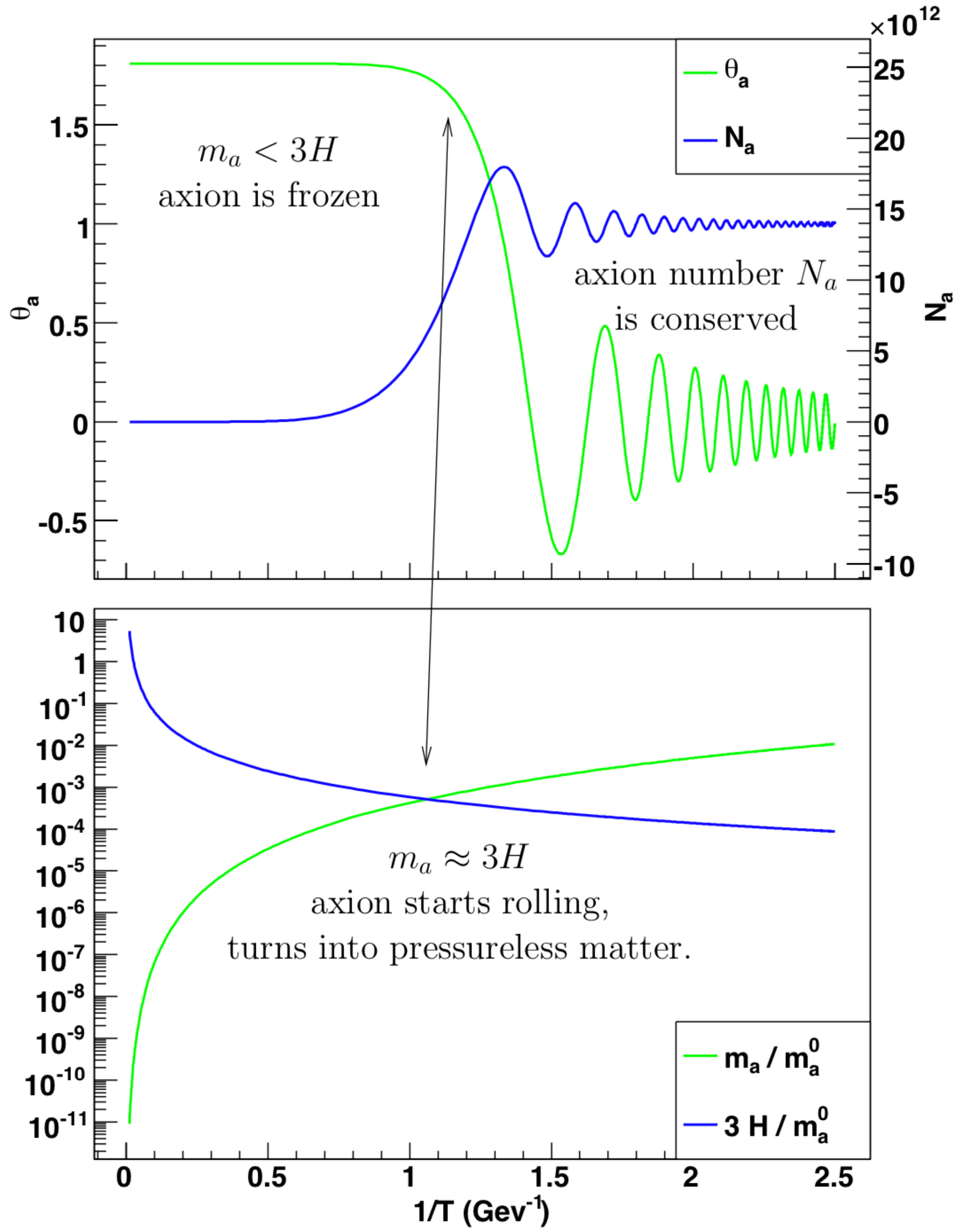
s(s = entropy)

$$\rho_a(\text{today}) = \rho_a(T)\frac{m_a(\text{today})}{m_a(T)}\frac{s(\text{today})}{s(T)}$$





Basic work: Wantz-Shellard 2009



Missing ingredient:
Exact knowledge of

$$\chi(T)$$

Needed to find freezout
Conditions

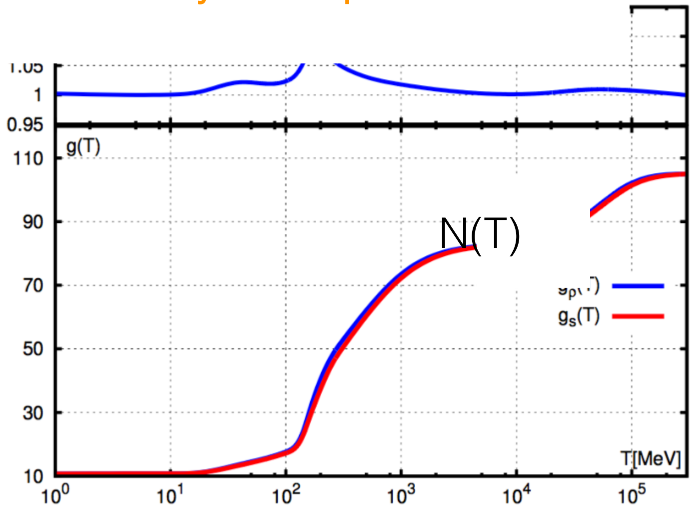
Time from Big Bang

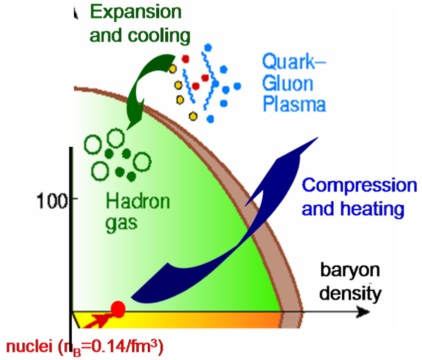
Temperatures

150 MeV < T < 500 MeV

..and beyond

Temperature and Time from BigBang
are linked by the Equation of State





T_c Peccei Quinn $\gtrsim 10^7$ -10^8 GeV

T_c Electroweak $\simeq 160$ GeV (SM)

Temperature

Time from Big Bang

Hubble parameter
$H(T) \simeq T^2/M_P$

Axions's freezout

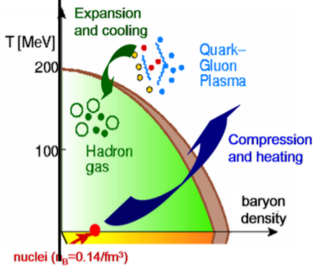
3H(T) = m_a(T)

