Andrea Simonelli

# Analytic Solutions of the DGLAP Evolution and Theoretical Uncertainties

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# DGLAP evolution overview

Flavor basis $\left\{f_{i/h}, \ldots, f_{g/h}\right\}$

$$\frac{\partial}{\partial \log Q^2} f_{i/h}(x,Q) = \sum_j \int_x^1 \frac{dz}{z} P_{i/j}(z, a_S(Q))\, f_{j/h}\left(\frac{z}{x}, Q\right) \equiv \sum_j P_{i/j} \otimes f_{j/h}(x,Q)$$

Coupled set of $2N_f + 1$ differential equations

Evolution basis $\left\{V, q_{ij}^{\pm}, \ldots, \begin{pmatrix}\Sigma \\ g\end{pmatrix}\right\}$

Non-Singlet Sector

- Completely decoupled set of $2N_f - 1$ differential equations.
- **Easy** to solve (just functions).

$$\frac{\partial}{\partial \log Q^2} V(x,Q) = P_V \otimes V(x,Q)$$

$$\frac{\partial}{\partial \log Q^2} q_{ij}^{\pm}(x,Q) = P_{\pm} \otimes q_{ij}^{\pm}(x,Q)$$

Singlet Sector

- Coupled pair of differential equation.
- **Difficult** to solve (2D matrices).

$$\frac{\partial}{\partial \log Q^2} \begin{pmatrix}\Sigma(x,Q) \\ g(x,Q)\end{pmatrix} = \begin{pmatrix}P_{qq} & P_{qg} \\ P_{gq} & P_{gg}\end{pmatrix} \otimes \begin{pmatrix}\Sigma(x,Q) \\ g(x,Q)\end{pmatrix}$$

# DGLAP evolution overview

Various methods to solve them. Two main strategies:

- **Numeric** approaches (mostly in x-space)
- **Analytic** approaches (mostly in Mellin-space)

$$f(N,Q) = \int_0^1 dx\, x^{N-1} f(x,Q)$$

Defining the **Evolution Operator**:

$$\begin{cases} \mathbf{q}_S(N,Q) = \mathbf{E}(N;Q_0,Q)\mathbf{q}_S(N,Q_0) \\ q_{\rm NS}(N;Q) = E_{\rm NS}(N;Q_0,Q)q_{\rm NS}(N;Q_0) \end{cases}$$

Expanded up to $N^n$LO

$$\begin{cases} \dfrac{\partial}{\partial \log Q^2}\mathbf{E}\left(N;Q_0,Q\right) = \mathbf{P}^{(n)}\left(N,a_S(Q)\right)\mathbf{E}\left(N;Q_0,Q\right) \\ \mathbf{E}\left(N;Q_0,Q\right) = \mathbf{1} \end{cases}$$

Or <u>equivalently</u> mapped to:

$$\begin{cases} \dfrac{\partial}{\partial a_S}\mathbf{E}\left(N;a_0,a_S\right) = \mathbf{R}^{(n)}\left(N,a_S\right)\mathbf{E}\left(N;a_0,a_S\right) \\ \mathbf{E}\left(N;a_0,a_0\right) = \mathbf{1} \end{cases}$$

$$\mathbf{R}^{(n)}\left(N,a_S\right) = -\frac{1}{a_S}\mathbf{R}_0(N) - \sum_{k=1}^{n} a_S^{k-1}\frac{1+\sum_{j=1}^{n-k} b_j\, a_S^j}{1+\sum_{j=1}^{n} b_j\, a_S^j}\mathbf{R}_k(N)$$

$$\mathbf{R}_k = \frac{\mathbf{P}_k}{\beta_0} - \sum_{j=1}^{k} b_j \mathbf{R}_{k-j}$$

# Analytic solutions to DGLAP evolution

Once the perturbative order $n$ for the splitting kernels $\mathbf{P}$ has been fixed, solutions can be catalogued as:

| | | | | |
|---|---|---|---|---|
| ❑ Exact | vs | Approximated | ← | $\frac{\partial \mathbf{E}^{\text{sol}}}{\partial a_S} - \mathbf{R}^{(n)}\mathbf{E}^{\text{sol}} \stackrel{?}{=} 0$ |
| ❑ Closed | vs | Iterated | ← | Finite/Infinite number of operators |
| ❑ Exponentiated | vs | Expanded | ← | (Product of) exponentials vs $a_S/a_0$ expansion |

**IDEAL GOAL**

Only achievable for **Non-Singlet** sector: $$\frac{\partial E_{\text{NS}}}{\partial a_S} = R^{(n)}_{\text{NS}} E_{\text{NS}} \Longrightarrow E_{\text{NS}}(N; a_0, a_S) = \exp\left\{\int_{a_0}^{a_S} da R^{(n)}_{\text{NS}}(N; a)\right\}$$

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Only achievable for **Non-Singlet** sector: $$\frac{\partial E_{NS}}{\partial a_S} = R^{(n)}_{NS} E_{NS} \Longrightarrow E_{NS}(N; a_0, a_S) = \exp\left\{\int_{a_0}^{a_S} da R^{(n)}_{NS}(N; a)\right\}$$

For **Singlet** instead: $$\frac{\partial \mathbf{E}}{\partial a_S} = \mathbf{R}^{(n)}\mathbf{E} \Longrightarrow \mathbf{E}(N; a_0, a_S) = \mathcal{T}\exp\left\{\int_{a_0}^{a_S} da \mathbf{R}^{(n)}(N; a)\right\}$$

Particularly **challenging** due to its *intrinsic matrix nature* and the splitting kernels do not commute beyond LO:

$$[\boldsymbol{R}_{k\geq 1}(N), \boldsymbol{R}_0(N)] \neq 0$$ !

