The η_c distribution amplitude from lattice QCD

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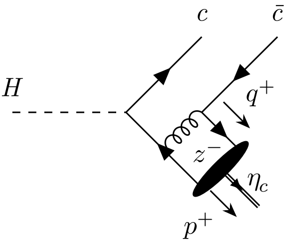
Distribution amplitudes

Matrix elements describing the hadronization of quarks or gluons



$\mathrm{I^G(J^{PC})} = 0^+(0^{-+})$

- Nonperturbative
- Light cone metric

parton-parton **distance** z^-

$$z^\mu z_\mu = 0 \quad z^\mu = (z^+, z^-, \bar{z})$$

Ioffe time $\nu \equiv pz = p^+ z^-$
quark **momentum fraction** $x = q^+/p^+$

For η_c $(A^+ = 0,\ z^2 \equiv z^\mu z_\mu)$

$$\underbrace{\phi(x)}_{z^2=0} = \int \frac{\mathrm{d}z^-}{2\pi} e^{\mathrm{i}x\nu} \underbrace{\langle \eta_c(p) | \bar{c}(-z/2)\gamma^+\gamma_5 c(z/2) | 0 \rangle \Big|_{z^+,\bar{z}=0}}_{M^+(\nu, z^2=0)}$$

Distribution amplitudes in Euclidean metric

Problem

We can only compute $z^\mu = (z_1, z_2, z_3, z_4) = (0, 0, 0, 0)$

Solution [3, 7, 10]

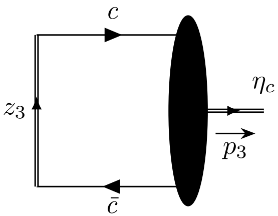
- Generalize $M^\mu(\nu, z^2)$ for $z^2 > 0$

$$M^\mu(p, z^2) = \left\langle \eta_c(p) \middle| \bar{c}(z)\gamma^\mu\gamma^5 W(z,0)c(0) \middle| 0 \right\rangle \Big|_{\bar{z}, z^4=0}$$

- Do a Lorentz decomposition

$$M^\mu(p, z^2) = 2p^\mu \mathcal{M}(p, z^2) + z^\mu \mathcal{M}'(p, z^2)$$

- Set $p^\mu = (0, 0, p^3, E)$
  $z^\mu = (0, 0, z^3, 0)$
- Choose $\mu = 4$ to isolate $\mathcal{M}(p, z^2)$



Distribution amplitudes in Euclidean metric

Form the renormalized quantity (shifted to $[z/2, -z/2]$) [1, 6, 8]

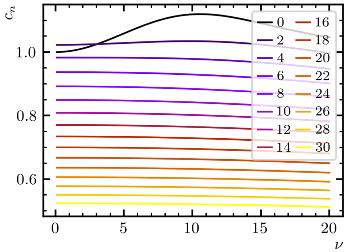
$$e^{-\mathrm{i}\nu/2} \frac{M^4(p, z^2) M^4(0,0)}{M^4(0, z^2) M^4(p, 0)} := \tilde{\phi}(\nu, z^2) + h.t.$$

Match to the $\overline{\mathrm{MS}}$ light-cone quantity at $\mu = 3\,\mathrm{GeV}$ [10]

$$\tilde{\phi}(\nu, z) = \int_0^1 \mathrm{d}w\, C(w, \nu, z\mu) \int_0^1 \mathrm{d}x \cos\left[w\nu\left(x - 1/2\right)\right] \phi(x, \mu)$$

Take the limit $z^2 \to 0$ + remove log divergences

$$c_n := \int_0^1 \mathrm{d}w\, C(w, \nu, z\mu) w^n$$



Parameterizing the DA

Expand the DA in a series of Gegenbauer polynomials [11]

$$\phi(x) = (1-x)^{\lambda-1/2} x^{\lambda-1/2} \sum_{n=0}^{\infty} d_{2n}^{(\lambda)} \tilde{G}_{2n}^{(\lambda)}(x)$$

The matching to the pseudo-DA in Ioffe time is [10]