m_a(T) = $\sqrt{\chi(T)}/f_a$

Quark Gluon Plasma: Topology

Axions' mass and density today

After freezout $\frac{n_a}{s}$ constant

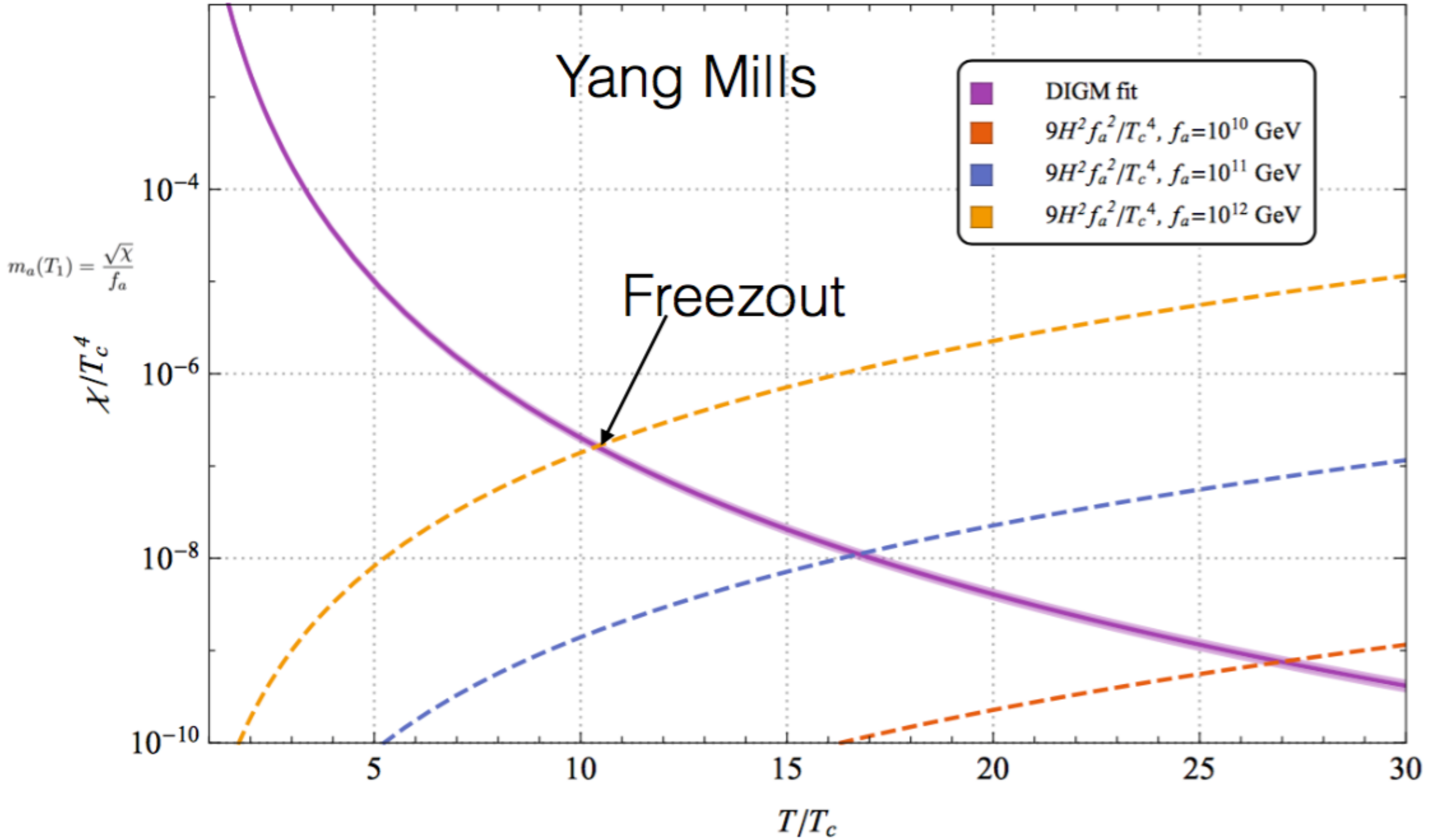


$$\rho_{a,0} = \frac{n_a}{s} m_a s_0$$

Wantz, Shellard 2010

Axion freezout : 3H(T) = m_a(T) = $\sqrt{\chi(T)}/f_a$

Berkowitz Buchoff Rinaldi 2015



Axion density at freezout controls axion density today

From exponent *d* to axion mass in three steps

$$\chi_{\rm top} \simeq A T^{-d}$$

$$d = (6.26, 6.88, 7.52, 7.48)$$

$$m_\pi = (470, 370, 260, 210) \text{ MeV}$$

Time from Big Bang

Temperature

Hubble parameter $H(T) \simeq T^2/M_P$

Axions's freezout

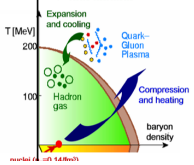
1. 3H(T) = m_a(T) $\quad$ $m_a(T) = \sqrt{\chi(T)}/f_a$

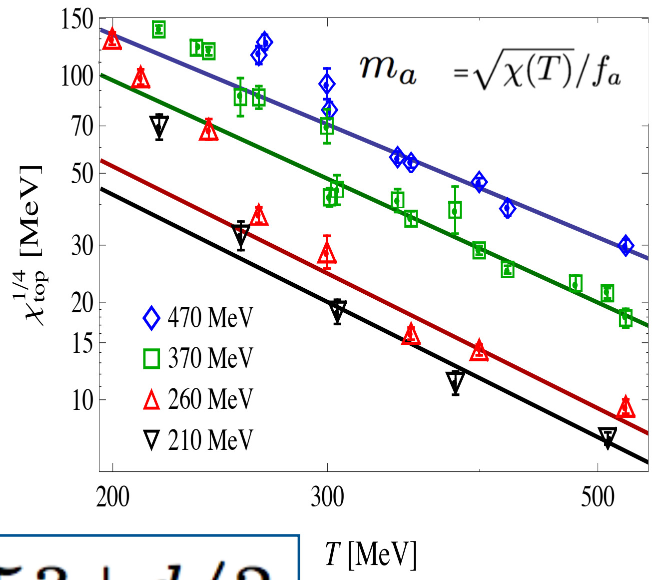
Quark Gluon Plasma: Topology

Axions' mass and density today

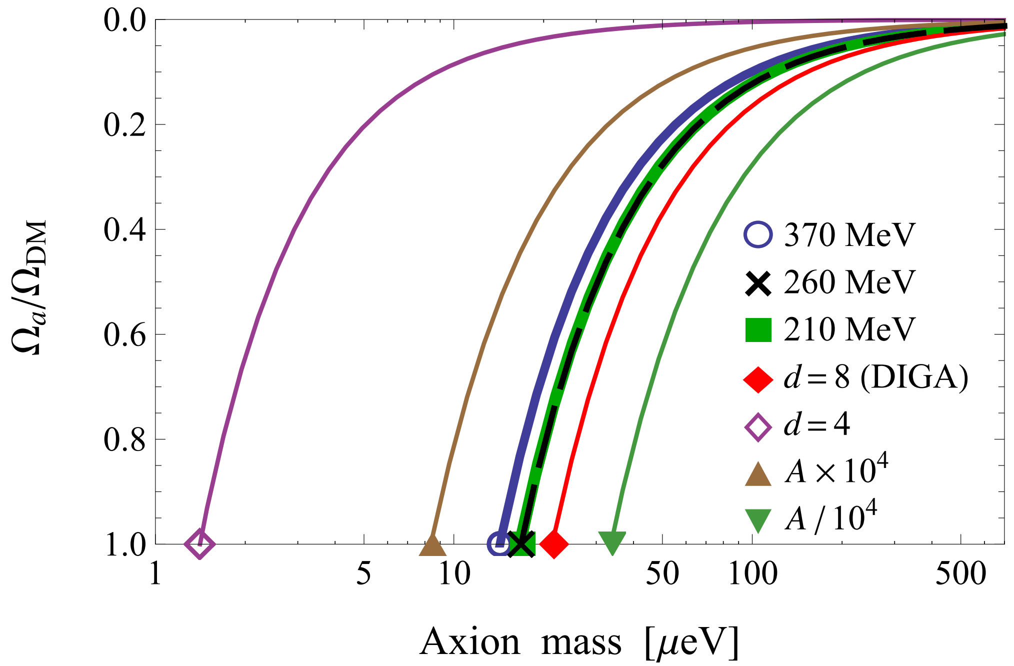
2. After freezout $\frac{n_a}{s}$ constant