# Analytic solutions for Singlet Sector

Once the perturbative order $n$ for the splitting kernels $\mathbf{P}$ has been fixed, solutions can be catalogued as:

| | | | | |
|---|---|---|---|---|
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"*The splitting function matrices $\mathbf{P}_k$ of different orders k do generally not commute [...] This prevents, already at NLO, writing the solution of [Singlet Evolution] in a closed exponential form.*"

➢ J. Blumlein and A. Vogt, Phys. Rev. D 58, 014020 (1998)

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- Closed vs Iterated ← Finite/**Infinite** number of operators
- Exponentiated vs Expanded ← (Product of) exponentials vs $\mathbf{a_S/a_0}$ **expansion**

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Commonly, the solution is obtained through the **U-matrices** approach:

$$\mathbf{E}\left(N; a_0, a_S\right) = \mathbf{U}\left(N; a_S\right) \exp\left(h_1(a_0, a_S)\mathbf{R}_0(N)\right) \mathbf{U}^{-1}\left(N; a_0\right)$$

$$\mathbf{U}\left(N; a_S\right) = 1 + \sum_{k=1}^{\infty} a_S^k \mathbf{U}_k(N)$$

$\mathbf{U}_k(N)$: Obtained iteratively

- A. J. Buras, Rev. Mod. Phys. 52, 199 (1980)
- W. Furmanski and R. Petronzio, Z. Phys. C 11, 293 (1982)
- •
- •
- •
- J. Blumlein and A. Vogt, Phys. Rev. D 58, 014020 (1998)
- A. Vogt, Comput. Phys. Commun. 170, 65 (2005)

⇓ A benchmark over 40 years of QCD

# Analytic solutions for Singlet Sector

Once the perturbative order n for the splitting kernels $\mathbf{P}$ has been fixed, solutions can be catalogued as:

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"*The splitting function matrices* $\mathbf{P}_k$ *of different orders k do generally not commute [...] ~~This prevents, already at NLO, writing the solution of [Singlet Evolution] in a closed exponential form.~~*"

➢ J. Blumlein and A. Vogt, Phys. Rev. D 58, 014020 (1998)

Here I present an **ALTERNATIVE** to the $\mathbf{U}$-matrices approach which provides **closed** and **exponentiated** solutions beyond LO

**Analytic Solutions of the DGLAP Evolution and Theoretical Uncertainties**
Andrea Simonelli (Jan 24, 2024)
e-Print: 2401.13663 [hep-ph]

# How to deal with Non-Commutative Operators?

An analogous problem: time evolution of quantum systems $\frac{\partial \widehat{U}(t)}{\partial t} = \widehat{H}(t)\widehat{U}(t)\,, \qquad \widehat{U}(0) = 1$

## Dyson time-ordered exponential (1949)

➢ F. J. Dyson, Phys. Rev. 75, 486 (1949)

$$\widehat{U}(t) = \mathcal{T}\exp\left(\int_0^t d\tau \widehat{H}(\tau)\right) = 1 + \int_0^t d\tau \widehat{H}(\tau) + \frac{1}{2}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \widehat{H}(\tau_1)\widehat{H}(\tau_2) + \ldots$$

- Exponentiation is implicit (defined by its expansion).
- Popular in QFT and particle physics (and hence QCD).
- Ultimately the foundation of the **U-matrix** approach, with some further assumptions.

## Magnus Expansion (1954)

➢ W. Magnus, Commun. Pure Appl. Math. 7, 649 (1954).

$$\widehat{U}(t) = e^{\widehat{\Omega}(t)}, \quad \text{with } \widehat{\Omega}(t) = \sum_{k\geq 1} \widehat{\Omega}_k(t)$$

$$\widehat{\Omega}_1(t) = \int_0^t d\tau \widehat{H}(\tau),$$

$$\widehat{\Omega}_2(t) = \frac{1}{2}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \left[\widehat{H}(\tau_1), \widehat{H}(\tau_2)\right]$$

- Exponentiation is **explicit**.
- Many applications during the years, but never popular as the Dyson approach.
- The **alternative** presented here is based on this approach.

# Sketch of the strategy

Suppose: $\widehat{H}(t) = \widehat{H}_0(t) + \varepsilon \widehat{H}_1(t)$ , and set: $\widehat{S}_k(t) = \int_0^t d\tau H_k(\tau)$

$$\begin{cases} \dfrac{\partial \widehat{U}(t)}{\partial t} = \widehat{H}(t)\widehat{U}(t) \\ \widehat{U}(0) = 1 \end{cases}$$

1. Interaction Picture:

$$\widehat{U}(t) = \widehat{G}(t)\widehat{U}_{\rm int}(t)\widehat{G}(0)^{-1}, \quad \text{with } \widehat{G}(t) = \exp\left(\int_0^t d\tau \widehat{H}_0(\tau)\right)$$

$$\Rightarrow \quad \frac{\partial \widehat{U}_{\rm int}(t)}{\partial t} = \varepsilon\, \widehat{H}_{\rm int}(t)\widehat{U}_{\rm int}(t), \quad \text{where } \widehat{H}_{\rm int}(t) = \exp\left(-\widehat{S}_0(t)\right)\widehat{H}_1(t)\exp\left(\widehat{S}_0(t)\right)$$

$$\widehat{U}(t) = \exp\left(\widehat{S}_0(t)\right)\exp\left(\varepsilon\, \widehat{S}_1(t)\right)\mathcal{T}\exp\left(\varepsilon \int_0^t d\tau e^{-\varepsilon\, \widehat{S}_1(\tau)}\left(\widehat{H}_{\rm int}(\tau) - \widehat{H}_1(\tau)\right)e^{\varepsilon\, \widehat{S}_1(\tau)}\right)$$