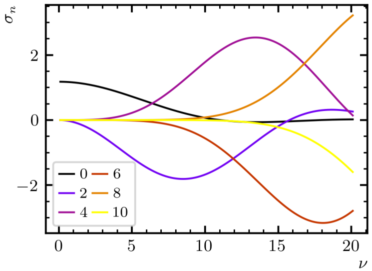
$$\tilde{\phi}(\nu, z) = \int_0^1 \mathrm{d}w\, C(w,\nu,z) \int_0^1 \mathrm{d}x \cos(w\nu x - w\nu/2)\,\phi(x)$$

It can be rewritten as

$$\tilde{\phi}(\nu, z) = \sum_{n=0}^{\infty} d_{2n}^{(\lambda)} \sigma_{2n}^{(\lambda)}(\nu, z),$$

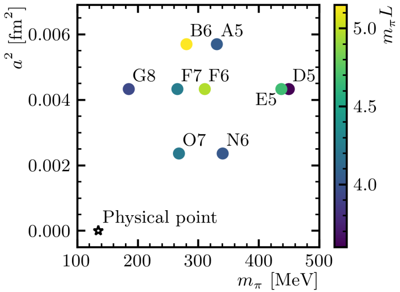
$$d_0^{(\lambda)} = \frac{4^\lambda}{B\left(\frac{1}{2}, \frac{1}{2}+\lambda\right)}$$

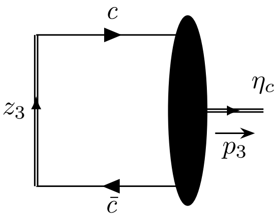
Energy scale $\mu = 3\,\mathrm{GeV}$



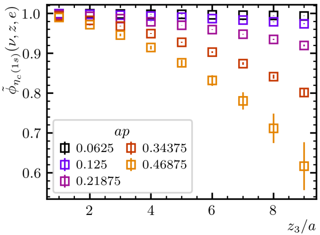
$N_f = 2$ Coordinated Lattice Simulations [2, 4]

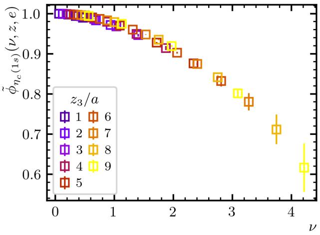
- $\mathcal{O}(a)$-improved Wilson quarks
- $\kappa_u = \kappa_d \coloneqq \kappa_\ell$
- No electromagnetism
- Wilson gauge action
- No Symanzik program for $M^\mu(p, z) \to \mathcal{O}(a)$ lattice artifacts
- Between 1000 and 2000 measurements per ensemble



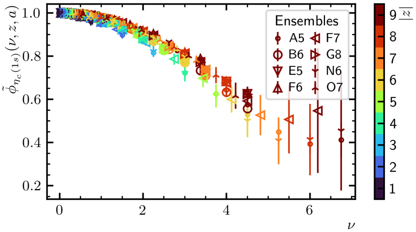


Ensemble O7





Entire dataset



Continuum extrapolation

Make all terms dimensionless with Λ_{QCD}

$$\tilde{\phi}(\nu, z, e) = \tilde{\phi}(\nu, z) + aB_1(\nu) + z^2 C_1(\nu) + \frac{a}{|z|}\left(A_1(\nu) + \left(m_{\eta_c} - m_{\eta_c,\text{phy}}\right) D_1(\nu) + \left(m_\pi^2 - m_{\pi,\text{phy}}^2\right) E_1(\nu)\right)$$

The main ingredients are the

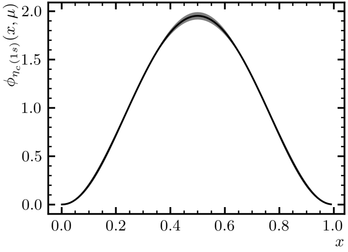
- **leading-twist** (continuum) $\tilde{\phi}(\nu, z)$