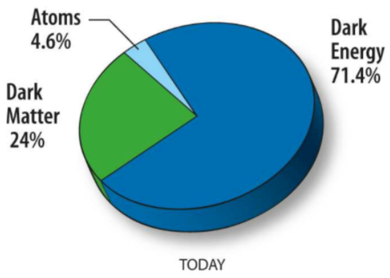
3. $\rho_{a,0} = \frac{n_a}{s} m_a s_0$



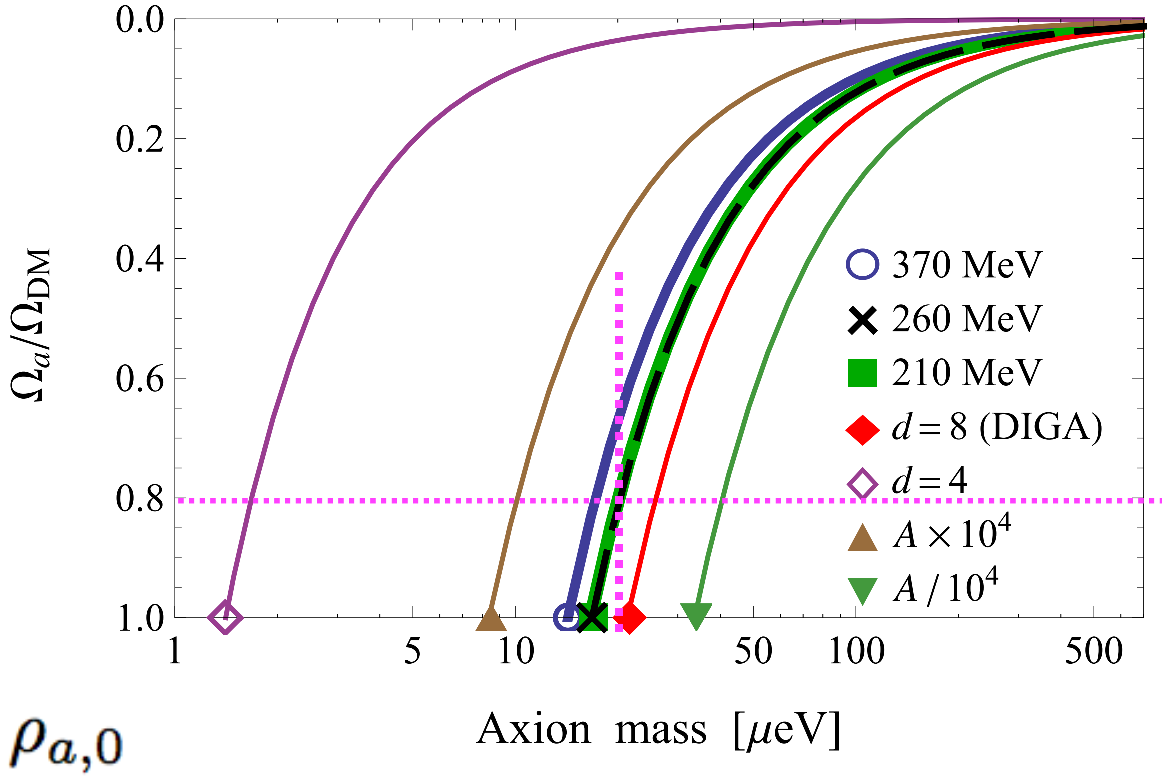


$$\rho_a(m_a) \propto m_a^{-\frac{3.053+d/2}{2.027+d/2}}$$





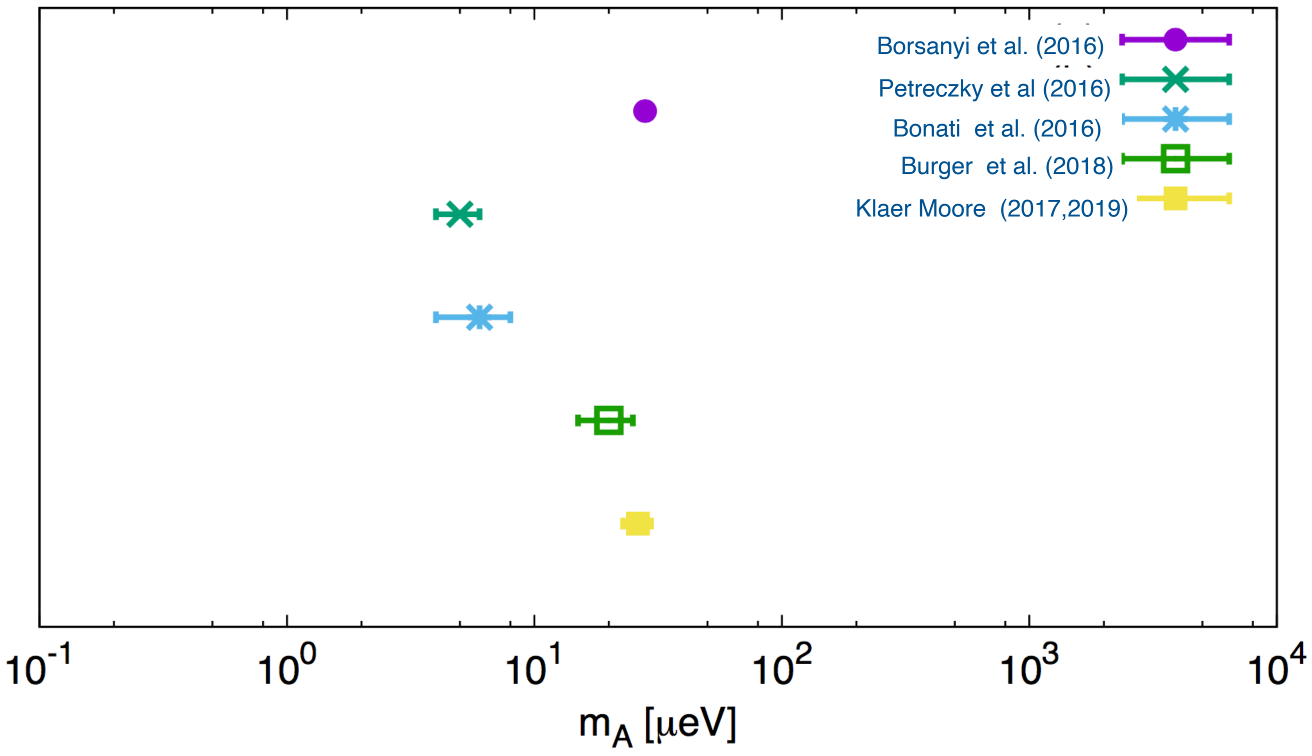
$$\Omega_a = \frac{\rho_{a,0}}{\rho_c},$$

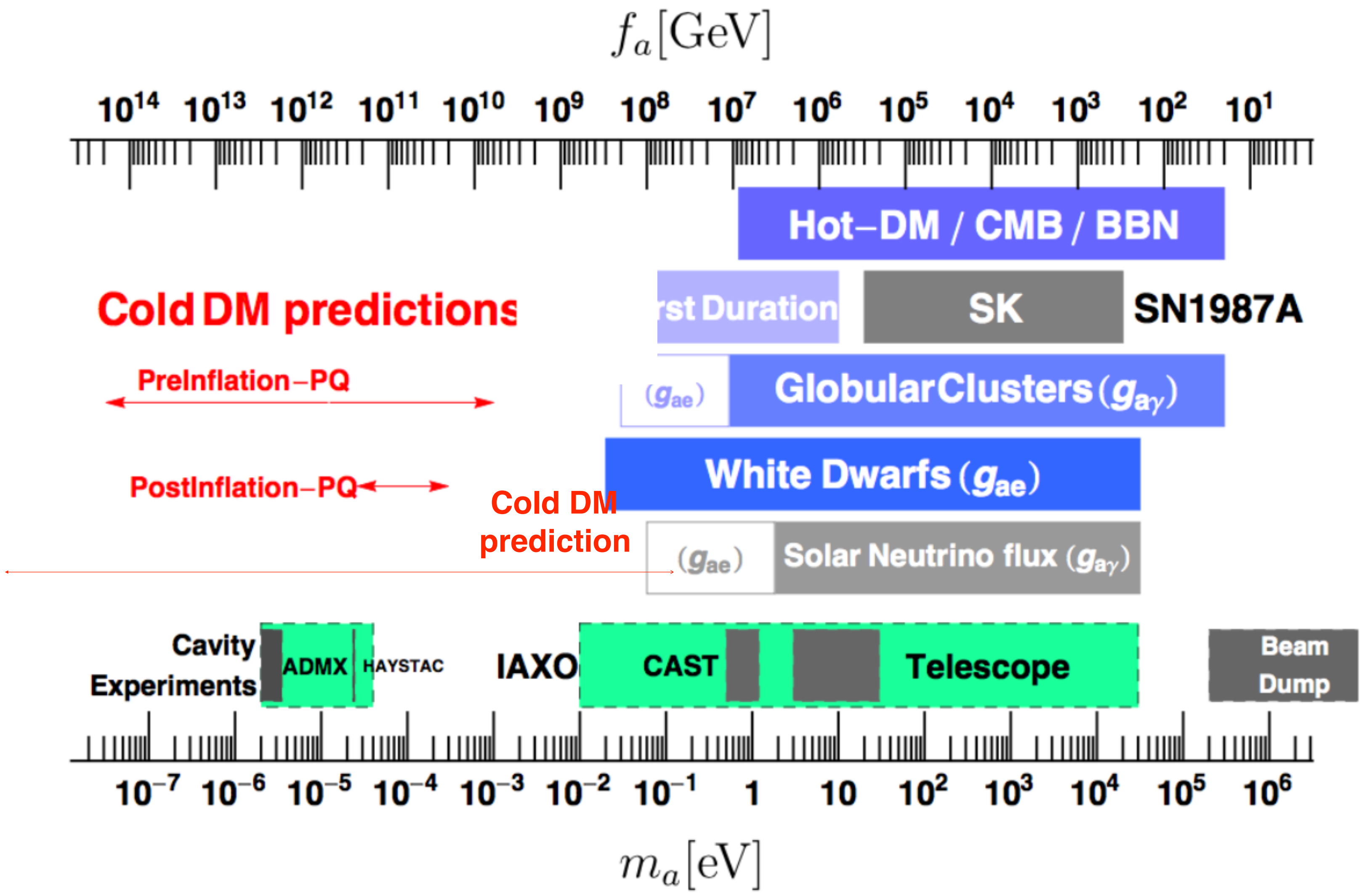


$$\Omega_a = \frac{\rho_{a,0}}{\rho_c},$$

Example: if axions constitute 80% DM, our results give a lower bound for the axion mass of $\simeq 30\mu\text{eV}$

