Recent progress in the calculation of the N3LO splitting functions

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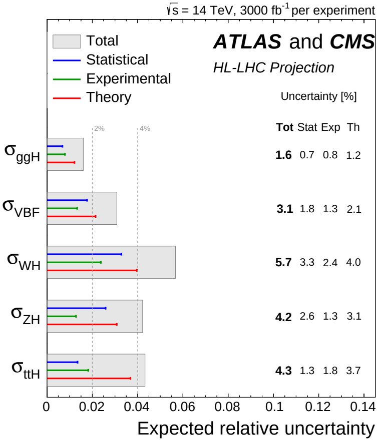
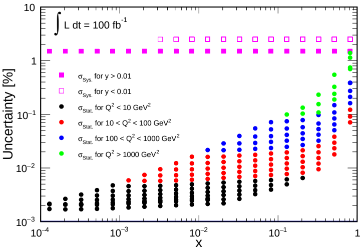
31st International Workshop on Deep Inelastic Scattering, Grenoble - 9th Apr 2024

A forthcoming era of precision physics



(CERN Yellow Reports 2019)



(EIC Yellow Report 2021)

Experimental errors O(1%)

- Higgs production at the HL-LHC
- Inclusive DIS at the EIC

The need for N^3LO QCD corrections

	Q [GeV]	$\delta\sigma^{\text{N}^3\text{LO}}$	δ(scale)	δ(PDF-TH)
NCDY	100	-2.1%	$^{+0.66\%}_{-0.79\%}$	$\pm 2.5\%$
CCDY(W^+)	150	-2.0%	$^{+0.5\%}_{-0.5\%}$	$\pm 2.1\%$
CCDY(W^-)	150	-2.1%	$^{+0.6\%}_{-0.5\%}$	$\pm 2.13\%$

J. Baglio, C. Duhr, B. Mistlberger, R. Szafron 2022

- N^3LO corrections $O(\%)$, scale uncertainties sub-percent.
- δ(PDF-TH): Additional error due missing N^3LO PDFs.

$$\delta(\text{PDF-TH}) = \frac{\sigma^{\text{NNLO}}(\text{NNLO PDF}) - \sigma^{\text{NNLO}}(\text{NLO PDF})}{2}$$

PDF evolution to N^3LO

$$\mu^2 \frac{d}{d\mu^2} f_{\rm i}\left(x,\mu^2\right) = \int_x^1 \frac{dy}{y}\, P_{\rm ij}(\alpha_s, y)\, f_{\rm j}\left(\frac{x}{y},\mu^2\right), \quad {\rm i,j = q,g}$$

- Flavour non-singlet PDFs

$$q_{\rm ns}^{\pm} = (f_{\rm i} \pm f_{\bar{\rm i}}) - (f_{\rm k} \pm f_{\bar{\rm k}}), \qquad {\rm i,k = u,d,s}$$

- Flavour-singlet PDFs

$$q_{\rm s}(x,\mu^2) = \sum_{\rm i=u,d,s} (f_{\rm i} + f_{\bar{\rm i}}), \qquad g(x,\mu^2) = f_g(x,\mu^2).$$

The N^3LO evolution requires the four-loop splitting functions.

$$P_{\rm ij}(\alpha_s, x) = \underbrace{a\, P_{\rm ij}^{(0)}}_{\rm LO} + \underbrace{a^2\, P_{\rm ij}^{(1)}}_{\rm NLO} + \underbrace{a^3\, P_{\rm ij}^{(2)}}_{\rm NNLO} + \underbrace{a^4\, P_{\rm ij}^{(3)}}_{\rm N^3LO}, \; a = \frac{\alpha_s}{4\pi}$$

Towards N^{3}LO splitting functions: exact results

- Flavour non-singlet PDFs in the **planar limit** (Ruijl,Ueda,Vermaseren,Vogt 2017)
- All the $P_{\rm ij}$ in the large-n_f limit (Gracey 1994,1996; Davies,Vogt,Ruijl,Ueda,Vermaseren 2016)
- n_f^2 term in $P^{(3)}_{\rm qq}$ (Gehrmann,von Manteuffel,Sotnikov,Yang 2023)
- n_f^2 term in $P^{(3)}_{\rm gq}$ (**GF**,Herzog,Moch,Vermaseren,Vogt 2023)
- Flavour non-singlet: $n_f C_F^3$ term (Gehrmann,von Manteuffel,Sotnikov,Yang 2023)

See the talk by **T.Z. Yang** on Thursday!