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So far, just a shift of the problem

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$$\begin{cases} \frac{\partial \mathbf{U}(t)}{\partial t} = \mathbf{H}(t)\mathbf{U}(t) \\ \mathbf{U}(0) = \mathbf{1} \end{cases}$$

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So far, just a shift of the problem

2. Huge simplifications in 2D:

$$\mathbf{H}_{\text{int}}(t) = \mathbf{H}_1(t) - \frac{\sinh\left(\Delta_{S_0}(t)\right)}{\Delta_{S_0}(t)}\left[\mathbf{S}_0(t), \mathbf{H}_1(t)\right] + \frac{\cosh\left(\Delta_{S_0}(t)\right) - 1}{\Delta_{S_0}^2(t)}\left[\mathbf{S}_0(t), \left[\mathbf{S}_0(t), \mathbf{H}_1(t)\right]\right]$$

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$$\mathbf{H}_{\text{int}}(t) = \mathbf{H}_1(t) - \frac{\sinh\left(\Delta_{S_0}(t)\right)}{\Delta_{S_0}(t)}[\mathbf{S}_0(t), \mathbf{H}_1(t)] + \frac{\cosh\left(\Delta_{S_0}(t)\right) - 1}{\Delta_{S_0}^2(t)}[\mathbf{S}_0(t), [\mathbf{S}_0(t), \mathbf{H}_1(t)]]$$

3. **Magnus Expansion + Zassenhaus formula**:

$$\mathbf{U}^{\text{appr}}(t) = \exp\left(\mathbf{S}_0(t)\right)\exp\left(\varepsilon \mathbf{S}_1(t)\right)\exp\left(\varepsilon \mathbf{T}_1(t)\right)\exp\left(\varepsilon \mathbf{T}_2(t)\right)$$

$$\mathbf{T}_1(t) = -\int_0^t d\tau \frac{\sinh\left(\Delta_{S_0}(\tau)\right)}{\Delta_{S_0}(\tau)}[\mathbf{S}_0(\tau), \mathbf{H}_1(\tau)],$$

$$\mathbf{T}_2(t) = \int_0^t d\tau \frac{\cosh\left(\Delta_{S_0}(\tau)\right) - 1}{\Delta_{S_0}(\tau)}[\mathbf{S}_0(\tau), [\mathbf{S}_0(\tau), \mathbf{H}_1(\tau)]]$$

# Closed and Exponentiated Solutions

$$\begin{cases} \dfrac{\partial \mathbf{U}(t)}{\partial t} = \mathbf{H}(t)\mathbf{U}(t) \\ \mathbf{U}(0) = \mathbf{1} \end{cases}$$

- **LO**

$$\mathbf{H}(t) = \mathbf{H}_0(t) \Longrightarrow \mathbf{U}^{\mathrm{appr}}(t) = \exp\left(\mathbf{S}_0(t)\right)$$

1 Operator

- **NLO**

$$\mathbf{H}(t) = \mathbf{H}_0(t) + \varepsilon \mathbf{H}_1(t) \Longrightarrow \mathbf{U}^{\mathrm{appr}}(t) = \exp\left(\mathbf{S}_0(t)\right) \exp\left(\varepsilon\, \mathbf{S}_1(t)\right) \prod_{i=1}^{2} \exp\left(\varepsilon\, \mathbf{T}_i(t)\right)$$

4 Operators

- **NNLO**

$$\mathbf{H}(t) = \mathbf{H}_0(t) + \varepsilon \mathbf{H}_1(t) + \varepsilon^2 \mathbf{H}_2(t) \Longrightarrow$$

$$\mathbf{U}^{\mathrm{appr}}(t) = \exp\left(\mathbf{S}_0(t)\right) \exp\left(\varepsilon\, \mathbf{S}_1(t)\right) \prod_{i=1}^{2} \exp\left(\varepsilon\, \mathbf{T}_i(t)\right) \exp\left(\varepsilon^2\, \mathbf{S}_2(t)\right) \prod_{i=1}^{2} \exp\left(\varepsilon^2\, \mathbf{Q}_i(t)\right) \prod_{i} \exp\left(\frac{1}{2}\varepsilon^2\, \mathbf{W}_i(t)\right)$$

13 Operators

- Etc...

Order by order it is possible to describe the evolution using a **well-defined finite set of operators**

# Analytic Solution of DGLAP Evolution

$$\begin{cases} \frac{\partial}{\partial a_S}\mathbf{E}\left(N;a_0,a_S\right) = \mathbf{R}^{(n)}\left(N,a_S\right)\mathbf{E}\left(N;a_0,a_S\right) \\ \mathbf{E}\left(N;a_0,a_0\right) = \mathbf{1} \end{cases}$$

**1 Operator**

- **LOWEST ORDER**:
  the "Hamiltonian" is: $\mathrm{H}(a_S) = -\frac{1}{a_S}\mathrm{R}_0(N)$

$$\mathbf{E}^{\mathrm{LO}}\left(N;a_0,a_S\right) = \exp\left(h_1(a_0,a_S)\mathbf{R}_0(N)\right)$$

$$h_1\left(a_0,a_S\right) = -\log\left(\frac{a_S}{a_0}\right)$$

Known since the dawn of QCD

$$\boldsymbol{q}_{\mathrm{LO}}(N,a_{\mathrm{s}},N) = \left(\frac{a_{\mathrm{s}}}{a_0}\right)^{-\boldsymbol{R}_0(N)}\boldsymbol{q}(N,a_0) \equiv \boldsymbol{L}(N,a_{\mathrm{s}},a_0)\,\boldsymbol{q}(N,a_0)$$

- A. Vogt, Comput. Phys. Commun. 170, 65 (2005)

It is the only exact result for the Singlet Sector.