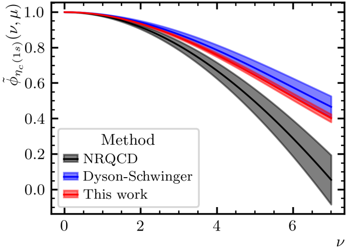
$$\tilde{\phi}(\nu, z) = \frac{4^\lambda \sigma_0^{(\lambda)}(\nu, z)}{B\left(\frac{1}{2}, \frac{1}{2} + \lambda\right)}$$

- **higher-twist** (continuum) C_1
- z-dependent A_1 and global B_1 **lattice artifacts**
- mass-dependent corrections D_1 and E_1

Results on the light cone

We obtain $\lambda = 2.73 \pm 0.12 \pm 0.12 \pm 0.06$





We compare to alternative determinations [5, 9]

	This work	Dyson-Schwinger	NRQCD
$\langle \xi^2 \rangle$	0.134(6)	0.118(18)	0.171(23)
$\langle \xi^4 \rangle$	0.043(4)	0.036(9)	0.018 808(19)

Conclusions and outlook

We compute the η_c DA with $N_f = 2$ CLS ensembles and obtain

$$\phi(x) = \frac{4^{\lambda}(1-x)^{\lambda-1/2}x^{\lambda-1/2}}{B\left(1/2, 1/2+\lambda\right)}$$

defined at $\mu = 3\,\text{GeV}$ and

$$\lambda = 2.73 \pm 0.12 \pm 0.12 \pm 0.06$$

In this analysis, we have seen that

- The comparison with Dyson-Schwinger is good
- Analysis choices yield sizable systematic uncertainties
- Finite-size effects are negligible

In the future, we will tackle

- Missing sea-quarks with $N_f = 2+1+1$ ensembles

Complete set of CLS ensembles

id	β	a [fm]	L/a	m_π [MeV]	κ_ℓ	κ_c
A5	5.2	0.0755(9)(7)	32	331	0.13594	0.12531
B6			48	281	0.13597	0.12529
D5	5.3	0.0658(7)(7)	24	450	0.13625	0.12724
E5			32	437	0.13625	0.12724
F6			48	311	0.13635	0.12713
F7			48	265	0.13638	0.12713
G8			64	185	0.136417	0.12710
N6	5.5	0.0486(4)(5)	48	340	0.13667	0.13026
O7			64	268	0.13671	0.13022

The matching kernel

Objective: Compute the matching integrals

$$\tilde{\phi}(\nu, z) = \int_0^1 \mathrm{d}w \, C(w, \nu, z\mu) \int_0^1 \mathrm{d}x \cos\left[w\nu\left(x - 1/2\right)\right] \phi(x, \mu)$$

Definitions: The DA matching kernel is [10]

$$C(w, \nu, z\mu) = \delta(w-1) - \frac{\alpha_s C_F}{2\pi}\left[\log\left(\frac{\mu^2}{\mu_0^2}\right) B(w, \nu) + L(w, \nu)\right]$$

where the scale μ_0 contains the z^2 dependence

$$\frac{1}{\mu_0^2} \equiv \frac{z^2 \; e^{2\gamma_E + 1}}{4}$$

we take $\mu = 3\,\mathrm{GeV}$

The matching kernel

The contribution $B(w,\nu)$ is [10]

$$B(w,\nu) = \left[\frac{2w}{1-w}\right]_+ \cos\left(\frac{(1-w)\nu}{2}\right) + \frac{2}{\nu}\sin\left(\frac{(1-w)\nu}{2}\right) - \frac{1}{2}\delta(w-1)$$

And the contribution $L(w,\nu)$ is [10]

$$\begin{aligned} L(w,\nu) =& 4\left[\frac{\log(1-w)}{1-w}\right]_+ \cos\left(\frac{(1-w)\nu}{2}\right) \\ & - 2\left(\frac{2}{\nu}\sin\left(\frac{(1-w)\nu}{2}\right) - \frac{1}{2}\delta(w-1)\right) \end{aligned}$$