Lower limits on post-infl. axion mass from lattice QCD

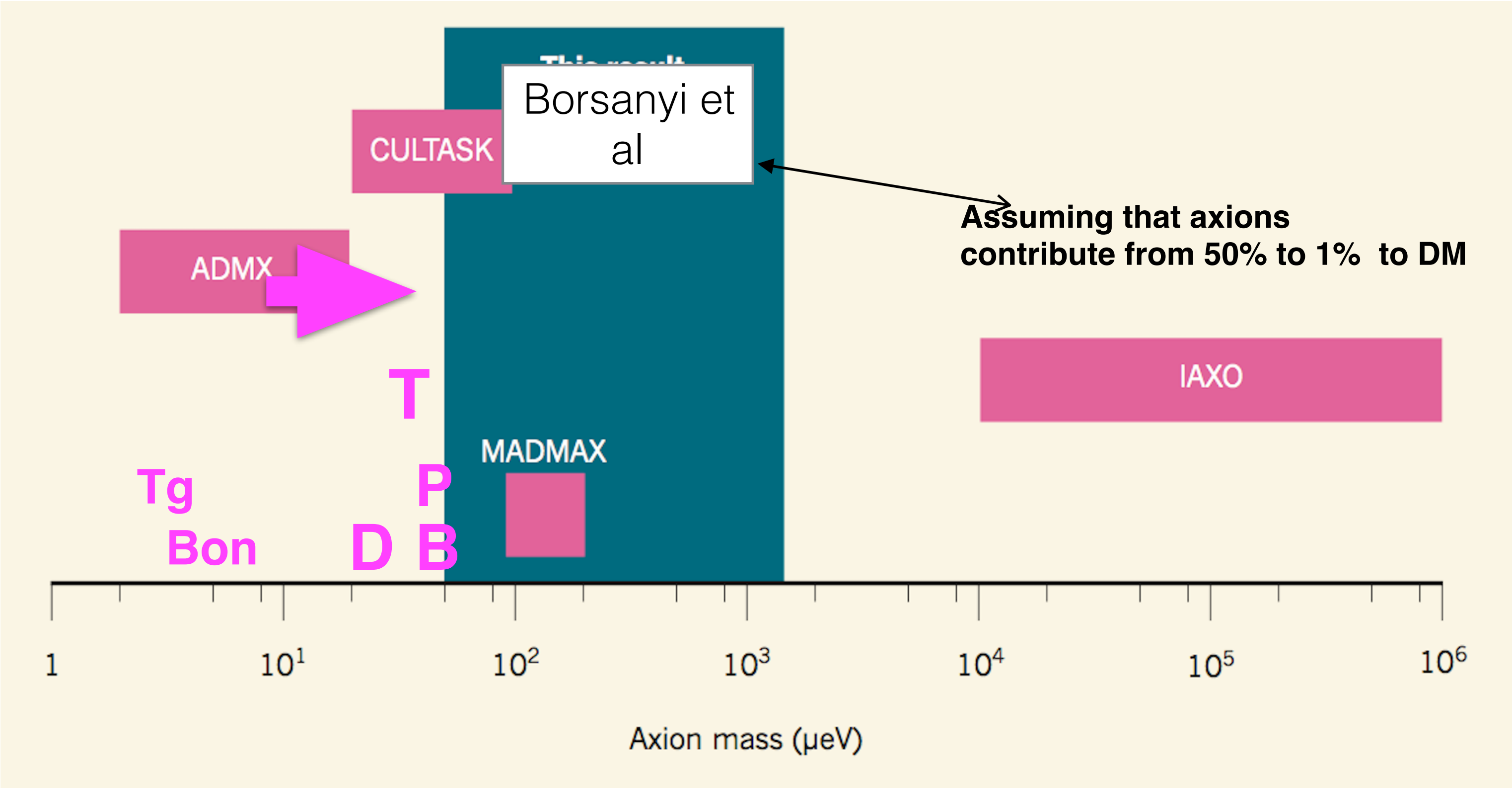




f_a[GeV]
10^14 10^13 10^12 10^11 10^10 10^9 10^8 10^7 10^6 10^5 10^4 10^3 10^2 10^1
Hot–DM / CMB / BBN
Cold DM predictions
rst Duration
SK
SN1987A
PreInflation–PQ
(g_{ae})
GlobularClusters ($g_{a\gamma}$)
PostInflation–PQ
White Dwarfs (g_{ae})
Cold DM prediction
(g_{ae})
Solar Neutrino flux ($g_{a\gamma}$)
Cavity Experiments
ADMX
HAYSTAC
IAXO
CAST
Telescope
Beam Dump
10^-7 10^-6 10^-5 10^-4 10^-3 10^-2 10^-1 1 10 10^2 10^3 10^4 10^5 10^6
m_a[eV]

Lower limits on the axion mass assuming that axions make 100% of DM:

Tg: TWEXT gluonic; Bon: Bonati et al.; D: DIGA, B: Borsanyi et al.,
P: Petreczky et al., T: TWEXT, fermionic



Updated from Nature N&V

Operator Improvement -

Smoothing a rough landscape

CERN-PH-TH/2010-143

Properties and uses of the Wilson flow in lattice QCD

Martin Lüscher

CERN, Physics Department, 1211 Geneva 23, Switzerland

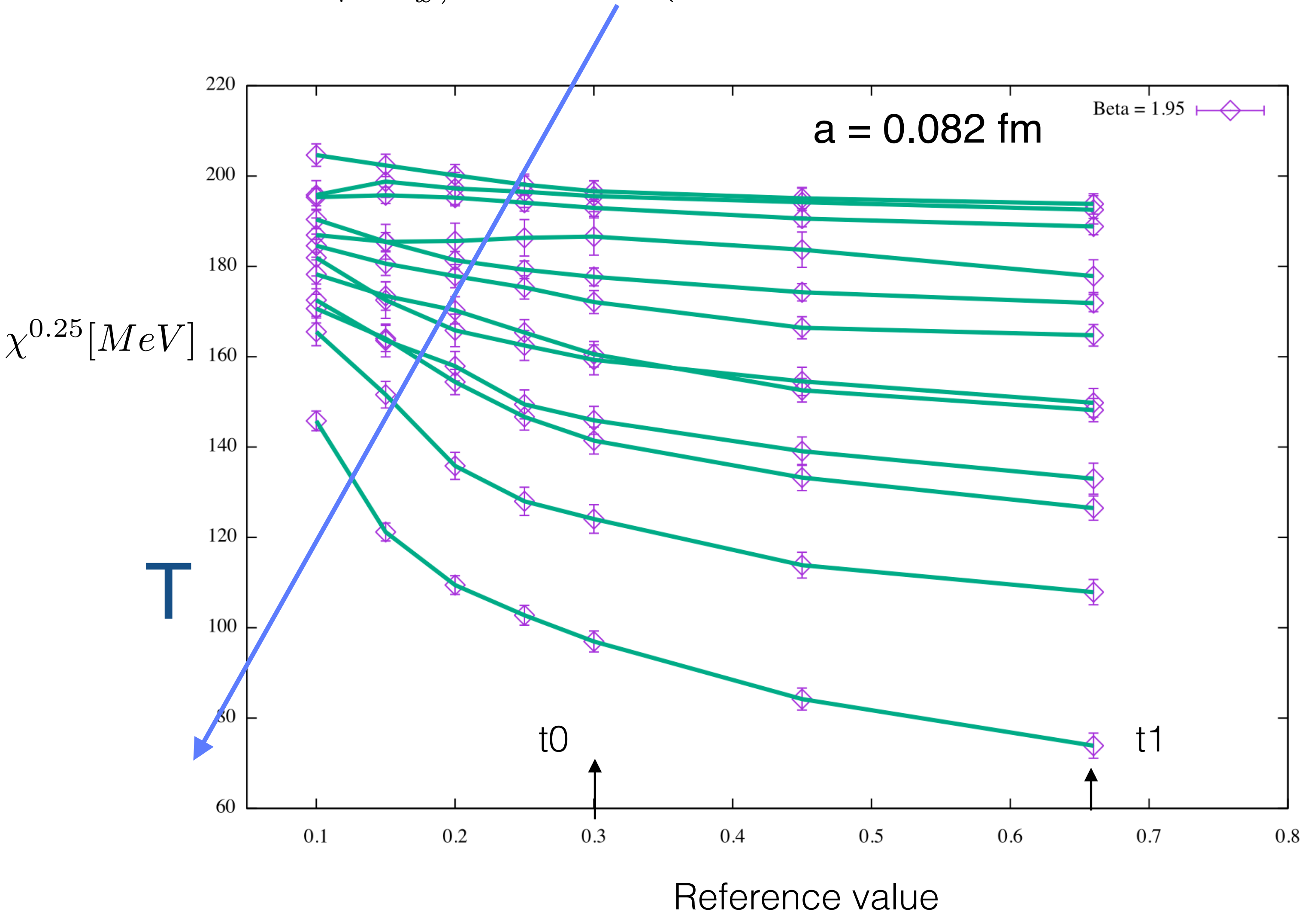
In a nutshell:

Evolves gauge fields towards minimum of the Action in fictitious time τ_F

Gaussian smearing over sphere with flow radius $\sqrt{8\tau_F}$

Flowing towards the plateau

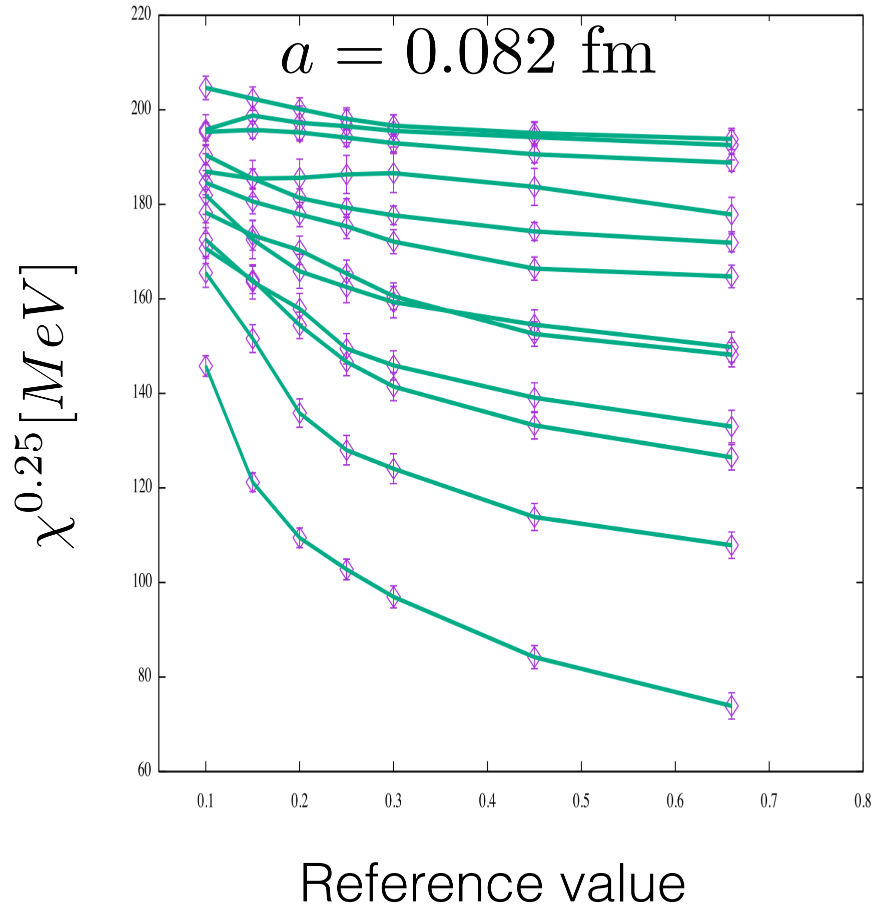
$$t^2 < E > |_{t=t_x, x=0-6} = (0.3, 0.66, 0.1, 0.15, 0.2, 0.25, 0.45)$$