Approximate results

- The **moments** of the splitting functions

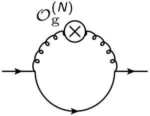
$$\int_0^1 dx\, x^{N-1}\, P_{\mathrm{ij}}^{(N)}(x, \alpha_s) = -\gamma_{\mathrm{ij}}^{(N)}$$

  govern the evolution of the moments of the PDFs.

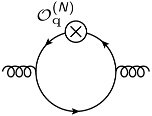
- Approximate N^3LO PDFs using the results of a fixed number of moments
  (McGowan,Cridge,Harland-Lang,Thorne 2022;
  Cridge,Harland-Lang,Thorne 2023 & 2024; Hekhorn,Magni 2023;
  NNPDF 2024)
  Talks by **L. Harland-Lang**, **R. Thorne**, **J. Rojo**, **J. Cruz Martinez**

The OPE method (Gross,Wilczek 1974)

Compute the moments from the ϵ-poles of 2-point functions



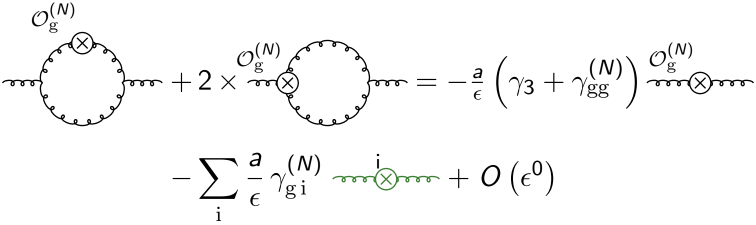
$$= -\frac{a}{\epsilon}\left(\gamma_\psi + \gamma_{\mathrm{gq}}^{(N)}\right) \mathcal{O}_{\mathrm{q}}^{(N)} + \mathcal{O}(\epsilon^0)$$



$$= -\frac{a}{\epsilon}\left(\gamma_3 + \gamma_{\mathrm{qg}}^{(N)}\right) \mathcal{O}_{\mathrm{g}}^{(N)} + \mathcal{O}(\epsilon^0)$$

- $\mathcal{O}_{\mathrm{g}}^{(N)}$ ($\mathcal{O}_{\mathrm{q}}^{(N)}$) are **gauge-invariant** gluon (quark) operators
- Two-point integrals known to 4 loops: `Forcer` (Ruijl,Ueda,Vermaseren 2017)
- Moments up to $N = 16$ of the **non-singlet** splitting functions (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017)

Troubles in the singlet sector



- Gauge variant **alien** operators mix with the physical ones.
- Long time to carry out the **two-loop calculation**:
    - Basis of aliens at 2 loops (Dixon,Taylor 1974)
    - Result for the physical operators (Hamberg,van Neerven 1991)

Operator structure to four loops

- Systematic construction of the aliens for every fixed N (**GF**,Herzog 2022)

$$\widetilde{\mathcal{L}} = \underbrace{\mathcal{L}_0 + c_{\rm g}\,\mathcal{O}_{\rm g}^{(N)} + \mathcal{O}_{\rm EOM}^{(N)}}_{\mathcal{L}_{\rm GGI}} + \underbrace{\mathbf{s'}\left[\bar{c}^a\left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a\right)\right]}_{\text{Gauge fixing + ghost}}$$

$$\mathcal{O}_{\rm EOM}^{(N)} = (D^\mu F_\mu)^a\Big[\underbrace{\eta\,\partial^{N-2}A^a}_{\mathcal{O}_A^I} + g f^{aa_1a_2}\sum_{i_1+i_2=N-3}\underbrace{\kappa_{i_1i_2}(\partial^{i_1}A^{a_1})(\partial^{i_2}A^{a_2})}_{\mathcal{O}_A^{II}}$$
$$+ g^2\sum_{\substack{i_1+i_2+i_3\\N-4}}\underbrace{\left(\kappa^{(1)}_{i_1i_2i_3} f^{aa_1z}f^{a_2a_3z} + \kappa^{(2)}_{i_1i_2i_3} d_4^{aa_1a_2a_3} + \kappa^{(3)}_{i_1i_2i_3} d_{\widehat{4ff}}^{aa_1a_2a_3}\right)(\partial^{i_1}A^{a_1})..(\partial^{i_3}A^{a_3})}_{\mathcal{O}_A^{III}} + \ldots$$

- Agreement with general theory (Joglekar,Lee 1974)
- Agreement with explicit results at three loops (Gehrmann,von Manteuffel,Yang 2023).