# Analytic Solution of DGLAP Evolution

$$\begin{cases} \frac{\partial}{\partial a_S}\mathbf{E}(N;a_0,a_S) = \mathbf{R}^{(n)}(N,a_S)\,\mathbf{E}(N;a_0,a_S) \\ \mathbf{E}(N;a_0,a_0) = \mathbf{1} \end{cases}$$

**4 Operators**

NEW

- **NEXT-LOWEST ORDER**:
  the "Hamiltonian" is: $\mathrm{H}(a_S) = -\frac{1}{a_S}\mathbf{R}_0(N) - \frac{1}{1+b_1 a_S}\mathbf{R}_1(N)$

$$\mathbf{E}^{\mathrm{NLO}}(N;a_0,a_S) = \exp\left(h_1(a_0,a_S)\mathbf{R}_0(N)\right)\exp\left(h_2(a_0,a_S)\mathbf{R}_1(N)\right)$$
$$\times\exp\left(h_3(\Delta_0(N);a_0,a_S)\left[\mathbf{R}_0(N),\mathbf{R}_1(N)\right]\right)\exp\left(h_4(\Delta_0(N);a_0,a_S)\left[\mathbf{R}_0(N),\left[\mathbf{R}_0(N),\mathbf{R}_1(N)\right]\right]\right)$$

$$h_1(a_0,a_S) = -\log\left(\frac{a_S}{a_0}\right);$$

$$h_2(a_0,a_S) = -\frac{1}{b_1}\log\left(\frac{1+b_1 a_S}{1+b_1 a_0}\right);$$

$$h_3(\Delta;a_0,a_S) = -\frac{1}{2\Delta}\left(a_S F_-(\Delta;a_0,a_S) - a_0 F_-(\Delta;a_0,a_0)\right);$$

$$h_4(\Delta;a_0,a_S) = -\frac{1}{2\Delta^2}\left(a_S F_+(\Delta;a_0,a_S) - a_0 F_+(\Delta;a_0,a_0) + 2h_2(a_0,a_S)\right)$$

$$F(\Delta;a_0,a_S) = \left(\frac{a_S}{a_0}\right)^{\Delta}\frac{1}{1+\Delta}\,{}_2F_1\left(1,1+\Delta;2+\Delta,-b_1 a\right)$$

$$F_+(\Delta;a_0,a_S) = F(\Delta;a_0,a_S) + F(-\Delta;a_0,a_S);$$

$$F_-(\Delta;a_0,a_S) = F(\Delta;a_0,a_S) - F(-\Delta;a_0,a_S),$$

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# Comparing approaches  *vs* 

Current available closed solution at NLO is the **truncated** solution from U-matrices approach:

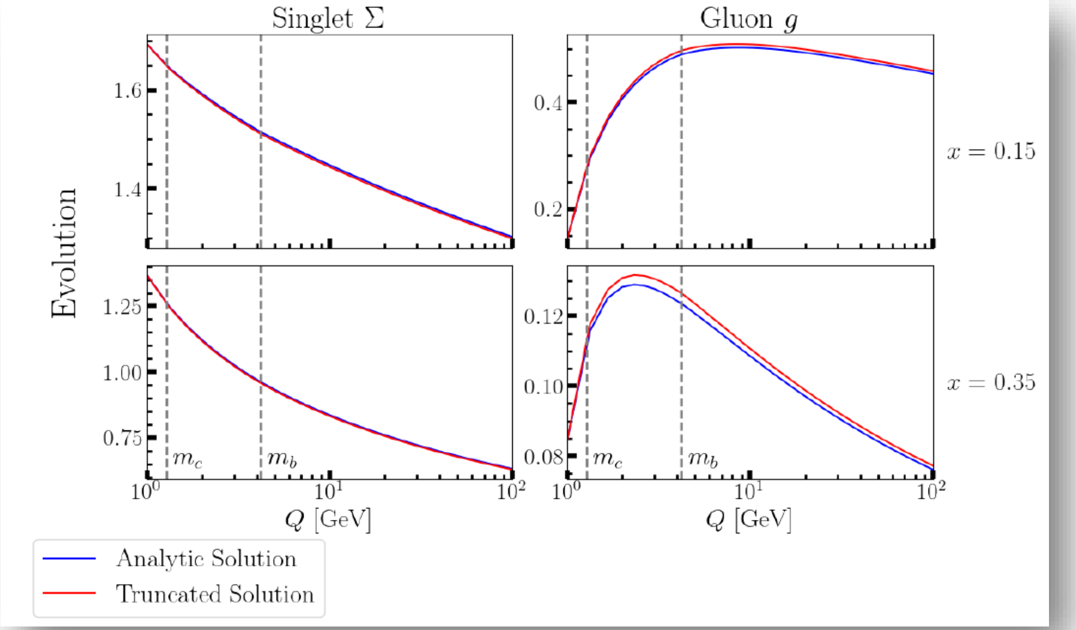
$$\mathbf{E}^{\mathrm{NLO}}_{\mathrm{tr.}}\left(N; a_0, a_S\right) = e^{h_1(a_0,a_S)\mathbf{R}_0(N)} + a_S \mathbf{U}_1(N)\, e^{h_1(a_0,a_S)\mathbf{R}_0(N)} - a_0\, e^{h_1(a_0,a_S)\mathbf{R}_0(N)}\mathbf{U}_1(N)$$