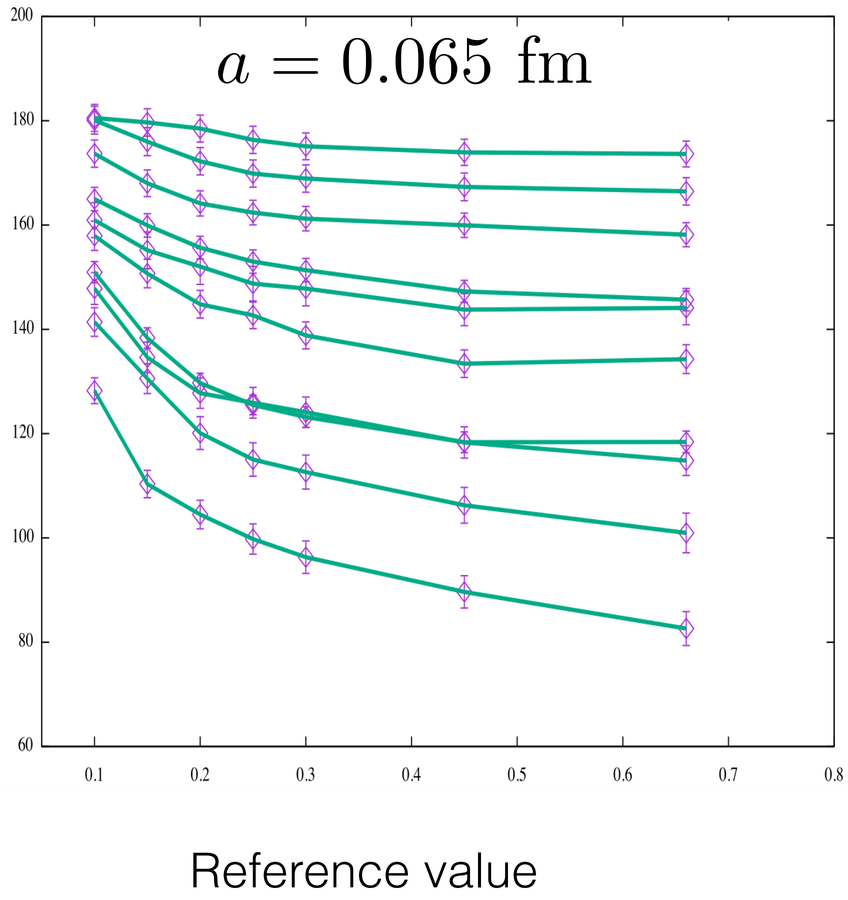


We focus
on
t(0.3) = t0
and
t(0.66)= t1

On finer lattices, plateau is almost reached:

Gradient method coincides with cooling





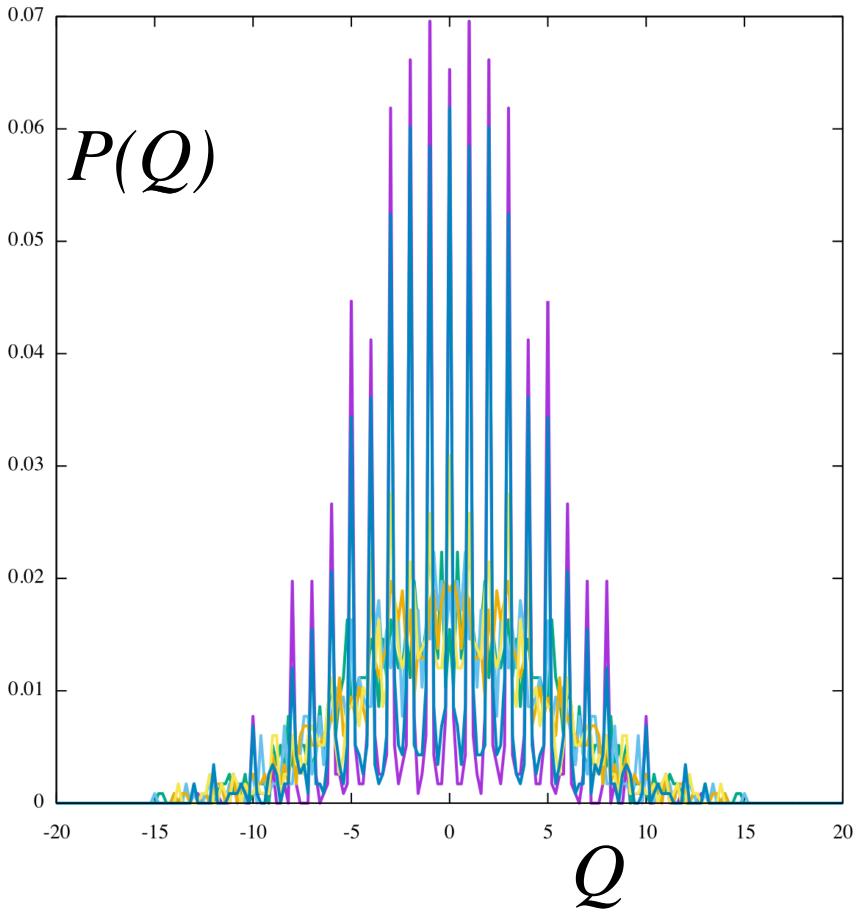
QCD:

Distribution of the topological charge *P(Q)*

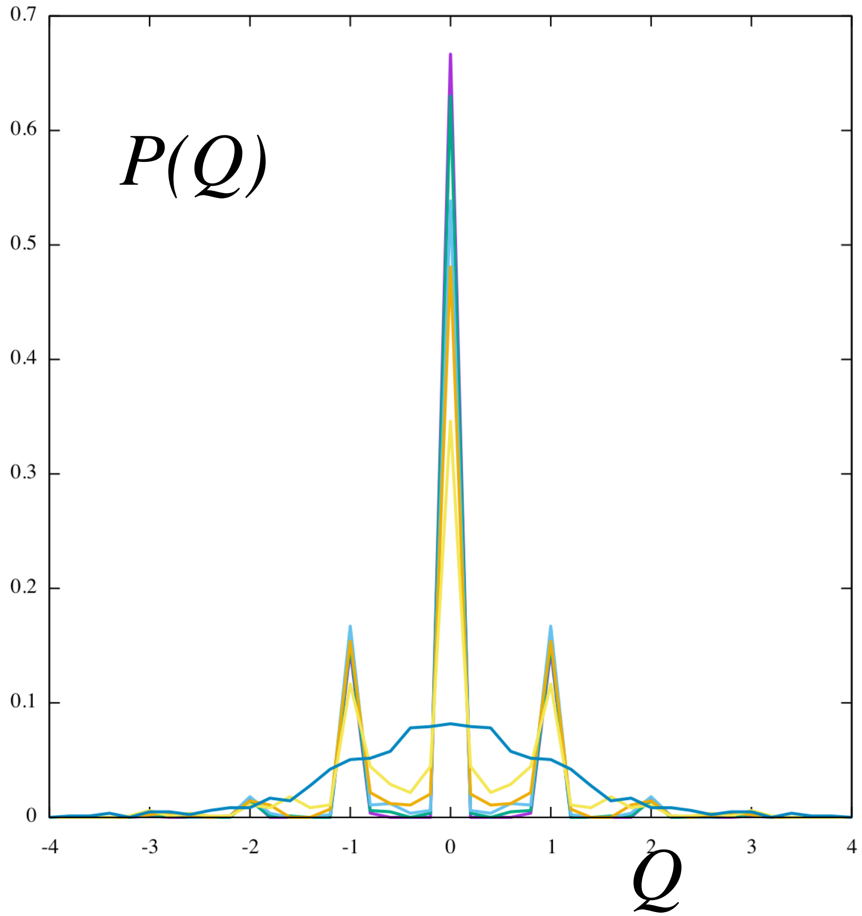
cluster around integers as cooling proceeds

(results for a = 0.06 fm)

T=153 MeV



T=253 MeV



CERN-PH-TH/2010-143

Different smoothing methods

Properties and uses of the Wilson flow in lattice QCD

Martin Lüscher

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PROCEEDINGS OF SCIENCE

String tension from smearing and Wilson flow methods

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Recently, we proposed a new method to extract the string tension from 4-dimensionally smeared Wilson loops. In this talk, we first show that the results obtained using this smearing method are identical to those obtained by Wilson flow, once the time step is sufficiently small. We then demonstrate the practical advantage of our method by applying it to the calculation of string tension in SU(3) Yang-Mills theory.

Comparison of the gradient flow with cooling in $SU(3)$ pure gauge theory

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Dipartimento di Fisica dell'Università di Pisa and INFN - Sezione di Pisa,
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The gradient (Wilson) flow has been introduced recently in order to provide a solid theoretical framework for the smoothing of ultraviolet noise in lattice gauge configurations. It is interesting to ask how it compares with other, more heuristic and numerically cheaper smoothing techniques, such as standard cooling. In this study we perform such a comparison, focusing on observables related to topology. We show that, already for moderately small lattice spacings, standard cooling and the gradient flow lead to equivalent results, both for average quantities and configuration by configuration.

The topological susceptibility of the pure SU(3) Yang–Mills vacuum on the lattice ☆

M. Campostrini, A. Di Giacomo, Y. Gündüç [1,2], M.P. Lombardo,
H. Panagopoulos and R. Tripiccione

INFN, Sezione di Pisa and Dipartimento di Fisica dell'Università, I-56100 Pisa, Italy

Received 2 August 1990

Using a "field theoretic" approach, we compute the topological susceptibility χ of the pure gauge SU(3) theory on the lattice. We also apply an algorithm of gradual cooling, and use these two approaches as a cross-check on each other. The final value we find for χ confirms results found earlier using an abrupt-cooling algorithm.

