Towards four-loop splitting functions in QCD
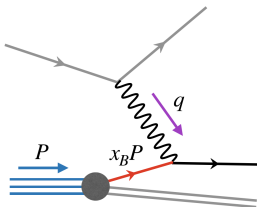
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DIS2024, Grenoble, 11 April 2024

Universität Zürich UZH

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European Research Council
Established by the European Commission

Swiss National
Science Foundation

Parton densities and splitting functions

- Bjorken variable

$$x_B = \frac{-q^2}{2P \cdot q}$$

- Factorization

$$\sigma \sim \sum_a f_{a|N}(x_B) \otimes \hat{\sigma}_a(x_B)$$

- Quark parton density in axial gauge

$$f_{q|N}(x_B) = \int \frac{dt}{2\pi} e^{-i x_B t \Delta \cdot p} \langle N(P)|\bar{\psi}(t\Delta)\frac{\slashed{\Delta}}{2}\psi(0)|N(P)\rangle , \ \Delta^2 = 0$$

- Splitting functions (SFs) govern the DGLAP evolutions of PDFs

$$\frac{df_{i|N}}{d\ln\mu} = 2\sum_k P_{ik} \otimes f_{k|N}$$

Motivations for four-loop splitting functions

- Several $\hat{\sigma}$ are available at N^{3}LO, to be consistent, need N^{3}LO PDFs
- The effect of N^{3}LO PDFs can be large
- The fields in fitting N^{3}LO PDFs are active
  - ▶ MSHT20 aN3LO talk by R. Thorne
  - ▶ NNPDF aN3LO talk by J. Rojo
  - ▶ CT is planning talk by P. Nadolsky
- The evolution of N^{3}LO PDFs requires four-loop splitting functions
- $P_{ij}^{(3)}$ is a crucial ingredient in N^{4}LL resummation

Splitting functions & Anomalous dimensions (A.D.)

- Mellin transformation

$$f_q(n) = -\int_0^1 dz\, z^{n-1} f_q(z)\,, \quad \gamma_{ij}(n) = -\int_0^1 dz\, z^{n-1} P_{ij}(z)$$

- DGLAP evolution in n-space

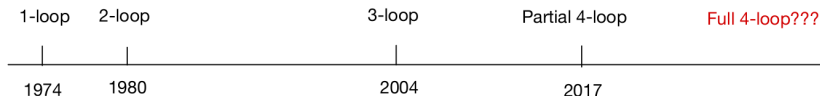$$\frac{d}{d\ln\mu} f_q(n, \mu^2) = -2\sum_j \gamma_{qj}(n)\, f_j(n, \mu^2)$$

- PDFs in n-space are hadronic operator matrix elements (OMEs)

$$f_q(n) \sim \langle N(P)|\, \bar{\psi}\slashed{\Delta}(\Delta \cdot D)^{n-1}\psi\, |N(P)\rangle = \langle N(P)|\, O_q\, |N(P)\rangle$$

$$f_g(n) \sim \langle N(P)|\Delta_{\mu_1} G_{a,\mu}^{\mu_1}(\Delta \cdot D)_{ab}^{n-2}\Delta_{\mu_n} G_b^{\mu_n\mu}\, |N(P)\rangle = \langle N(P)|\, O_g\, |N(P)\rangle$$

Towards four-loop splitting functions

- Timeline of the calculation of splitting functions

| 1-loop	2-loop		3-loop	Partial 4-loop	Full 4-loop???
1974	1980		2004	2017	

- Fixed moments See the talk by G. Falconi
  - Non-singlet $\gamma_{ns}^{(3)}$ with $n \leq 16$ [S. Moch et al. 2017]
  - Pure-singlet $\gamma_{ps}^{(3)}$ with $n \leq 20$ [G. Falconi et al. 2023]
  - $\gamma_{qg}^{(3)}$ and $\gamma_{gq}^{(3)}$ with $n \leq 20$ [G. Falconi et al. 2023, 2024]
  - $\gamma_{gg}^{(3)}$ with $n \leq 10$ [S. Moch et al. 2023]
- Exact results with all-n dependence
  - All $\gamma_{ij}^{(3)}$ in the large-N_f limit [Gracey 1994,1996; Davies et al. 2016]
  - $\gamma_{ns}^{(3)}$ with leading color [S. Moch et al. 2017]
  - N_f^2 term for $\gamma_{qg}^{(3)}$ [G. Falconi et al. 2023, 2024]
  - N_f^2 term for $\gamma_{ps}^{(3)}$ and $N_f C_F^3$ term for $\gamma_{ns}^{(3)}$ This talk
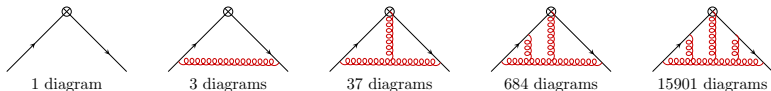
DIS method vs OME method