Given two functions $f(x)$ and $g(x)$ defined in a certain domain, **the plus prescription** is

$$\left[\frac{f(x)}{1-x}\right]_+ g(x) = \frac{f(x)}{1-x}(g(x) - g(1))$$

The matching kernel

Method: Rewrite the relation between $\tilde{\phi}(\nu, z)$ and $\phi(x, \mu)$

$$\tilde{\phi}(\nu, z) = \int_0^1 \mathrm{d}x \, K(x, \nu, \mu z)\phi(x, \mu)$$

write the kernel as a series of Gegenbauer polynomials

$$K(x, \nu, \mu z) = \sum_{n=0}^{\infty} \frac{\sigma_{2n}^{(\lambda)}(\nu, z\mu)}{A_{2n}^{(\lambda)}} \tilde{G}_{2n}^{(\lambda)}(x)$$

and every coefficient in the series is given by

$$\sigma_n^{(\lambda)}(\nu, z\mu) = \sum_{k=0}^{\infty} \left(-\frac{\nu^2}{4}\right)^k \frac{c_{2k}(\nu, z\mu)}{\Gamma(2k+1)} I(n, k, \lambda)$$

See [11] for a similar analysis of PDFs

The matching kernel

The λ-dependent function is the Mellin transform of the Gegenbauer polynomials

$$I(n,k,\lambda) \equiv \int_{-1}^{+1} \mathrm{d}g\, g^{2k}(1-g^2)^{\lambda-1/2} G_n^{(\lambda)}(g)$$
$$= \frac{2\pi}{4^{\lambda+k} n!} \frac{\Gamma(1+2k)\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma\left(\lambda+\frac{n+2k+2}{2}\right)\Gamma\left(1+k-\frac{n}{2}\right)}$$

The n-th moment of the kernel is given by

$$c_n(\nu, z\mu) = \int_0^1 \mathrm{d}w\, C(w,\nu,z\mu) w^n$$
$$= 1 - \frac{\alpha_s C_F}{2\pi}\left[\log\left(\frac{\mu^2}{\mu_0^2}\right) b_n(\nu) + l_n(\nu)\right]$$

The matching kernel

$$I(0,k,\lambda) = B(\lambda + \frac{1}{2}, k + \frac{1}{2})$$

$$I(2,k,\lambda) = 2\lambda k B(\lambda + \frac{3}{2}, k + \frac{1}{2})$$

$$I(4,k,\lambda) = \frac{2}{3}(\lambda+1)\lambda k(k-1)B(\lambda + \frac{5}{2}, k + \frac{1}{2})$$

$$I(6,k,\lambda) = \frac{4}{45}(2+\lambda)(1+\lambda)\lambda k(k-1)(k-2)B(\lambda + \frac{7}{2}, k + \frac{1}{2})$$

$$I(8,k,\lambda) = \frac{2}{315}(3+\lambda)(2+\lambda)(1+\lambda)\lambda k(k-1)(k-2)(k-3) B(\lambda + \frac{9}{2}, k + \frac{1}{2})$$

The matching kernel

The moments of $B(w)$ are given by

$$b_n(\nu) = -\sum_{j=0}^{n-1} \frac{2}{j+2} {}_1F_2\left(1, \frac{j+3}{2}, \frac{j+4}{2}, -\frac{\nu^2}{16}\right) - \frac{\nu^2}{24} {}_2F_3\left(1, 1, 2, 2, 5/2, -\frac{\nu^2}{16}\right) - \frac{1}{2} + \frac{1}{(n+2)(n+1)} {}_1F_2\left(1, \frac{n+3}{2}, \frac{n+4}{2}, -\frac{\nu^2}{16}\right)$$

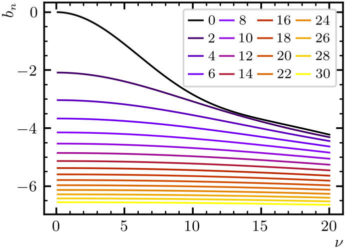
Note all hypergeometric functions ${}_pF_q$ have $p \leq q \leftrightarrow$ Converge for all ν values [12]