Quark evolution: quark-to-quark splitting

- Only $\mathcal{O}^{I}$ and $\mathcal{O}^{II}$ mix with **pure singlet** $\gamma_{\text{ps}}(N)$

$$\gamma_{\mathrm{qq}}(N) = \gamma_{\text{ps}}(N) + \gamma_{\text{ns}}^{+}(N)$$

- Results for the first ten moments ($N = 20$) (**GF**,Herzog,Moch,Vogt 2023)

$$\gamma_{\mathrm{ps}}^{(3)}(N\!=\!2) = -691.5937093\, n_f + 84.77398149\, n_f^{\,2} + 4.466956849\, n_f^{\,3}\,,$$
$$\dots$$
$$\gamma_{\mathrm{ps}}^{(3)}(N\!=\!20) = -0.442681568\, n_f + 0.805745333\, n_f^{\,2} - 0.020918264\, n_f^{\,3}\,.$$

- Checks with moments up to $N = 12$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2023), large n_f limit (Davies,Moch,Ruijl,Ueda,Vermaseren,Vogt 2016), complete n_f^2 colour factor (Gehrmann,von Manteuffel,Sotnikov,Yang 2023).

Quark evolution: gluon-to-quark splitting

The same approach applied to $\gamma_{\rm qg}(N)$

- Results for ten moments (**GF**,Herzog,Moch,Vogt 2023)

$$\begin{aligned}
\gamma^{(3)}_{\rm qg}(N\!=\!2) &= -654.4627782\, n_f + 245.6106197\, n_f^2 - 0.924990969\, n_f^3\,, \\
\gamma^{(3)}_{\rm qg}(N\!=\!4) &= 290.3110686\, n_f - 76.51672403\, n_f^2 - 4.911625629\, n_f^3\,, \\
&\dots \\
\gamma^{(3)}_{\rm qg}(N\!=\!20) &= 52.24329555\, n_f - 109.3424891\, n_f^2 - 2.153153725\, n_f^3\,.
\end{aligned}$$

- Agreement with moments up to $N = 10$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2021,2023)
- Agreement with the large-n_f limit (Davies,Vogt,Ruijl,Ueda,Vermaseren 2016)

Approximations of the splitting functions

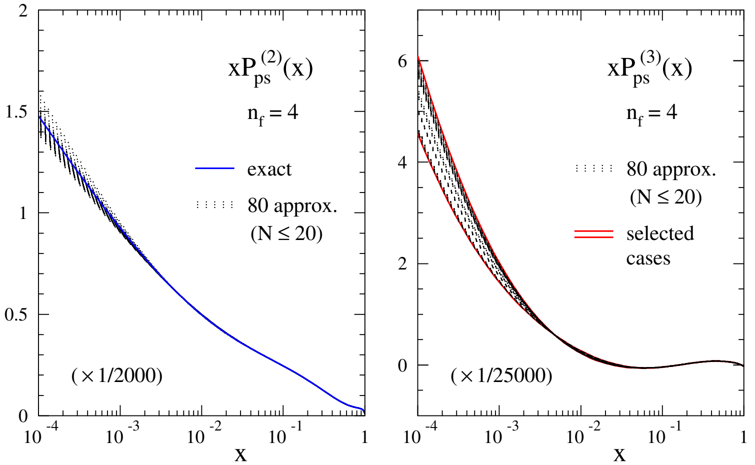
Candidate functions matching the moments up to $N = 20$ and

- Small-x limits
  - Coefficient of $\frac{\log^2 x}{x}$ (Catani,Hautmann 1994)
  - Coefficients of $\log^k x$ with $k = 6, 5, 4$ (Davies,Kom,Moch,Vogt 2022)

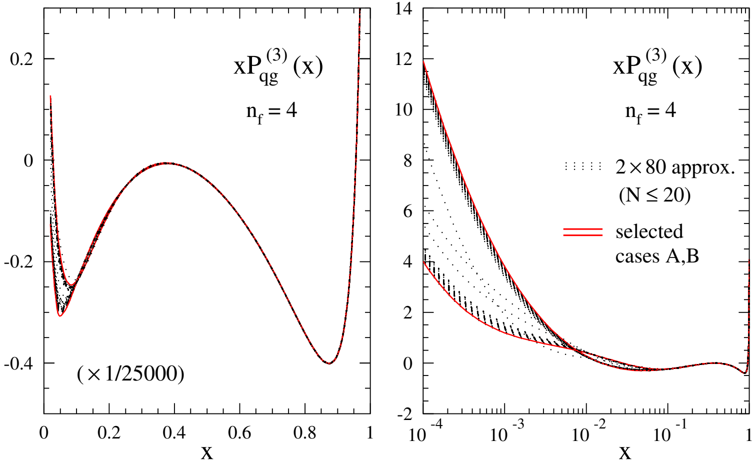
- Large-x limits (Soar,Moch,Vermaseren,Vogt 2010; Vogt 2010; Almasy,Soar,Vogt 2011)
  - P_{ps}: $(1-x)^j \log^k(1-x)$ with $k = 4, 3$ and $\forall j \geq 1$
  - P_{qg}: $\log^k(1-x)$ with $k = 6, 5, 4$
    $(1-x)\log^k(1-x)$ with $k = 6, 5, 4$