- Forward DIS (gauge invariant)
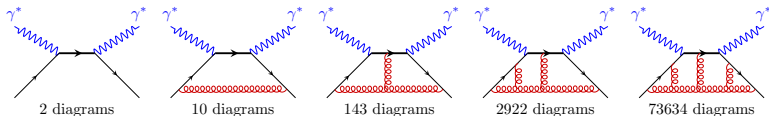
- Shrinking the heavy lines into effective vertices
- Partonic off-shell OME (fewer diagrams, easier integrals)

- Off-shell OMEs are not gauge invariant, physical operators mix with gauge-variant (GV) operators
- Main goal: find all GV operators or their Feynman rules

Computation of off-shell OMEs with all-n dependence

- Non-standard terms appearing in the Feynman rules
- Example: Feynman rules for O_q at lowest order

$$\to \slashed{\Delta}(\Delta \cdot p_1)^{n-1}$$

- How to retain all-n dependence?
  - Sum non-standard term into a linear propagator using a tracing parameter x[J. Ablinger et al. 2012]

$$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} x^n (\Delta \cdot p)^{n-1} = \frac{x}{1 - x\Delta \cdot p}$$

  - Parameter-x space $\to$ n-space, for example

$$H(1, 1; x) = \sum_{n=1}^{\infty} x^n \left(-\frac{1}{n^2} + \frac{S(1, n)}{n} \right)$$

Significant efforts in deriving GV operators

- [D.J. Gross, F. Wilczek, 1974] pointed out mixing with GV operators

- [J.A. Dixon and J.C. Taylor, 1974] constructed order g_s GV operators, not clear how to generalize to higher order

- [Joglekar and Lee, 1975] gave a general theorem about the renormalization of gauge invariant operators No explicit results were given

- [G. Falcioni and F. Herzog, 2022] constructed the GV operators for fixed n

- This talk: A new framework to derive all-n GV counterterm Feynman rules to any loop orders[T. Gehrmann, A. v. Manteuffel,TZY,2023]

A new framework of deriving GV operators

Generalize the 2×2 mixing matrix

$$\left(\begin{array}{c} O_q \\ O_g \end{array} \right)^{\text{R, naive}} = \left(\begin{array}{cc} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{array} \right) \left(\begin{array}{c} O_q \\ O_g \end{array} \right)^{\text{B}}$$

to

$$\left(\begin{array}{c} O_q \\ O_g \\ O_{ABC} \end{array} \right)^{\text{R}} = \left(\begin{array}{ccc} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ 0 & 0 & Z_{AA} \end{array} \right) \left(\begin{array}{c} O_q \\ O_g \\ O_{ABC} \end{array} \right)^{\text{B}} + \left(\begin{array}{c} [ZO]_q^{\text{GV}} \\ [ZO]_g^{\text{GV}} \\ [ZO]_A^{\text{GV}} \end{array} \right)^{\text{B}}$$

$$Z_{gA} = \mathcal{O}(a_s), \ Z_{qA} = \mathcal{O}(a_s^2), \ [ZO]_{g,q}^{\text{GV}} = \sum_{l=2,3}^{\infty} a_s^l \, [ZO]_{g,q}^{\text{GV}, (l)}$$

- $O_{ABC} = O_A + O_B + O_C$, O_A(gluon fields only), O_B(quark+gluon fileds), O_C (ghost + gluon fields)
- $[ZO]_{g,q}^{\text{GV}, (l)}$: collection of counterterms