Smearing

Build a sequence of operators

$$\varphi^{(s)}(n) \to \varphi^{(s+1)}(n) = \varphi^{(s)}(n) + \epsilon \sum{}' \varphi^{(s)}(n')$$

Evolution in ‘smearing time $\tau = s\epsilon$

$$\partial\varphi/\partial\tau = \nabla^2 \varphi$$

In momentum space :

$$\exp(-Wk^2) \qquad \approx (W+\tau)^{-5/2}$$

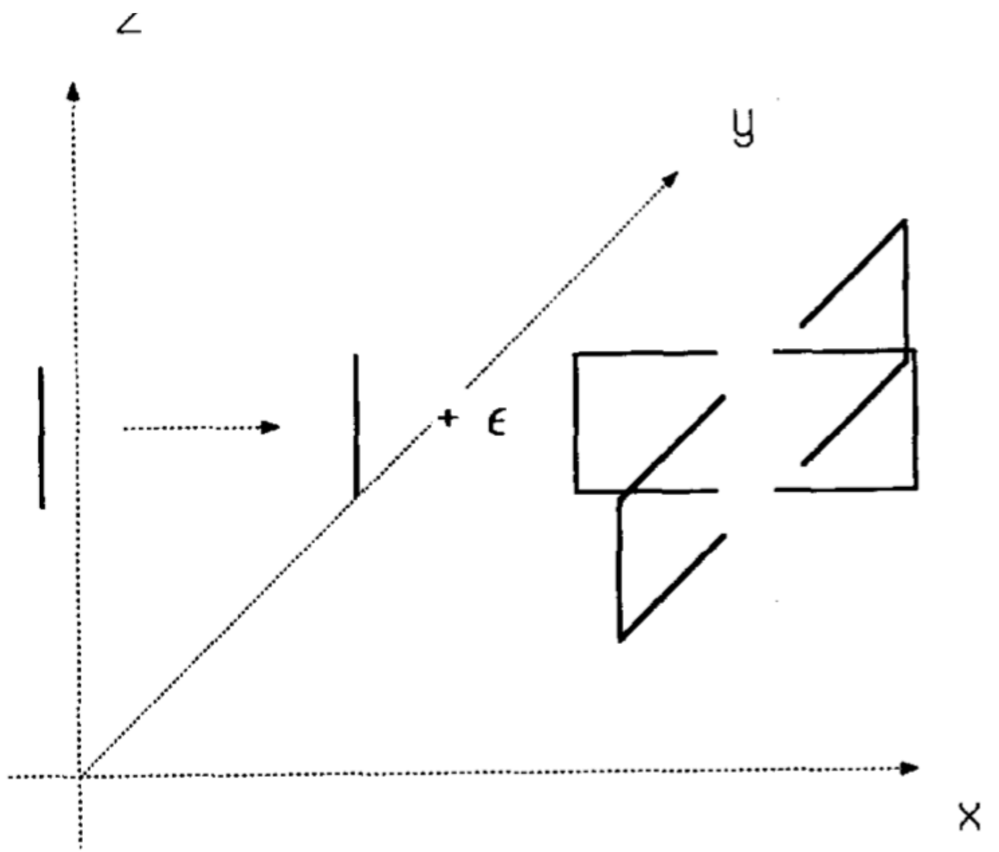


Fig. 1. The smearing procedure. We substitute a link with itself plus ϵ times the sum of the incomplete neighboring space like

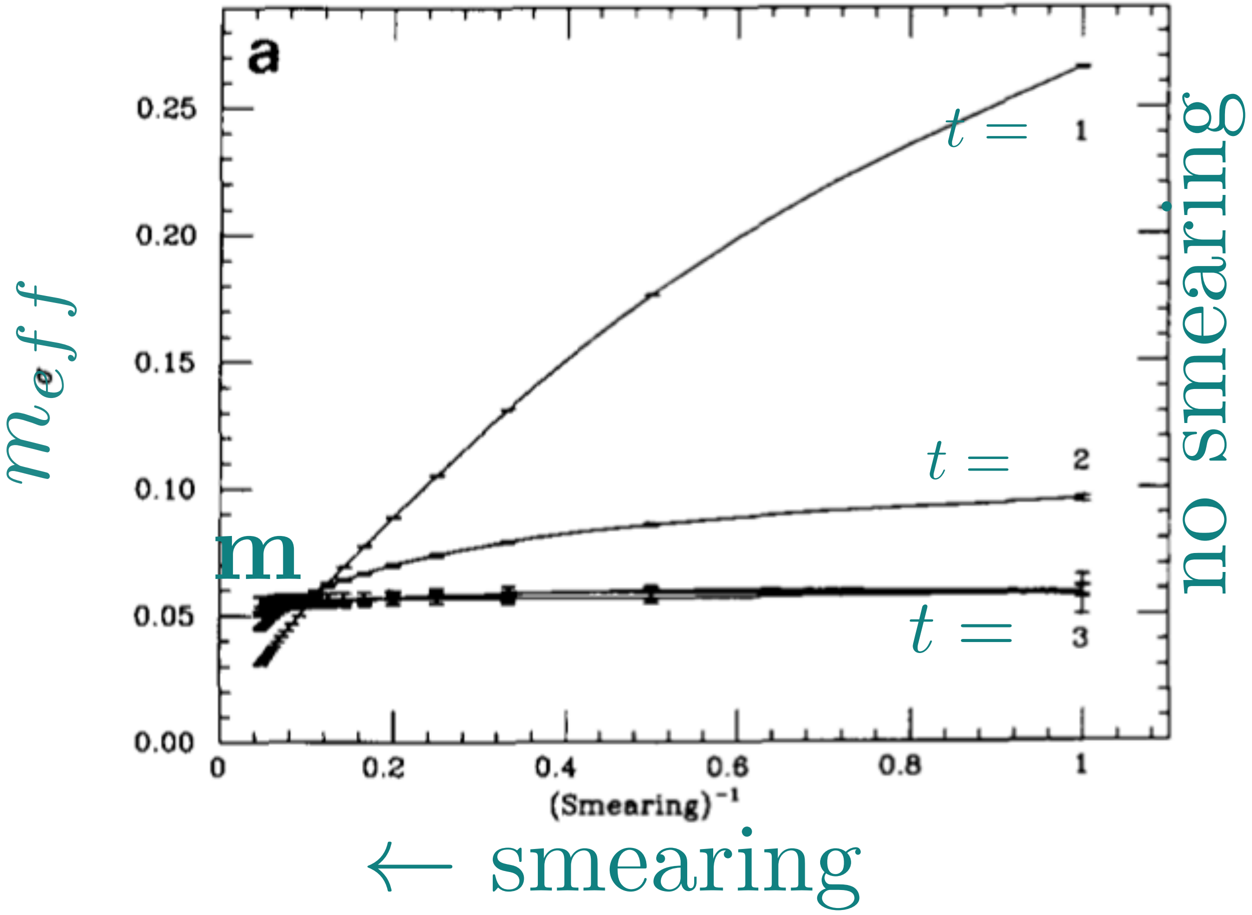
Smearing at work

$$G(t)_{t\to\infty} \to e^{-\mathbf{m}t}$$

Define

$$m_{eff}(t) \equiv -\frac{d\ln G(t)}{dt}$$

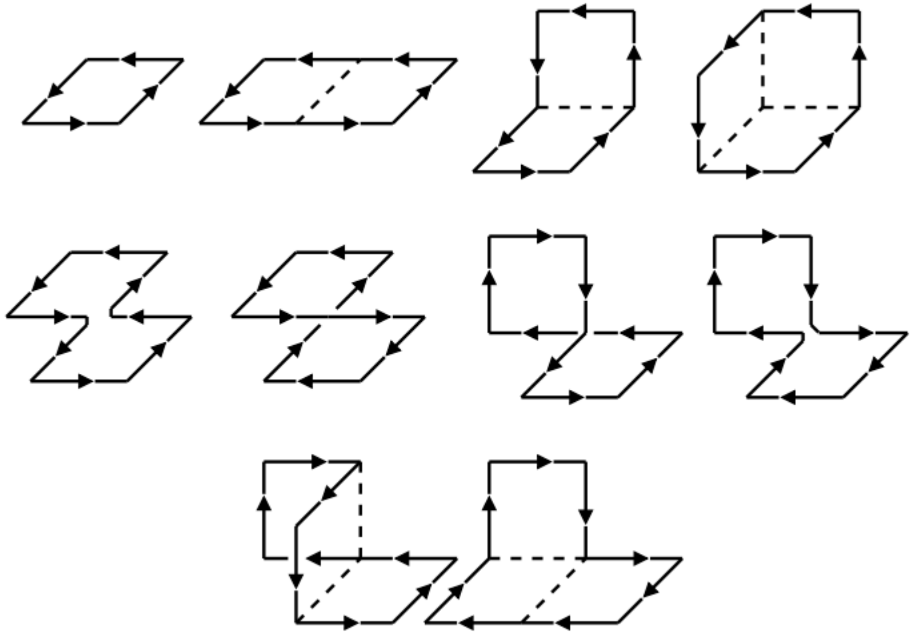
$$m_{eff}(t)_{t\to\infty} \to \mathbf{m}$$



Glueball masses

$$G(t) \equiv \langle O(t) \cdot O(0) \rangle - \langle O(t) \rangle \cdot \langle O(0) \rangle$$

$$\approx \exp(-mt) ,$$ Note: *t* Euclidean time!