    $\log^3(1-x) \ldots \log(1-x)$ unknown

Approximations of $P_{ps}^{(3)}(x)$



Approximations of $P^{(3)}_{qg}(x)$



Evolution of the quark PDF

$$\dot{q}_\mathrm{s} \equiv \mu^2 \frac{dq_\mathrm{s}}{d\mu^2} = P_\mathrm{qq} \otimes q_\mathrm{s} + P_\mathrm{qg} \otimes g$$

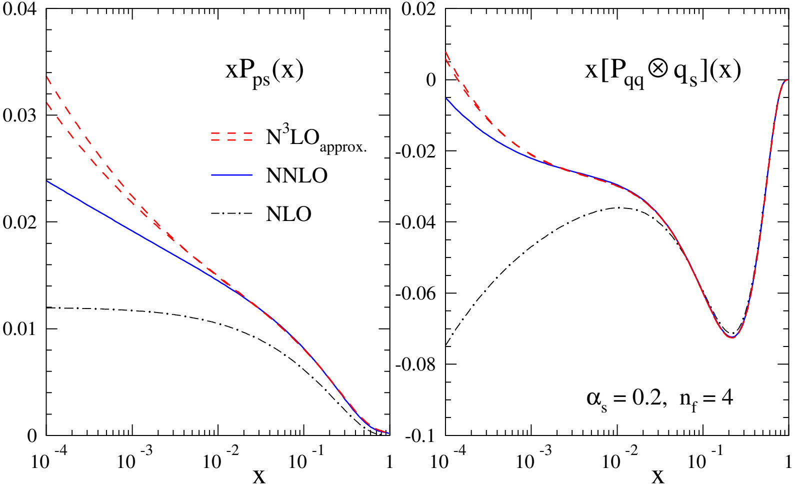
In the convolution $P(x \to 0)$ multiplies $f(x \to 1)$

$$P \otimes f = \int_x^1 \frac{dz}{z} P(z)\, f\left(\frac{x}{z}\right)$$

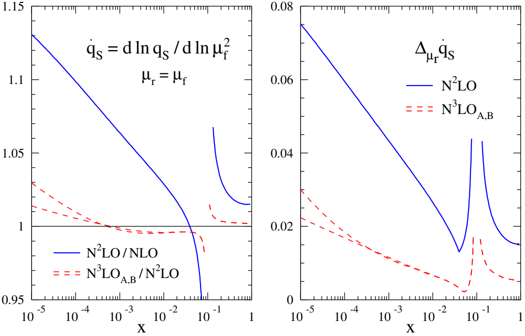
Large-x suppression of the PDFs: model PDFs

$$\begin{aligned} x\, g(x) &= 1.6\, x^{-0.3}\, (\mathbf{1}-\mathbf{x})^{\mathbf{4.5}}\, (1-0.6x^{0.3}), \\ x\, q_\mathrm{s}(x) &= 0.6\, x^{-0.3}\, (\mathbf{1}-\mathbf{x})^{\mathbf{3.5}}\, (1+5.0x^{0.8}) \end{aligned}$$

Example: $P_{\mathrm{qq}} \otimes q_{\mathrm{s}}$



Scale evolution of the quark distribution



Stability under variations of the renormalisation scale

$$\Delta_{\mu_r} \dot{q}_{\rm s} = \frac{1}{2} \frac{\max[\dot{q}_{\rm s}(\mu_r^2 = \lambda \mu_f^2)] - \min[\dot{q}_{\rm s}(\mu_r^2 = \lambda \mu_f^2)]}{\text{average}[\dot{q}_{\rm s}(\mu_r^2 = \lambda \mu_f^2)]}, \qquad \lambda = \frac{1}{4} \ldots 4$$

Towards QCD evolution to %-level

- The OPE approach allows us to compute efficiently moments up to $N = 20$ of $P_{\mathrm{qq}}(x)$ and $P_{\mathrm{qg}}(x)$.
- We constructed approximate N^{3}LO splitting functions
  - Uncertainties grow at small-x.
- Scale evolution of the **quark singlet** at high precision
  - N^{3}LO corrections below (2 ± 1)% up to $x = 10^{-5}$.
  - Improved scale stability $\lesssim 2\%$ up to $x = 10^{-5}$.

