The idea of this exercice is to extend the Standard Model to 4 quark generations and `propagate' the CKM hierarchy described by the Wolfenstein parametrization where the Cabibbo angle λ is small and all other parameters are assumed to be of order 1.

Wolfenstein parametrization of the CKM matrix in four dimensions

Based on Botella and Chau Phys. Lett.B168:97,1986

Here is 3D CKM (embedded in 4D) in PDG parametrization. Notation is as usual cij=Cos[θij], sij=Sin[θij]

In[ ]:=
```
CKM3D = {{c12 * c13, s12 * c13, s13 Exp[-ⅈ δ]},
   {-s12 * c23 - c12 * s23 * s13 Exp[ⅈ δ], c12 * c23 - s12 * s23 * s13 Exp[ⅈ δ], s23 * c13},
   {s12 * s23 - c12 * c23 * s13 Exp[ⅈ δ], -c12 * s23 - s12 * c23 * s13 Exp[ⅈ δ], c23 * c13}};
CKM3D = Normal @ SparseArray [Flatten @
     {Table[{i, j} → CKM3D⟦i, j⟧, {i, 3}, {j, 3}], {4, 4} → 1}];
MatrixForm[
 CKM3D]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} c12\ c13 & c13\ s12 & e^{-i\delta}\ s13 & 0 \\ -c23\ s12 - c12\ e^{i\delta}\ s13\ s23 & c12\ c23 - e^{i\delta}\ s12\ s13\ s23 & c13\ s23 & 0 \\ -c12\ c23\ e^{i\delta}\ s13 + s12\ s23 & -c23\ e^{i\delta}\ s12\ s13 - c12\ s23 & c13\ c23 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Following Eq. (6) in Botella & Chau one introduces complex rotations matrices to implement the mixing with the 4th generation

In[ ]:=
```
Mu = Normal @
   SparseArray [{{1, 1} → 1, {2, 2} → 1, {3, 3} → cu, {3, 4} → su, {4, 3} → -su, {4, 4} → cu}];
Mv = Normal @ SparseArray [{{1, 1} → 1, {2, 2} → cv, {2, 4} → sv Exp[-ⅈ ϕ3],
     {3, 3} → 1, {4, 2} → -sv Exp[ⅈ ϕ3], {4, 4} → cv}];
Mw = Normal @ SparseArray [{{1, 1} → cw, {1, 4} → sw Exp[-ⅈ ϕ2], {2, 2} → 1,
     {3, 3} → 1, {4, 1} → -sw Exp[ⅈ ϕ2], {4, 4} → cw}];
```

The 4D CKM matrix is thus

In[ ]:=
```
MatrixForm [Mu].MatrixForm [Mv].MatrixForm [Mw].MatrixForm [CKM3D]
```

Out[ ]=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cu & su \\ 0 & 0 & -su & cu \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cv & 0 & e^{-i\phi3}\ sv \\ 0 & 0 & 1 & 0 \\ 0 & -e^{i\phi3}\ sv & 0 & cv \end{pmatrix} . \begin{pmatrix} cw & 0 & 0 & e^{-i\phi2}\ sw \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -e^{i\phi2}\ sw & 0 & 0 & cw \end{pmatrix} .$$

$$\begin{pmatrix} c12\ c13 & c13\ s12 & e^{-i\delta}\ s13 & 0 \\ -c23\ s12 - c12\ e^{i\delta}\ s13\ s23 & c12\ c23 - e^{i\delta}\ s12\ s13\ s23 & c13\ s23 & 0 \\ -c12\ c23\ e^{i\delta}\ s13 + s12\ s23 & -c23\ e^{i\delta}\ s12\ s13 - c12\ s23 & c13\ c23 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Wolfenstein parameters in 3D are

In[ ]:=
```
wolf3D = {s12 → λ, s23 → A λ^2, Exp[ⅈ δ] → A λ^3 (ρ - ⅈ η) / s13, Exp[-ⅈ δ] → A λ^3 (ρ + ⅈ η) / s13}
```

Out[ ]= $\left\{s12 \to \lambda,\ s23 \to A\,\lambda^2,\ e^{i\,\delta} \to \frac{A\,\lambda^3\,(-i\,\eta+\rho)}{s13},\ e^{-i\,\delta} \to \frac{A\,\lambda^3\,(i\,\eta+\rho)}{s13}\right\}$

In 4D we have 5 new parameters: su, sv, sw and $\phi 2$, $\phi 3$; one can get a natural extension of 3D CKM with new parameters by factorizing some power of λ in the new couplings, so that the Wolfenstein hierarchy is propagated. Since 12 couplings are λ, 23 are λ^2, 13 are λ^3, an obvious possibility is 34 of order λ^3, 24 of order λ^4, 14 of order λ^4. Then one introduces A24, (ρ14, η14), (ρ24, η24) which can potentially be considered of O(1) with respect to the λ expansion:

In[ ]:=
```
wolf4D = {su → A24 λ^3, Exp[ⅈ ϕ3] → A24 λ^4 (ρ24 - ⅈ η24) / sv, Exp[-ⅈ ϕ3] → A24 λ^4 (ρ24 + ⅈ η24) / sv,
   Exp[ⅈ ϕ2] → A24 λ^5 (ρ14 - ⅈ η14) / sw, Exp[-ⅈ ϕ2] → A24 λ^5 (ρ14 + ⅈ η14) / sw}
```

Out[ ]= $\left\{su \to A24\,\lambda^3,\ e^{i\,\phi 3} \to \frac{A24\,\lambda^4\,(-i\,\eta 24+\rho 24)}{sv},\ e^{-i\,\phi 3} \to \frac{A24\,\lambda^4\,(i\,\eta 24+\rho 24)}{sv},\ e^{i\,\phi 2} \to \frac{A24\,\lambda^5\,(-i\,\eta 14+\rho 14)}{sw},\ e^{-i\,\phi 2} \to \frac{A24\,\lambda^5\,(i\,\eta 14+\rho 14)}{sw}\right\}$

The Euler angles can be taken in the first quadrant, so that both sines and cosines are positive

In[ ]:= $angles = \left\{c12 \to \sqrt{1 - s12^2},\ c23 \to \sqrt{1 - s23^2},\ c13 \to \sqrt{1 - A^2\,\lambda^6\,(\rho^2 + \eta^2)},\ cu \to \sqrt{1 - su^2},\ cv \to \sqrt{1 - A24^2\,\lambda^8\,(\rho 24^2 + \eta 24^2)},\ cw \to \sqrt{1 - A24^2\,\lambda^{10}\,(\rho 14^2 + \eta 14^2)}\right\};$

Q

expand the 4D CKM matrix at order λ^5 and discuss the relative size of New Physics contributions with respect to the Standard Model.

answer