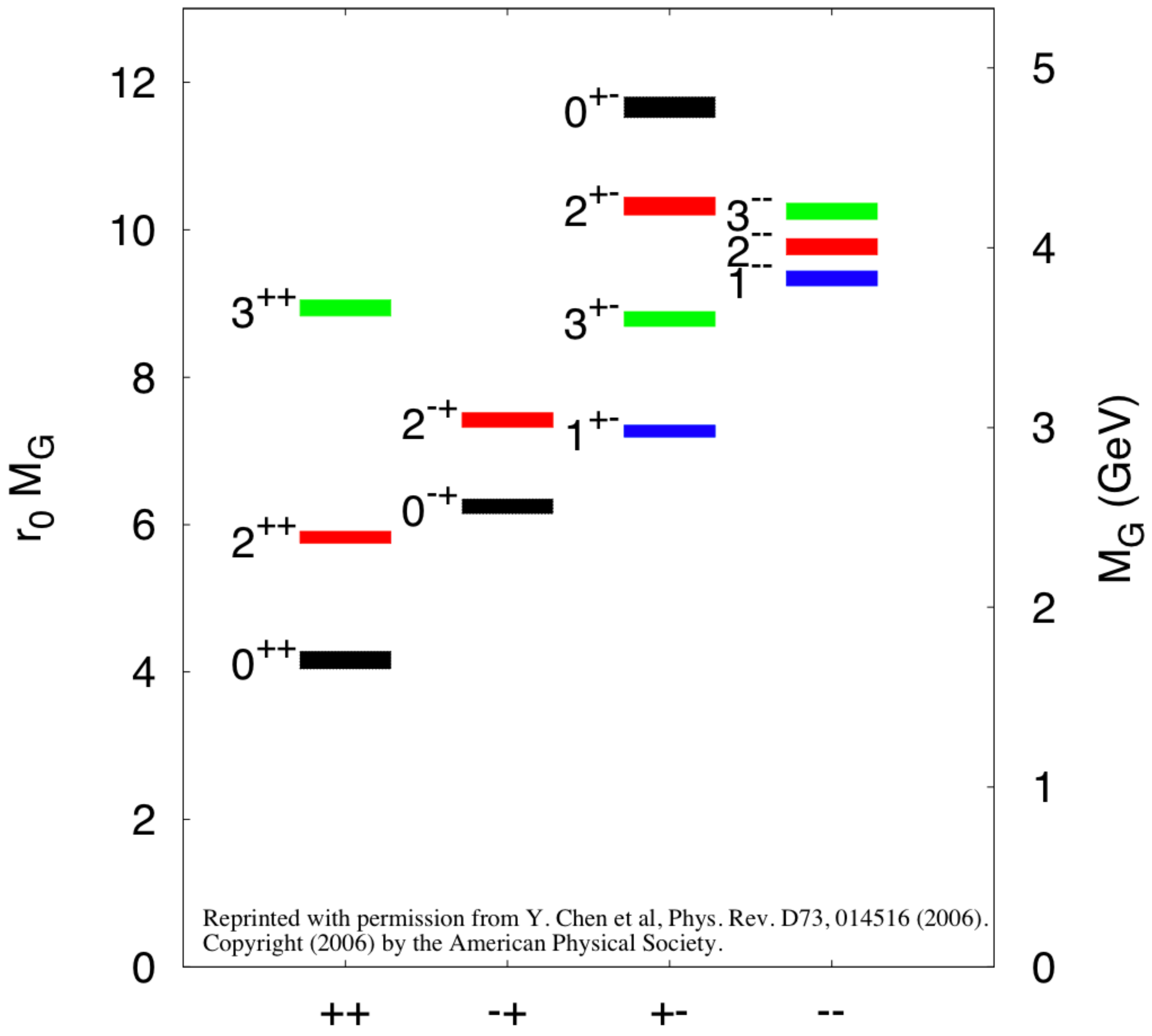
$$G(t) = \int \delta(M-\omega) e^{-\omega t} \propto e^{-Mt}$$

Smearing

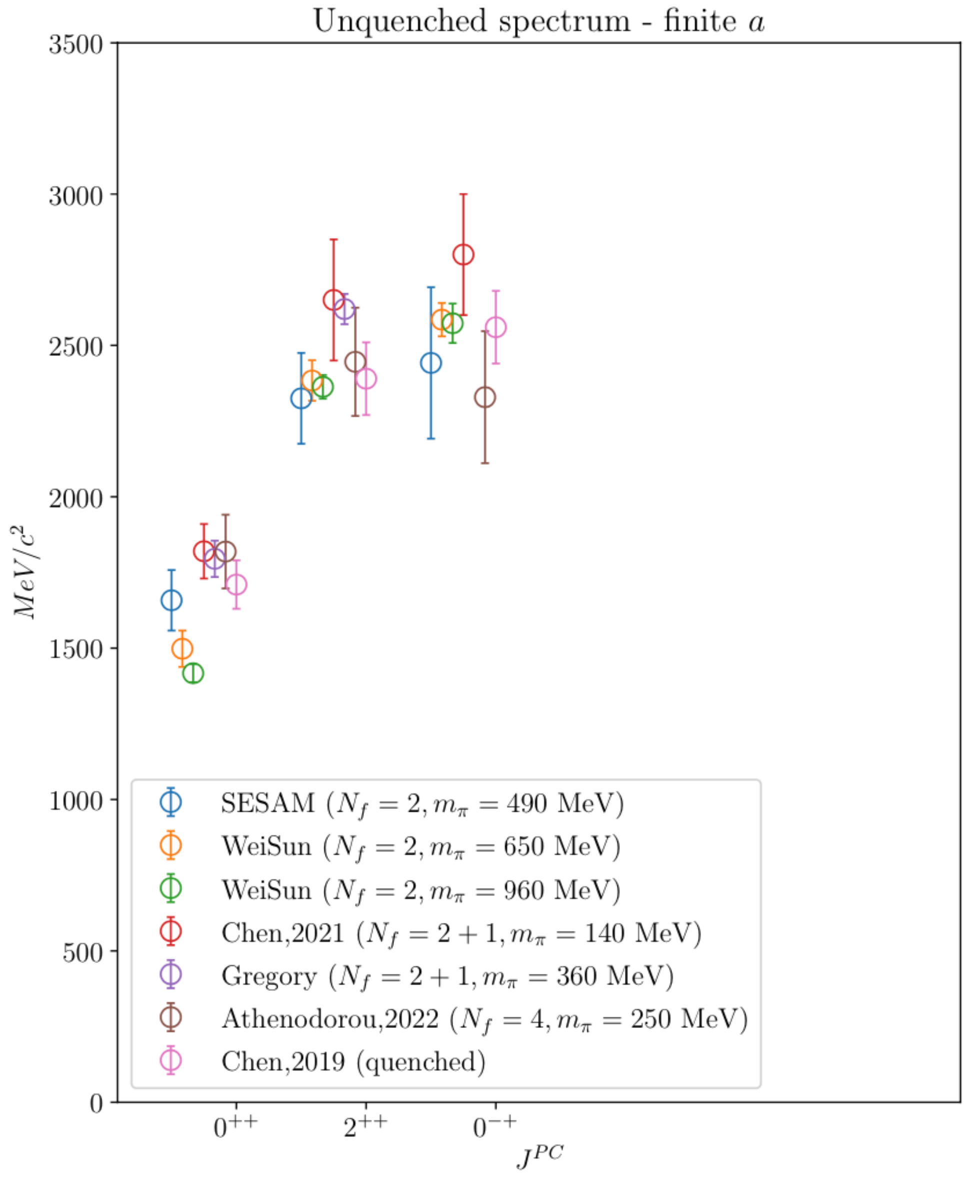
$$G_0^{(s)}(t) \equiv O^{(s)}(x,y) + O^{(s)}(y,z) + O^{(s)}(x,z)$$ 0++

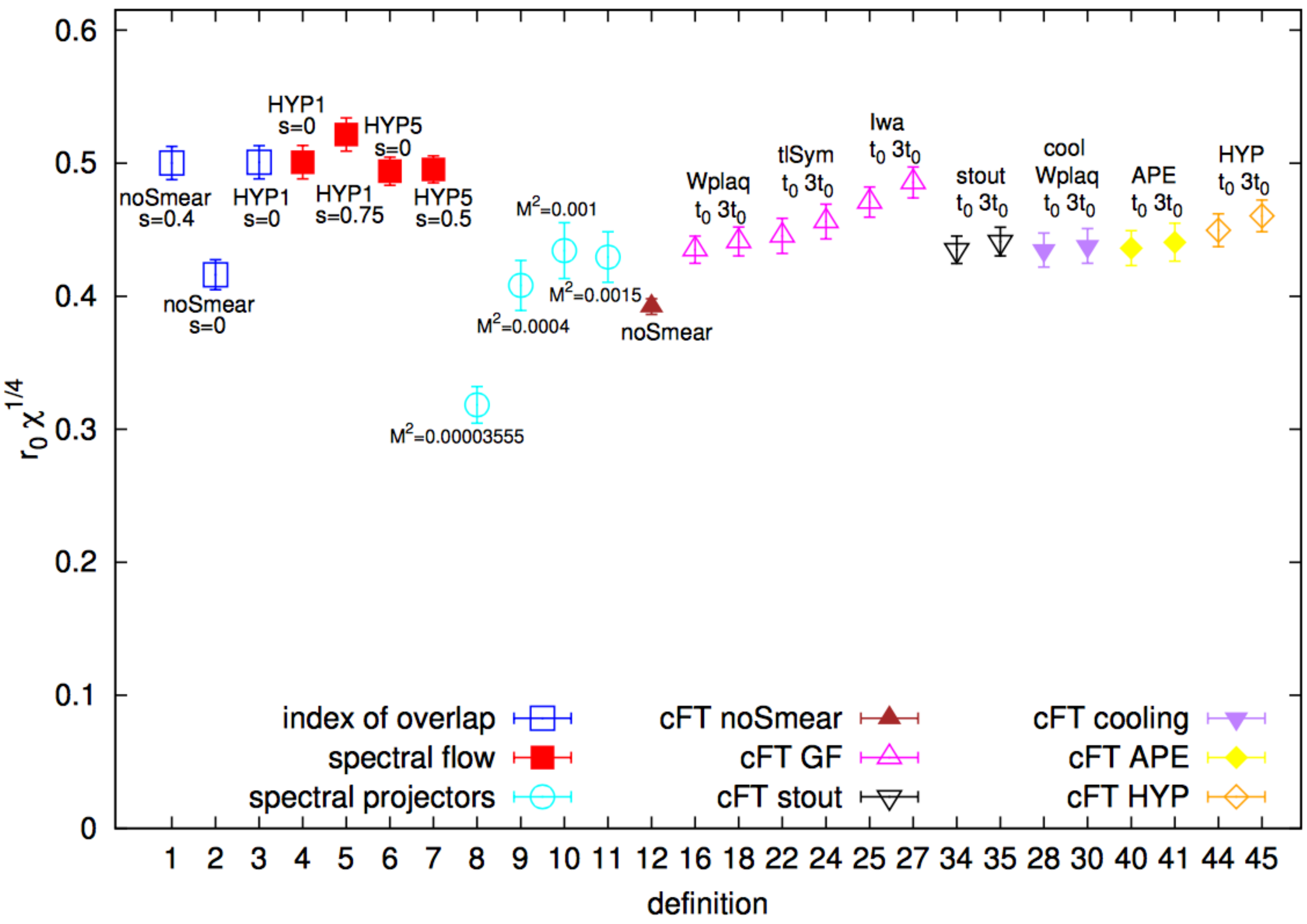
$$G_2^{(s)}(t) \equiv -2O^{(s)}(x,y) + O^{(s)}(y,z) + O^{(s)}(x,z) .$$ 2++