Outlook: P_{gq} New results!

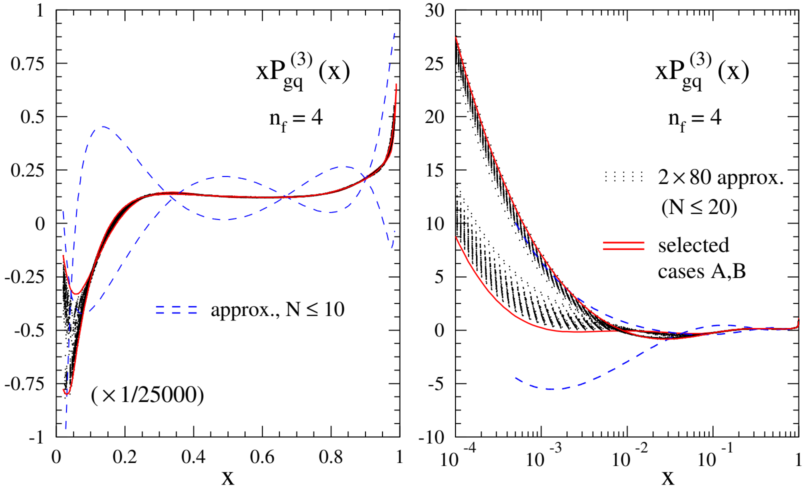
The large-n_f limit features simpler diagrams and fewer aliens

- Calculation of n_f^2 terms to $N = 60$, only class I aliens.
- Reconstruction of the complete N dependence.
  (**GF**,Herzog,Moch,Vermaseren,Vogt 2023)
- Beyond n_f^2 terms $\mathcal{O}^I$, $\mathcal{O}^{II}$ and $\mathcal{O}^{III}$ are relevant.
- Results for the moments up to $N = 20$ ← New!
  (**GF**,Herzog,Pelloni,Moch,Vogt 2024)

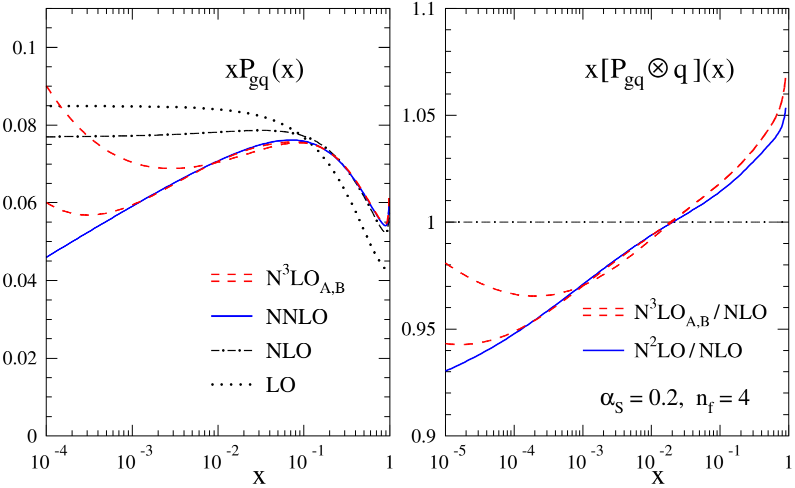
$$
\begin{aligned}
\gamma_{\mathrm{gq}}^{(3)}(N=2) &= -16663.2 + 4439.14 n_f - 202.555 n_f^2 - 6.37539 n_f^3, \\
\gamma_{\mathrm{gq}}^{(3)}(N=4) &= -6565.75 + 1291.01 n_f - 16.1462 n_f^2 - 0.839763 n_f^3, \\
&\cdots \\
\gamma_{\mathrm{gq}}^{(3)}(N=20) &= -1054.26 + 105.498 n_f + 2.39223 n_f^2 + 0.199385 n_f^3.
\end{aligned}
$$

- Checks: moments up to $N = 10$, large-n_f limit.

Approximate P_{gq} evolution: improving upon $N = 10$ New!



Approximate P_{gq} at N^{3}LO and convolutions New!



Thank you!

Quark singlet evolution: gluon-to-quark contribution