Derive Feynman rules from off-shell OMEs

- Key idea: derive Feynman rules instead of GV operators themselves
- Consider all-off-shell OMEs with $2j + m$-gluon external states

$$\langle j|O_g^{\mathsf{R}}|j+m\,g\rangle_{1\mathrm{PI}}^{\mu_1\cdots\mu_m} = \langle j|(Z_{gq}O_q^{\mathsf{B}} + Z_{gg}O_g^{\mathsf{B}})|j+m\,g\rangle_{1\mathrm{PI}}^{\mu_1\cdots\mu_m}$$
$$+ \langle j|Z_{gA}O_{ABC}^{\mathsf{B}}|j+m\,g\rangle_{1\mathrm{PI}}^{\mu_1\cdots\mu_m} + \langle j|\,[ZO]_g^{\mathsf{GV}}\,|j+m\,g\rangle_{1\mathrm{PI}}^{\mu_1\cdots\mu_m}\,,\ j = q, g \text{ or } \mathsf{c}$$

- Expand OMEs order by order in loops and legs

$$\langle j|O|j+m\,g\rangle^{\mu_1\cdots\mu_m} = \sum_{l=1}^{\infty}\left[\langle j|O|j+m\,g\rangle^{\mu_1\cdots\mu_m,\,(l),\,(m)}\right]\left(\frac{\alpha_s}{4\pi}\right)^l g_s^m$$

- Left: UV renormalized and IR finite $\rightarrow$ no poles in ϵ
- Right: Each term is UV divergent, but the sum should be finite
- Requirement of finiteness $\longrightarrow$ counterterm Feynman rules order by order in α_s

OMEs required for deriving four-loop splitting functions

- 2-point OMEs are used to extract splitting functions
- Multi-point (≥ 3) OMEs: determine counterterm Feynman rules of GV operators
- Order: from right to left, from bottom to top

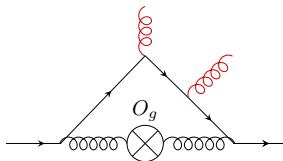| Loops \ Legs	2	3	4	5	6
0	A.D.	$[ZO]_g^{\mathrm{GV},\,(3)}$	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},\,(3)}$	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	O_g	
2	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	O_g		
3	O_{ABC}	O_g			
4	O_q, O_g				

- 3-loop splitting functions (done)
- 4-loop two-point OMEs: focus on $\langle q|O_q|q\rangle^{(4)}$

Sample results: Feynman rules for O_B
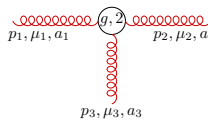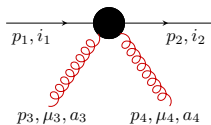

$$\rightarrow \frac{1 + (-1)^n}{-8} g_s^2 \Delta^{\mu_3} \Delta^{\mu_4} \left(T^{a_3} T^{a_4} - T^{a_4} T^{a_3}\right)_{i_2 i_1} \slashed{\Delta} \sum_{j_1=0}^{n-3} \Bigg($$

$$3 \left(\Delta \cdot (p_1 + p_2)\right)^{-j_1+n-3} \Big[\left(-\Delta \cdot p_3\right)^{j_1} - \left(-\Delta \cdot p_4\right)^{j_1} \Big]$$

$$- \left(-\Delta \cdot p_4\right)^{j_1} \left(\Delta \cdot p_3\right)^{-j_1+n-3} \Bigg)$$

extracted from

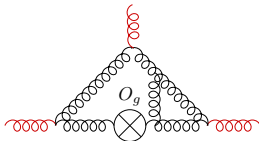$$\sum_n^\infty x^n (\text{all-}n \text{ Feynman rules}) \rightarrow \text{linear propagator in } x\text{-space}$$

Complexities from two-loop counterterm Feynman rules

$$\rightarrow 2ig_s C_A^2 f^{a_1 a_2 a_3} \frac{1+(-1)^n}{256 n(n-1)} \frac{(\Delta \cdot p_1)^{n-2}}{\Delta \cdot p_2} \left(\Delta^{\mu_2} \Delta^{\mu_3} p_1^{\mu_1} \Delta \cdot p_1 \right.$$

$$\left. + \cdots \right) \left\{ \frac{F_{-2,0}(\xi, z_1, n)}{\epsilon^2} + \frac{F_{-1,0}(\xi, z_1, n)}{\epsilon} \right\}, \ z_1 = \frac{\Delta \cdot p_2}{\Delta \cdot p_1}$$

extracted from

- $F_{-1,0}$ contains generalized harmonic sums, for example

$$S_{1,1}\left(1, z_1 + 1; n\right) = \sum_{t_1=1}^{n} \frac{1}{t_1} \sum_{t_2=1}^{t_1} \frac{(1+z_1)^{t_2}}{t_2}$$

$$\sum_n^{\infty} x^n (\text{all-}n \text{ Feynman rules}) \rightarrow \text{polylogarithms in } x\text{-space}$$
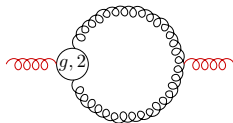
- IBP reductions with polylogarithms? (not feasible) *Need new idea*