Closed, but *not* exponentiated

***vs***

$$\mathbf{E}^{\mathrm{NLO}}\left(N; a_0, a_S\right) = \exp\left(h_1(a_0,a_S)\mathbf{R}_0(N)\right)\exp\left(h_2(a_0,a_S)\mathbf{R}_1(N)\right) \times \exp\left(h_3(\Delta_0(N); a_0, a_S)\left[\mathbf{R}_0(N), \mathbf{R}_1(N)\right]\right)\exp\left(h_4(\Delta_0(N); a_0, a_S)\left[\mathbf{R}_0(N), \left[\mathbf{R}_0(N), \mathbf{R}_1(N)\right]\right]\right)$$

Closed *and* exponentiated




$$\mathbf{q}_S^{\mathrm{sol}}(x,Q) = \int \frac{dN}{2\pi i} x^{-N}\, \mathbf{E}^{\mathrm{sol}}\left(N; a_0, a_S(Q)\right) \mathbf{q}_S(x, Q_0) \quad \text{with: } \mathbf{q}_S = \begin{pmatrix}\Sigma \\ g\end{pmatrix}$$

Singlet PDF at input scale. A simple (proton) model is:

$$u_V(N;Q_0) = 2\frac{B(\alpha_u + N, \beta_u + 1)}{B(\alpha_u + 1, \beta_u + 1)}$$

$$d_V(N;Q_0) = \frac{B(\alpha_d + N, \beta_d + 1)}{B(\alpha_d + 1, \beta_d + 1)}$$

$$g(N;Q_0) = \gamma_g\, B(\alpha_g + N, \beta_g + 1)$$

$$q_{\mathrm{sea}}(N;Q_0) = \gamma_{\mathrm{sea}}\, B(\alpha_{\mathrm{sea}} + N, \beta_{\mathrm{sea}} + 1) \quad \text{with: } \gamma_{\mathrm{sea}} = \frac{1 - u_V(2;Q_0) - d_V(2;Q_0) - g(2,Q_0)}{6\, B(\alpha_{\mathrm{sea}} + 2, \beta_{\mathrm{sea}} + 1)}$$

# Theoretical Uncertainties: how far from exact solution?

Defining the **Violation Operator**:

$$\mathbf{V}^{\rm sol}\left(N; a_0, a_S(Q)\right) = \frac{\partial \mathbf{E}^{\rm sol}\left(N; a_0, a_S(Q)\right)}{\partial \log Q^2} - \left(a_S(Q)\mathbf{P}_0(N) + a_S^2(Q)\mathbf{P}_1(N)\right)\mathbf{E}^{\rm sol}\left(N; a_0, a_S(Q)\right)$$

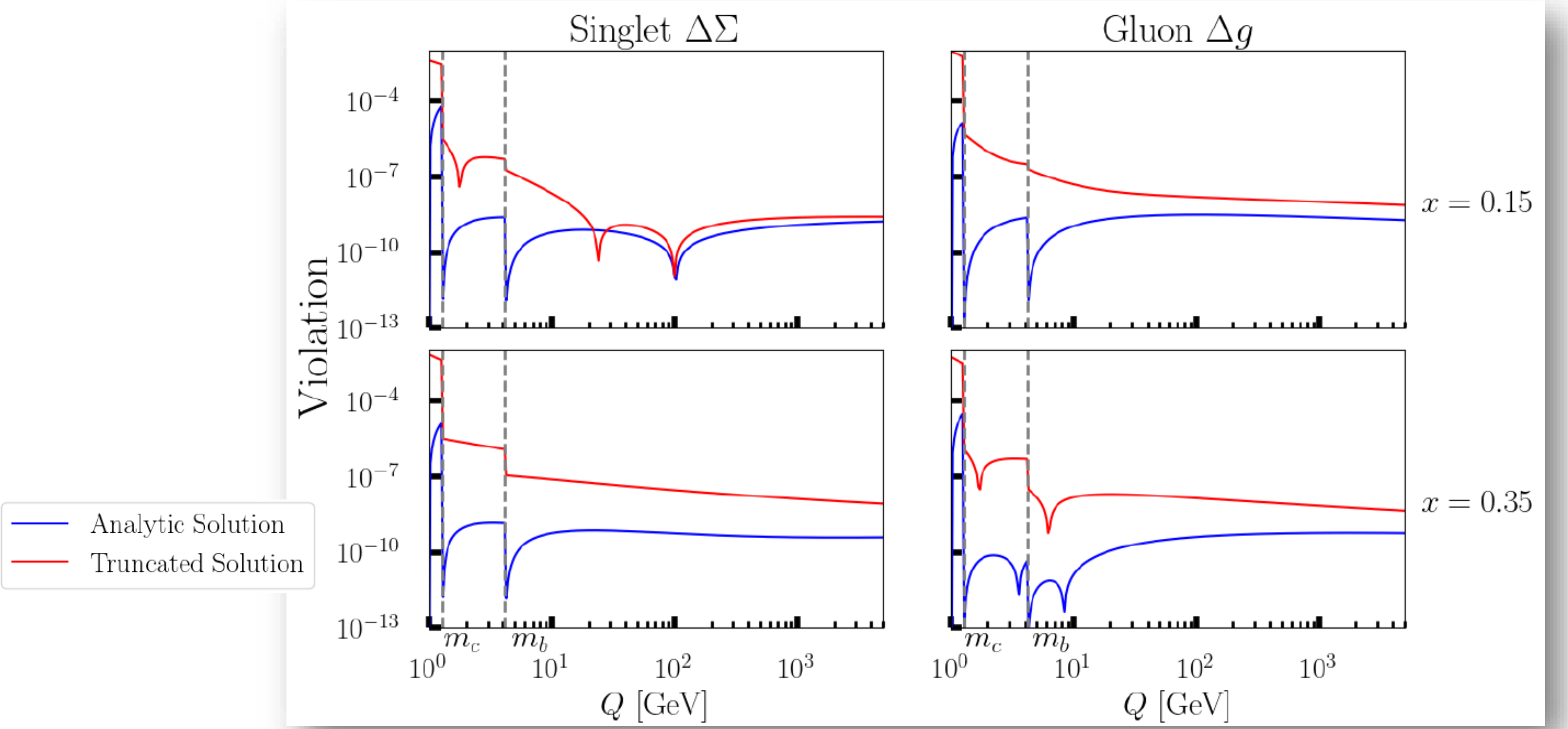
$$\text{exact sol.} \Longrightarrow \mathbf{V}^{\rm sol} \equiv 0$$