2006 data, From PDG 2022



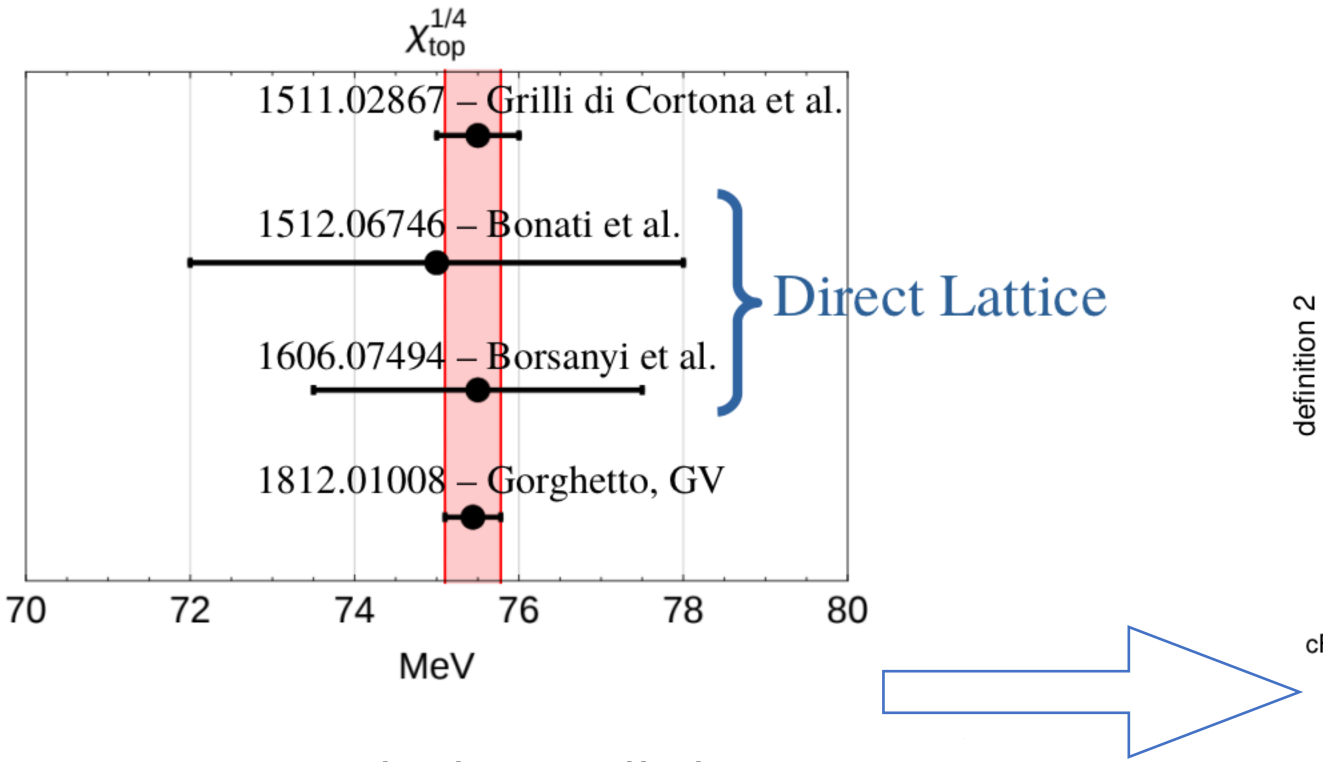
2022 compilation, Vadacchino@lat22



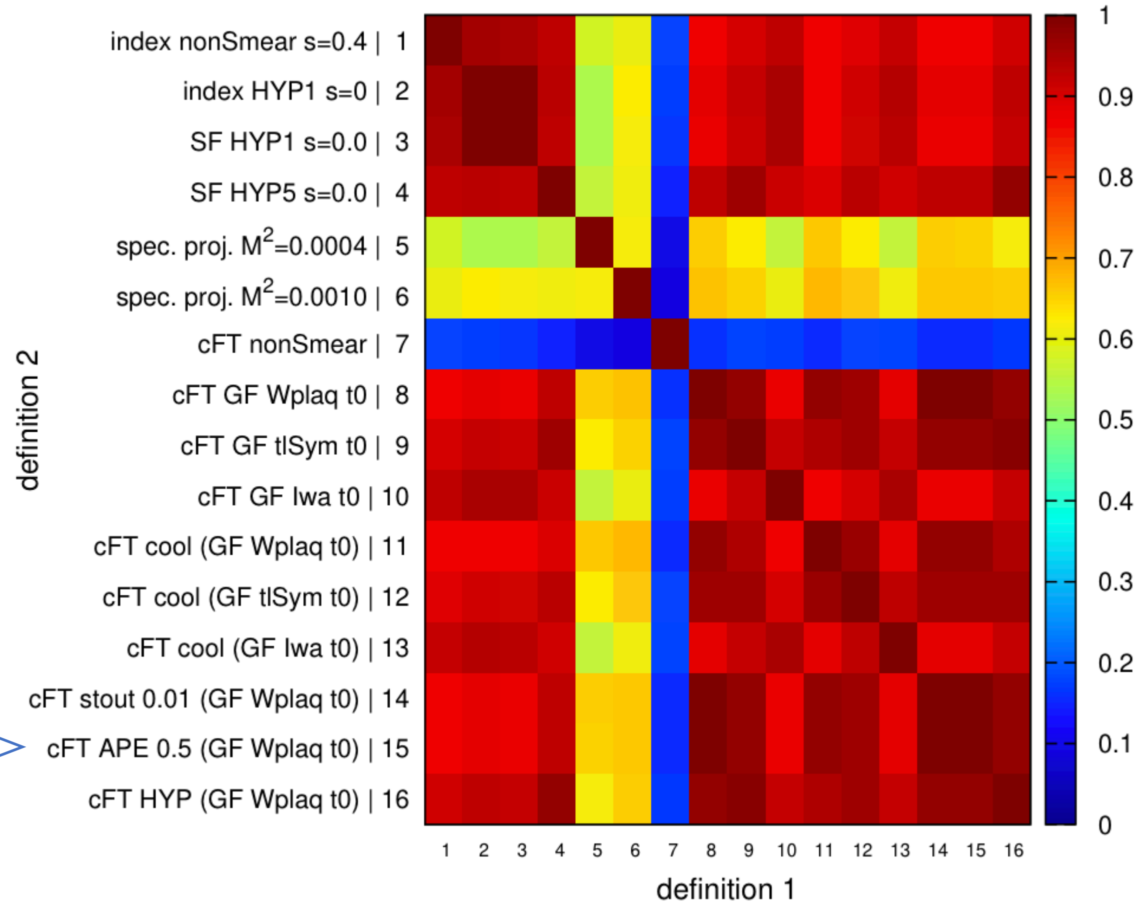


0.6
0.5
0.4
0.3
0.2
0.1
0
$r_0 \chi^{1/4}$
1 2 3 4 5 6 7 8 9 10 11 12 16 18 22 24 25 27 34 35 28 30 40 41 44 45
definition
noSmear s=0.4
noSmear s=0
HYP1 s=0
HYP1 s=0
HYP5 s=0
HYP1 s=0.75
HYP5 s=0.5
M^2=0.00003555
M^2=0.0004
M^2=0.001
M^2=0.0015
noSmear
Wplaq t_0 $3t_0$
tlSym t_0 $3t_0$
Iwa t_0 $3t_0$
stout t_0 $3t_0$
cool Wplaq t_0 $3t_0$
APE t_0 $3t_0$
HYP t_0 $3t_0$
index of overlap
spectral flow
spectral projectors
cFT noSmear
cFT GF
cFT stout
cFT cooling
cFT APE
cFT HYP

Many varieties of methods and smearing/smoothing/flowing/cooling.. good agreement at T=0



Plot by G. Villadoro



ETMC collaboration, 2017

Rather than a summary:

More on Euclidean formulation

Light quarks - chiral symmetry (and confinement)

Theta term, strong CP problem, topology, axion (brief recap)

Topology

Heavy quarks

Lattice QCD and axion cosmology