Two-point OMEs with two-loop counterterm insertions

- Consider a general term of two-loop counterterms with 3-gluon vertex

$$i g_s f^{a_1 a_2 a_3} C_A^2 \Delta^{\mu_1} \Delta^{\mu_2} \Delta^{\mu_3} p_1^2$$

$$\times \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} + \cdots$$

where a_{mn} is known for any fixed m, n

- *New idea*: replace a_{mn} by another tracing parameter t

$$h(x, t) = \sum_{n=3}^{\infty} x^n \sum_{m=0}^{n-3} t^m (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} = \frac{x^3}{(1 - x\, t\Delta \cdot p_1)(1 - x\Delta \cdot p_2)}$$

- Evaluate OMEs to any fixed n efficiently
- Compute OMEs to $n = 500$ and reconstruct the all-n result to ϵ^0

Computations of the four-loop two-point OMEs

- Focus on specific color structures of four-loop $\langle q|O_q^{\mathsf{B}}|q\rangle$
  - Non-singlet: $N_f^3 C_F, N_f^2 C_A C_F, N_f^2 C_F^2, N_f^2 (d^{abc})^2/N_c, N_f C_F^3$
  - Pure-singlet: $N_f^3 C_F, N_f^2 C_A C_F, N_f^2 C_F^2$
- Working in Feynman gauge with $\xi = 1$
- IBP reductions and derivation of DEs in parameter-x space
  - Syzygy + finite-field sampling + denominator guessing + function reconstruction implemented in `Finred` by A. v. Manteuffel
- All DEs can be turned into canonical form[J. Henn, 2013] by `CANONICA` and `Libra`

Sample Feynman diagrams for N_f^2 contributions
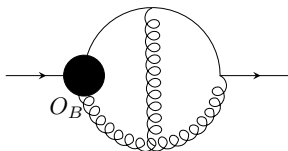
- OMEs with physical operator insertions
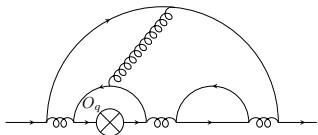
$\langle q|O_q^{\mathsf{B}}|q\rangle^{(4)}$
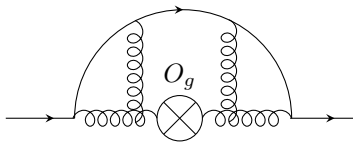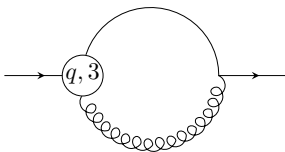
$\langle q|O_g^{\mathsf{B}}|q\rangle^{(3)}$

- OMEs with GV operator or counterterm insertion

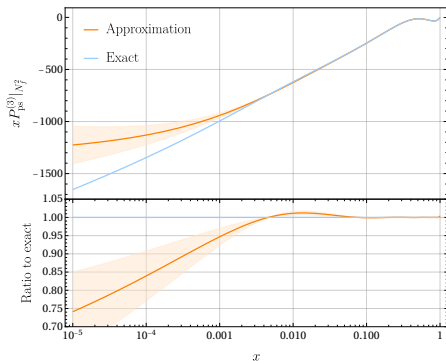$\langle q|O_B^{\mathsf{B}}|q\rangle^{(2)}$

$\langle q|\,[ZO]_q^{\mathrm{GV},\,(3)}\,|q\rangle^{(1)} = 0$

New results for N_f^2 pure-singlet contributions

- New exact result, agree with the $n \leq 20$ results[G. Falcioni et al. 2023]
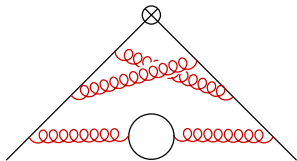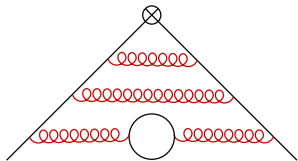- Extract small-x_B result

$$P_{\mathsf{ps}}^{(3)}(x)\big|_{N_f^2} = \frac{\log(x)^2}{x} \times 0 + \frac{\log(x)}{x}(\text{New results}) + \cdots$$