The discrepancy from the exact solution can be determined as:

$$\Delta\mathbf{q}_S^{\rm sol}(x,Q) = \int \frac{dN}{2\pi i} x^{-N}\, \mathbf{V}^{\rm sol}\left(N; a_0, a_S(Q)\right) \mathbf{q}_S(x,Q_0) \quad \text{with: } \Delta\mathbf{q}_S = \begin{pmatrix} \Delta\Sigma \\ \Delta g \end{pmatrix}$$

The bigger is the size of $\Delta\mathbf{q}_S^{\rm sol}(x,Q)$, the bigger are the theoretical errors

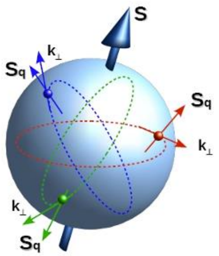
# Theoretical Uncertainties: how far from exact solution?




The analytic solution is **systematically** more precise than the truncated solution!

The improvement is particularly evident at low energies (several orders of magnitude)

⟹ It might be relevant for TMD physics



# **Bonus**: Consistent Log-counting and Improved Accuracy

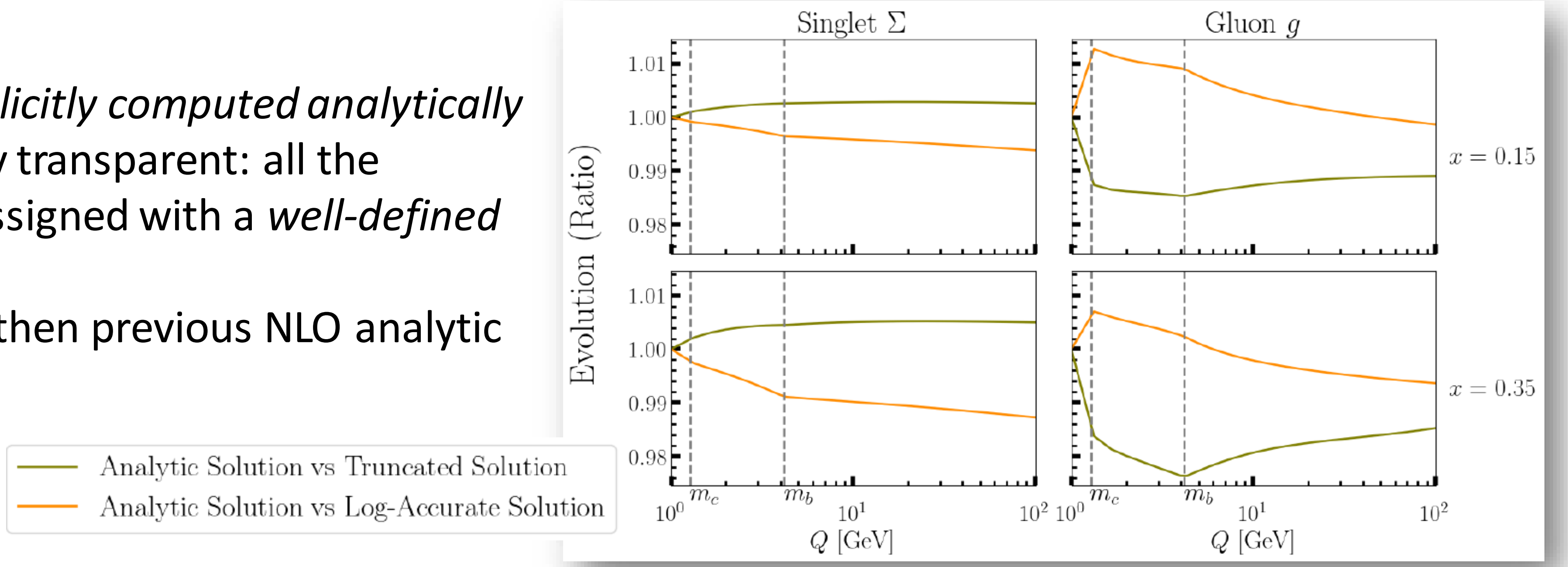
$$\frac{d}{d\log Q^2}a_S(Q) = \beta\left(a_S(Q)\right) \Longrightarrow a_0 \Leftrightarrow a_S,\ L = \log\left(\frac{Q}{Q_0}\right)$$