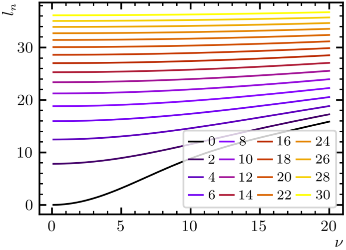
The moments of $L(w)$ are given by

$$
\begin{aligned}
l_n(\nu) = \ & 4\sum_{j=0}^{n-1} \binom{n}{j+1} \frac{(-1)^j}{(j+1)^2} {}_2F_3\left(\frac{j+1}{2}, \frac{j+1}{2}, \frac{1}{2}, \frac{j+3}{2}, \frac{j+3}{2}, -\frac{\nu^2}{16}\right) \\
& + \frac{\nu^2}{8} {}_3F_4\left(1, 1, 1, \frac{3}{2}, 2, 2, 2, -\frac{\nu^2}{16}\right) \\
& + 1 - \frac{2}{(n+2)(n+1)} {}_1F_2\left(1, \frac{n+3}{2}, \frac{n+4}{2}, -\frac{\nu^2}{16}\right)
\end{aligned}
$$

Note all hypergeometric functions ${}_pF_q$ have $p \leq q \leftrightarrow$ Converge for all ν values [12]

The matching kernel





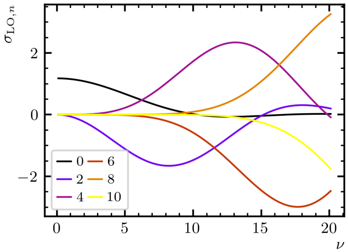
Nuisance functions

The nuisance functions are parametrized just like $\phi(x, \mu)$

$$A_r^{(\lambda)}(x) = (1-x)^{\lambda-1/2} x^{\lambda-1/2} \sum_{s=0}^{S_{a,r}} a_{r,2s}^{(\lambda)} \tilde{G}_{2s}^{(\lambda)}(x)$$

Fourier transform to ν space,

$$A_r^{(\lambda)}(\nu) = \int_0^1 \mathrm{d}x \, A_r^{(\lambda)}(x) \cos(x\nu - \nu/2) = \sum_{s=0}^{S_{A,r}} a_{r,2s}^{(\lambda)} \sigma_{\mathrm{LO},2s}^{(\lambda)}(\nu)$$



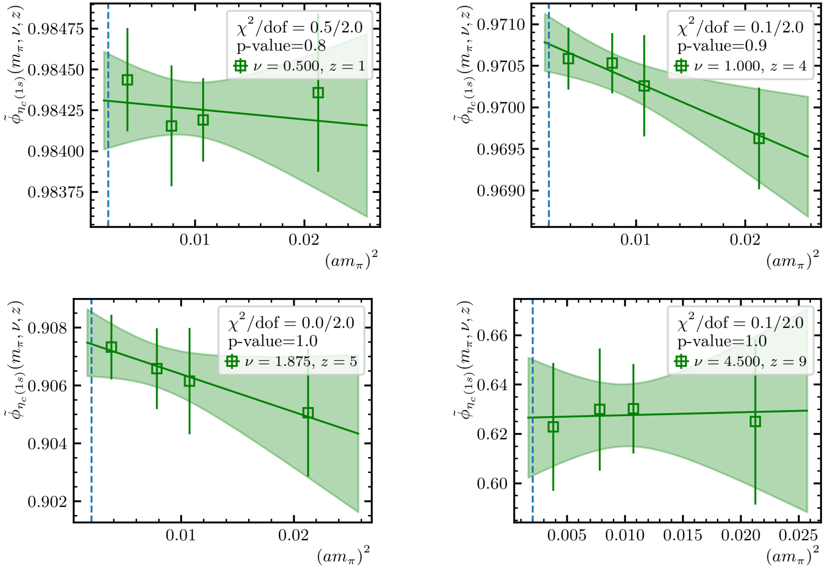
Nuisance effects vanish at $\nu = 0$

$$a_{r,0}^{(\lambda)} = 0 \longleftrightarrow \tilde{\phi}(\nu = 0, z) = 1$$

Enough to consider $S_{A_r} = 1$

$a_{1,2}$, $b_{1,2}$, $c_{1,2}$, $d_{1,2}$, $e_{1,2}$

Pion mass dependence



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