- Approximation: fitted from $n \leq 20$ results and previously known limits
- Large x-region: agree well with the exact result
- $x \sim 10^{-4}$, derivation $\sim 15\%$

New $N_f\, C_F^3$ contributions to 4-loop P_{ns}

- No mixture with non-physical operators
- Sample Feynman diagrams

- IBP reduction: 54 thousand integrals $\longrightarrow$ 658 master integrals
- Solve the master integrals by DE method analytically
- New exact resut, agree with $n \leq 16$ results from[S. Moch et al. 2017]
- Sample fixed n results with $n = 1000$

$$\gamma_{\mathsf{ns}}^{(3)}(n = 1000)\big|_{N_f\, C_F^3} = -312.9876980$$

- Large-x limit $\rightarrow$ a new analytic result for 4-loop rapidity A.D. $\gamma_3^{\mathsf{R}}\big|_{N_f\, C_F^3}$

Summary

- For off-shell OMEs, renormalization of physical operators mix with unknown GV operators
- Developed a new framework to infer splitting functions
  - Two-point OMEs are used to extract splitting functions
  - Multi-point (≥ 3) OMEs are required to determine counterterm Feynman rules of the GV operators
- Applied it to derive 3-loop singlet splitting functions and recovered the well-known results in the literature
- Proof of concept: get *exact results* for $\gamma_{\mathsf{ps}}^{(3)}\big|_{N_f^2}$ and $\gamma_{\mathsf{ns}}^{(3)}\big|_{N_f C_F^3}$

Thanks for your attention!

Renormalization of O_q to four loops in $q \to q$ channel

- Renormalization of two-point OMEs

$$\langle q|O_q^{\mathrm{R}}|q\rangle = Z_{qq}\,\langle q|O_q^{\mathrm{B}}|q\rangle + Z_{qg}\,\langle q|O_g^{\mathrm{B}}|q\rangle$$
$$+ Z_{qA}\,\langle q|O_{ABC}^{\mathrm{B}}|q\rangle + \textcolor{red}{\langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle},$$
$$Z_{qg} = \mathcal{O}(a_s), Z_{qA} = \mathcal{O}(a_s^2), [ZO]_q^{\mathrm{GV}} = \mathcal{O}(a_s^3)$$

- $\langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle = \mathcal{O}(a_s^5)$
  - Only $\langle q|\,[ZO]_q^{\mathrm{GV},\,(4)}\,|q\rangle^{(0)}$ and $\langle q|\,[ZO]_q^{\mathrm{GV},\,(3)}\,|q\rangle^{(1)}$ are relevant
  - Other operators (O_q, O_g, O_A, O_B) give all possible Lorentz structures of $q\bar{q}$, gg, $q\bar{q}g$ vertex Feynman rules
  - $\to \langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle^{(0)} = 0$, $\langle g|\,[ZO]_q^{\mathrm{GV}}\,|g\rangle^{(0)} = 0$,

  $$\langle q|\,[ZO]_q^{\mathrm{GV}}\,|qg\rangle^{(0)} = 0$$

Decomposition of splitting functions

- The general structure of quark splitting functions,

$$P_{q_i q_k} = \delta_{ik} P_{qq}^V + P_{qq}^S, \quad P_{q_i \bar{q}_k} = \delta_{ik} P_{q\bar{q}}^V + P_{q\bar{q}}^S$$

- Non-singlet and singlet splitting functions

Non-singlet: $P_{\mathrm{ns}}^\pm = P_{qq}^V \pm P_{q\bar{q}}^V, \quad P_{\mathrm{ns}}^V = P^- + \overbrace{n_f(P_{qq}^S - P_{q\bar{q}}^S)}^{P_{\mathrm{ns}}^s}$

Singlet: $P_{qq} = P^+ + \underbrace{n_f(P_{qq}^S + P_{q\bar{q}}^S)}_{P_{\mathrm{ps}}}, P_{qg}, P_{gq}, P_{gg}$