The exponents can be ordered in descendent powers of L

$$\mathbf{E}^{\mathrm{NLL}}(N;Q_0,Q) = \exp\left(\left(\tilde{f}_1(\lambda) + \frac{1}{L}\tilde{f}_2^{(1)}(\lambda)\right)\mathbf{P}_0(N)\right)\exp\left(\frac{1}{L}\tilde{f}_2^{(2)}(\lambda)\mathbf{P}_1(N)\right)$$
$$\times \exp\left(\frac{1}{L}\tilde{f}_3\left(\frac{\tilde{\Delta}_0(N)}{\beta_0},\lambda\right)[\mathbf{P}_0(N),\mathbf{P}_1(N)]\right)\exp\left(\frac{1}{L}\tilde{f}_4\left(\frac{\tilde{\Delta}_0(N)}{\beta_0},\lambda\right)[\mathbf{P}_0(N),[\mathbf{P}_0(N),\mathbf{P}_1(N)]]\right)$$

$$\lambda = 2a_S\beta_0 L$$

- Every ingredient is *explicitly computed analytically*
- Expansion is extremely transparent: all the neglected terms are assigned with a *well-defined scaling*
- Inevitably *less precise* then previous NLO analytic solution.

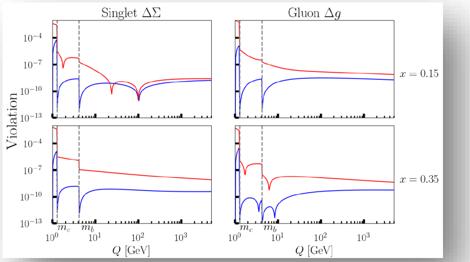


# Conclusions

- I presented an **alternative approach** to the usual strategy for solving the Singlet Sector of DGLAP evolution.
- Within this framework, I obtained the first **closed** and **exponentiated** solution at NLO.

$$\mathbf{E}^{\mathrm{NLO}}(N; a_0, a_S) = \exp\left(h_1(a_0, a_S)\mathbf{R}_0(N)\right)\exp\left(h_2(a_0, a_S)\mathbf{R}_1(N)\right) \times \exp\left(h_3(\Delta_0(N); a_0, a_S)\left[\mathbf{R}_0(N), \mathbf{R}_1(N)\right]\right)\exp\left(h_4(\Delta_0(N); a_0, a_S)\left[\mathbf{R}_0(N), \left[\mathbf{R}_0(N), \mathbf{R}_1(N)\right]\right]\right)$$

- This result is systematically more precise than its $\mathbf{U}$-matrices counterpart
- Log-accuracy follows quite straightforwardly:




$$\mathbf{E}^{\mathrm{NLL}}(N; Q_0, Q) = \exp\left(\left(\widetilde{f}_1(\lambda) + \frac{1}{L}\widetilde{f}_2^{(1)}(\lambda)\right)\mathbf{P}_0(N)\right)\exp\left(\frac{1}{L}\widetilde{f}_2^{(2)}(\lambda)\mathbf{P}_1(N)\right) \times \exp\left(\frac{1}{L}\widetilde{f}_3\left(\frac{\widetilde{\Delta}_0(N)}{\beta_0}, \lambda\right)\left[\mathbf{P}_0(N), \mathbf{P}_1(N)\right]\right)\exp\left(\frac{1}{L}\widetilde{f}_4\left(\frac{\widetilde{\Delta}_0(N)}{\beta_0}, \lambda\right)\left[\mathbf{P}_0(N), \left[\mathbf{P}_0(N), \mathbf{P}_1(N)\right]\right]\right)$$

# Future Perspectives

- Extension to NNLO (and beyond)
- Application to **QCD + QED**
- Compare performance with iterated solutions from $\mathbf{U}$-matrices approach
- PDF phenomenology
- TMD implementation and global fitting

**Thank You!**

# Back-up slides

# Detailed comparison with U-matrices approach

$$\mathbf{E}\left(N;a_0,a_S\right)=\mathbf{U}\left(N;a_S\right)\exp\left(h_1(a_0,a_S)\mathbf{R}_0(N)\right)\mathbf{U}^{-1}\left(N;a_0\right)$$

$$\mathbf{U}_k=-\frac{1}{k}\left(\mathbf{e}_-\widetilde{\mathbf{R}}_k\mathbf{e}_-+\mathbf{e}_+\widetilde{\mathbf{R}}_k\mathbf{e}_+\right)+\frac{\mathbf{e}_+\widetilde{\mathbf{R}}_k\mathbf{e}_-}{\Delta_0-k}-\frac{\mathbf{e}_-\widetilde{\mathbf{R}}_k\mathbf{e}_+}{\Delta_0+k}$$

- Iterated (x-space integration) $\mathbf{R}_k^{\mathrm{NLO}}=(-b_1)^{k-1}\mathbf{R}_1$

$$\mathbf{e}_-\mathbf{U}\left(N;a_S\right)\mathbf{e}_+=\mathbf{e}_-\mathbf{R}_1\Bigg[-\frac{\log\left(1+b_1\,a_S\right)}{b_1}-\frac{a_S}{1+\Delta_0}{}_2F_1\left(1,1+\Delta_0;2+\Delta_0;-b_1\,a_S\right)-$$
$$-\sum_{i=1}^{\infty}a_S^{1+i}\left(\frac{{}_2F_1\left(1,1+i;2+i;-b_1\,a_S\right)}{1+i}+\frac{{}_2F_1\left(1,1+i+\Delta_0;2+i+\Delta_0;-b_1\,a_S\right)}{1+i+\Delta_0}\right)\mathbf{U}_i\Bigg]\mathbf{e}_+$$

Iterative counterpart of the closed exponentiated solution!