- Evolution of PDFs

$$\frac{dT_i^\pm}{d\ln\mu} = 2P_{\mathrm{ns}}^\pm \otimes T_i^\pm, \frac{d\sum_{k=1}^{n_f} q_k^-}{d\ln\mu} = 2P_{\mathrm{ns}}^V \otimes \sum_{k=1}^{n_f} q_k^-, i = 3, 8, \cdots n_f^2 - 1$$

$$T_3^\pm = u^\pm - d^\pm, T_8^\pm = u^\pm + d^\pm - 2s^\pm, \cdots, q_k^\pm = q_k \pm \bar{q}_k,$$

$$\frac{d}{d\ln\mu} \begin{pmatrix} \Sigma \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \quad \Sigma = \sum_{k=1}^{n_f} q_k^+$$

Two-point OMEs with two-loop counterterm insertions

- For a fixed n, normal IBP, but need to reduce integrals with very high numerator degree

- All-n, IBP reduction with polylogarithms?

- Consider a general term of two-loop counterterms with 3-gluon vertex
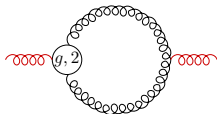
$$ig_s f^{a_1 a_2 a_3} C_A^2 \Delta^{\mu_1} \Delta^{\mu_2} \Delta^{\mu_3} p_1^2 \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} + \cdots$$

where a_{mn} is known only for fixed m, n

- New idea: replace a_{mn} by another tracing parameter t

$$h(x,t) = \sum_{n=3}^{\infty} x^n \sum_{m=0}^{n-3} t^m (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} = \frac{x^3}{(1 - x\, t\Delta \cdot p_1)(1 - x\Delta \cdot p_2)}$$

- Insert h into two-point diagrams: $\langle g | h(x,t) | g \rangle = \sum_{n=3}^{\infty} x^n \sum_{m=0}^{n-3} t^m c_{mn}$

- $\langle g | \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} | g \rangle = \sum_{m=0}^{n-3} a_{mn} c_{mn}$

Evaluate OMEs to any fixed n efficiently and reconstruct the full-n results

Three-loop results

- Two-point OMEs with two-loop counterterm insertions
  - Compute OMEs to $n = 500$ based on two tracing parameters x, t
  - Reconstruct all-n result to ϵ^0 using the data with n to 440
- Two-point OMEs with the insertion of O_q, O_g, O_{ABC}
  - Compute OMEs with all-x dependence based on a tracing paramter x
  - Expand HPLs to get all-n results in terms of HSs directly
- Results: confirm the ξ-independence explicitly and recover the well known results in the literature
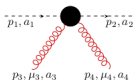 - non-singlet:
    $$\gamma_{\mathsf{ns}}^{(2)} - \gamma_{\mathsf{ns}}^{(2)}[\mathsf{MVV}] = 0$$
  - singlet:
    $$\gamma_{qq}^{(2)} - \gamma_{qq}^{(2)}[\mathsf{VMV}] = 0 \,, \quad \gamma_{qg}^{(2)} - \gamma_{qg}^{(2)}[\mathsf{VMV}] = 0$$
    $$\gamma_{gq}^{(2)} - \gamma_{gq}^{(2)}[\mathsf{VMV}] = 0 \,, \quad \gamma_{gg}^{(2)} - \gamma_{gg}^{(2)}[\mathsf{VMV}] = 0$$

All-n Feynman rules for O_C

$$\rightarrow \frac{1}{24}\frac{1+(-1)^n}{2}g_s^2\Delta^{\mu_3}\Delta^{\mu_4}\bigg\{f^{a_1a_3a}f^{a_2a_4a}\Big(6\,(-\Delta\cdot p_4)^{\,n-2}+6\,(\Delta\cdot p_3)^{\,n-2}$$

$$+6\,(\Delta\cdot(p_1+p_3))^{\,n-2}+6\,(\Delta\cdot(p_2+p_3))^{\,n-2}-\sum_{j_1=0}^{n-2}\Big[$$

$$+[\,(-\Delta\cdot p_3)^{j_1}+(-\Delta\cdot p_4)^{j_1}][3\,(\Delta\cdot p_1)^{\,n-j_1-2}$$

$$+3\,(\Delta\cdot p_2)^{\,n-j_1-2}+(\Delta\cdot(p_1+p_2))^{\,n-j_1-2}]$$

$$+9[(\Delta\cdot p_1)^{\,n-j_1-2}+(-\Delta\cdot p_2)^{\,n-j_1-2}][(\Delta\cdot(-p_2-p_3))^{j_1}+(\Delta\cdot(p_1+p_3))^{j_1}]\Big]$$

$$+13\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{j_1}\Big[(-\Delta\cdot p_2)^{j_1-j_2}\,(\Delta\cdot p_1)^{\,n-j_1-2}[(\Delta\cdot(-p_2-p_3))^{j_2}+(\Delta\cdot(p_1+p_3))^{j_2}]\Big]\Big)$$

$$+f^{a_1a_2a}f^{a_3a_4a}\Big(-6\,(\Delta\cdot p_3)^{\,n-2}-6\,(\Delta\cdot(p_2+p_3))^{\,n-2}$$

$$+\sum_{j_1=0}^{n-2}\Big[3\,(-\Delta\cdot p_4)^{j_1}\,(\Delta\cdot p_1)^{\,n-j_1-2}$$

$$+3\,(-\Delta\cdot p_3)^{j_1}\,(\Delta\cdot p_2)^{\,n-j_1-2}+[5\,(-\Delta\cdot p_3)^{j_1}-4\,(-\Delta\cdot p_4)^{j_1}]\,(\Delta\cdot(p_1+p_2))^{\,n-j_1-2}$$

$$+9[(\Delta\cdot p_1)^{\,n-j_1-2}+(-\Delta\cdot p_2)^{\,n-j_1-2}]\,(\Delta\cdot(-p_2-p_3))^{j_1}\Big]-3\Delta\cdot p_2\sum_{j_1=0}^{n-3}\Big[$$

$$3[(-\Delta\cdot p_3)^{j_1}-(-\Delta\cdot p_4)^{j_1}]\,(\Delta\cdot(p_1+p_2))^{\,n-j_1-3}-(-\Delta\cdot p_4)^{j_1}\,(\Delta\cdot p_3)^{\,n-j_1-3}\Big]$$

$$+\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{j_1}\Big[(-\Delta\cdot p_2)^{j_1-j_2}\,(\Delta\cdot p_1)^{\,n-j_1-2}[(\Delta\cdot(p_1+p_3))^{j_2}-14\,(\Delta\cdot(p_1+p_4))^{j_2}]\Big]\Big)$$

$$+\frac{6d_A^{a_1a_2a_3a_4}}{C_A}\Big(-\sum_{j_1=0}^{n-2}\Big[[(-\Delta\cdot p_3)^{j_1}+(-\Delta\cdot p_4)^{j_1}]\,(\Delta\cdot(p_1+p_2))^{\,-j_1+n-2}$$

$$+\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{j_1}\Big[(-\Delta\cdot p_2)^{j_1-j_2}\,(\Delta\cdot p_1)^{\,-j_1+n-2}[(\Delta\cdot(p_1+p_4))^{j_2}+(\Delta\cdot(p_1+p_3))^{j_2}]\Big]\Big)\Big\}.$$

Lorentz structures of a twist-two operator

- Based on the following two properties
  - A twist-two operator has spin-n and mass dimension $n + 2$
  - Propagator-type Feynman rules like $1/p^2$ can not appear in a vertex
- A twist-2 operator involving quarks or ghosts has one Lorentz structure only

$$\langle q|O|q + m\, g\rangle_{\text{1PI}}^{\mu_1 \cdots \mu_m,\,(0),\,(m)} = c_m \Delta^{\mu_1} \cdots \Delta^{\mu_m}$$