- Iterated $\mathbf{R}_{k\geq 2}^{\mathrm{NLO}}=0$

$$\mathbf{e}_-\mathbf{U}\left(N;a_S\right)\mathbf{e}_+=\mathbf{e}_-\mathbf{R}_1\left[-a_S\left(1+\frac{1}{\Delta_0+1}\right)-\sum_{k\geq 2}a_S^k\left(\frac{1}{k}+\frac{1}{\Delta_0+k}\right)\mathbf{U}_{k-1}\right]\mathbf{e}_+$$

- Truncated

# NNLO Operators

$$\mathbf{S}_2(t) = \int_0^t d\tau\, \mathbf{H}_2(t),$$

$$\mathbf{Q}_1(t) = -\int_0^t d\tau \frac{\sinh\left(\Delta_{S_0}(\tau)\right)}{\Delta_{S_0}(\tau)} \left[\mathbf{S}_0(\tau), \mathbf{H}_2(\tau)\right],$$

$$\mathbf{Q}_2(t) = \int_0^t d\tau \frac{\cosh\left(\Delta_{S_0}(\tau)\right) - 1}{\Delta^2_{S_0}(\tau)} \left[\mathbf{S}_0(\tau), \left[\mathbf{S}_0(\tau), \mathbf{H}_2(\tau)\right]\right],$$

$$\mathbf{W}_1(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \left[\mathbf{H}_1(\tau_1), \mathbf{H}_1(\tau_2)\right],$$

$$\mathbf{W}_1'(t) = -\left(\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 - \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1\right) \frac{\sinh \Delta_{S_0}(\tau_1)}{\Delta_{S_0}(\tau_1)} \left[\mathbf{H}_1(\tau_1), \left[\mathbf{S}_0(\tau_2), \mathbf{H}_1(\tau_2)\right]\right],$$

$$\mathbf{W}_1''(t) = \left(\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 - \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1\right) \frac{\cosh \Delta_{S_0}(\tau_1) - 1}{\Delta^2_{S_0}(\tau_1)} \left[\mathbf{H}_1(\tau_1), \left[\mathbf{S}_0(\tau_2), \left[\mathbf{S}_0(\tau_2), \mathbf{H}_1(\tau_2)\right]\right]\right],$$

$$\mathbf{W}_2(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \frac{\sinh \Delta_{S_0}(\tau_1)}{\Delta_{S_0}(\tau_1)} \frac{\sinh \Delta_{S_0}(\tau_2)}{\Delta_{S_0}(\tau_2)} \left[\left[\mathbf{S}_0(\tau_1), \mathbf{H}_1(\tau_1)\right], \left[\mathbf{S}_0(\tau_2), \mathbf{H}_1(\tau_2)\right]\right]$$

$$\mathbf{W}_2'(t) = -\left(\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 - \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1\right) \frac{\sinh \Delta_{S_0}(\tau_1)}{\Delta_{S_0}(\tau_1)} \frac{\cosh \Delta_{S_0}(\tau_2) - 1}{\Delta_{S_0}(\tau_2)} \times \left[\left[\mathbf{S}_0(\tau_1), \mathbf{H}_1(\tau_1)\right], \left[\mathbf{S}_0(\tau_2), \left[\mathbf{S}_0(\tau_2), \mathbf{H}_1(\tau_2)\right]\right]\right]$$

$$\mathbf{W}_3(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \frac{\cosh \Delta_{S_0}(\tau_1) - 1}{\Delta_{S_0}(\tau_1)} \frac{\cosh \Delta_{S_0}(\tau_2) - 1}{\Delta_{S_0}(\tau_2)} \left[\left[\mathbf{S}_0(\tau_1), \left[\mathbf{S}_0(\tau_1), \mathbf{H}_1(\tau_1)\right]\right], \left[\mathbf{S}_0(\tau_2), \left[\mathbf{S}_0(\tau_2), \mathbf{H}_1(\tau_2)\right]\right]\right]$$

# NLL Functions

$$f_1(\lambda) = -\log(1-\lambda);$$

$$f_2^{(1)}(\lambda) = -\frac{1}{2\beta_0}\frac{\beta_1}{\beta_0}\frac{\lambda}{1-\lambda}\log(1-\lambda);$$

$$f_2^{(2)}(\lambda) = \frac{1}{2\beta_0}\frac{\lambda^2}{1-\lambda};$$

$$f_3(\Delta,\lambda) = -\frac{1}{4\beta_0}\lambda\left(\frac{2}{(1-\Delta^2)}\frac{1}{1-\lambda} + \frac{1}{\Delta}\left(\frac{(1-\lambda)^{\Delta}}{1+\Delta} - \frac{(1-\lambda)^{-\Delta}}{1-\Delta}\right)\right);$$

$$f_4(\Delta,\lambda) = -\frac{1}{4\beta_0}\frac{\lambda}{1-\lambda}\frac{1}{\Delta^2}\left(-\frac{2\left(1-(1-\Delta^2)\lambda\right)}{1-\Delta^2} + \frac{(1-\lambda)^{\Delta}}{1+\Delta} + \frac{(1-\lambda)^{-\Delta}}{1-\Delta}\right)$$

$$\widetilde{f}_1(\lambda) = \frac{1}{\beta_0}f_1(\lambda);$$

$$\widetilde{f}_2^{(1)}(\lambda) = \frac{1}{\beta_0}\left(f_2^{(1)}(\lambda) - b_1 f_2^{(2)}(\lambda)\right);$$

$$\widetilde{f}_2^{(2)}(\lambda) = \frac{1}{\beta_0}f_2^{(2)}(\lambda);$$

$$\widetilde{f}_3(\Delta,\lambda) = \frac{1}{\beta_0^2}f_3(\Delta,\lambda);$$

$$\widetilde{f}_4(\Delta,\lambda) = \frac{1}{\beta_0^3}f_4(\Delta,\lambda).$$