- A twist-two operator involving only gluons
  - Only $1 + 3/2m(m-1)$ Lorentz structures for m-gluon Feynman rules
  - $m = 3$: $a_1 \Delta^{\mu_1} \Delta^{\mu_2} \Delta^{\mu_3} + a_2 \Delta^{\mu_1} \Delta^{\mu_2} p_1^{\mu_3} + \cdots + a_{10} \Delta^{\mu_3} g^{\mu_1 \mu_2}$
  - 19 for $m = 4$ and 31 for $m = 5$
- Count the mass dimension of a_i: $[a_i] = x_i[\Delta \cdot p_j] + y_i[p_j \cdot p_k](y_i \geq 0)$
  $[a_1] = {\color{blue}n - 3} + y_1[p_j \cdot p_k] = n + 2 - {\color{red}3} \to y_1 = 1({\color{red}\text{Linear}} \text{ in } p_1^2,\, p_1 \cdot p_2 \cdots)$
  $[a_2] + [p_1^{\mu_3}] = {\color{blue}n - 2} + y_2[p_j \cdot p_k] + 1 = n + 2 - {\color{red}3} \to y_2 = 0$
- Why not $a_{11}\Delta^{\mu_1} p_1^{\mu_2} p_2^{\mu_3}$

$$[a_{11}] + {\color{magenta}2} + {\color{red}3} \geq {\color{blue}n - 1} + y_{11}[p_j \cdot p_k] + {\color{magenta}2} + {\color{red}3} = n + 4 (\text{if } y_{11} = 0)$$

where ${\color{red}3}$ is mass dimension of the external 3 gluons. Twist-4 operators

Computations of single pole for one-loop multi-leg OMEs

- Set all Mandelstam variables p_1^2, $p_2^2 \cdots$ to numerical numbers and reconstruct their linear dependence
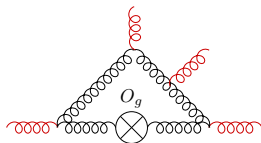
- Only two types of integrals are needed, other integrals are finite

$$I_1 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l - q_1)^2 l^2}, \quad I_2 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l - q_1)^2 l^2 \big(1 - x\Delta \cdot (l + q_2)\big)}$$

- At most x-dependent logarithms appear in the single pole

$$I_2 = \frac{1}{\epsilon} \left[\frac{\ln(1 - x\Delta \cdot q_1 - x\Delta \cdot q_2) - \ln(1 - x\Delta \cdot q_2)}{-x\Delta \cdot q_1} \right] + \mathcal{O}(\epsilon^0)$$

- Logarithms in x-space $\to$ n-space

$$\ln(1 - x\Delta \cdot p_1 - x\Delta \cdot p_2) = \sum_{n=1}^{\infty} x^n \left[\frac{-1}{n} (\Delta \cdot p_1 + \Delta \cdot p_2)^n \right]$$

- Factoring out the overall factor $Z_{gA}^{(1)} = -\frac{C_A}{\epsilon} \frac{1}{n(n-1)}$

Computations of two-loop three-leg OMEs

- Set all Mandelstam variables p_1^2, p_2^2 $\cdots$ to numerical numbers and

$$\Delta \cdot p_1 = 1, \ \Delta \cdot p_2 = z_1$$

- Derive DE with respect to x
- Difficult to solve DE in terms of special functions
- Expand DE to x^{100} in the limit of $x \to 0$, with the boundary conditions being two-loop three-leg integrals without operator insertions[T. G. Birthwright, E. W. N. Glover, and P. Marquard, 04]
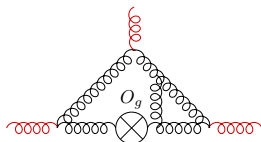
Reconstruct two-loop counterterm Feynman rules

- Obtain two-loop three-leg OMEs to x^{96} or $n = 96$
- For a fixed n, the result is a polynomial in z_1
- Construct full-x or full-n results from data to $n = 76$ based on ansatz
- Polylogarithms to weight-3, generalized Harmonic sums to weight-2

$$G(1, 1, 1/(1 + z_1); x) = \sum_{n=1}^{\infty} x^n \left[\frac{S_1(z_1 + 1; n)}{n^2} + \frac{S_2(z_1 + 1; n)}{n} \right.$$
$$\left. - \frac{S_{1,1}(1, z_1 + 1; n)}{n} - \frac{(z_1 + 1)^n}{n^3} \right]$$

where $S_{1,1}(1, z_1 + 1; n) = \sum_{t_1=1}^{n} \frac{1}{t_1} \sum_{t_2=1}^{t_1} \frac{(1+z_1)^{t_2}}{t_2}$

- Due to the generalized Harmonic sums, impossible to disentangle
  - renormalization constants (no z_1 dependence)
  - operator Feynman rules (no high-weight (≥ 1) functions)

A counterterm Feynman rule & infinite operator Feynman rules ($N_2 = \infty$)