# Pole Decomposition of BFKL eigenvalue

Alex Prygarin
Ariel University

DIS2024

April 9, 2024

# BFKL equation

The Balitsky-Fadin-Kuraev-Lipatov(BFKL '78) equation can be written as a stationary Schrodinger equation :

$$H\Psi_{\nu n} = E_{\nu n}\Psi_{\nu n}, \quad \Psi_{\nu n} \sim k^{i\nu+\frac{n}{2}} \ \bar{k}^{i\nu-\frac{n}{2}} \tag{1}$$

Leading order BFKL eigenvalue $\omega_0 = aE^{LO}_{\nu n}$, $\quad a = \frac{\alpha_s N_C}{2\pi}$

$$E^{LO}_{\nu n} = \psi\left(\frac{1}{2} + i\nu + \frac{|n|}{2}\right) + \psi\left(\frac{1}{2} - i\nu + \frac{|n|}{2}\right) - 2\psi(1) \tag{2}$$

In terms of digamma function:

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz} \tag{3}$$

In terms of harmonic sums (requires analytic continuation):

$$E^{LO}_{\nu n} = S_1\left(-\frac{1}{2} + i\nu + \frac{|n|}{2}\right) + S_1\left(-\frac{1}{2} - i\nu + \frac{|n|}{2}\right) \tag{4}$$

$$S_1(k) = \sum_{j=1}^{k} \frac{1}{k} \tag{5}$$

# NLO BFKL eigenvalue

Fadin and Lipatov ('98), Balitsky and Chirilli ('07)
NLO BFKL eigenvalue has a general structure

$$E_{\nu n}^{NLO} = [f_0(z) + f_0(\bar{z})]\,[f_1(z) + f_1(\bar{z})] + f_2(z) + f_2(\bar{z}) \qquad (6)$$

has no holomorphic separability in $z = -\frac{1}{2} + i\nu + \frac{n}{2}$ present at LO.

$f_0(z)$ has $\Psi(z+1)$ alternatively $S_1(z) = \sum_{k=1}^{z} \frac{1}{k}$

$f_1(z)$ has $\beta'(z+1) = -\sum_{j=0}^{\infty} \frac{(-1)^j}{(z+2)^2}$ alternatively $S_{-2}(z) = \sum_{j=0}^{z} \frac{(-1)^k}{k^2}$

$f_2(z)$ has $\sum_{k=0}^{\infty} \frac{\beta'(k+1) + (-1)^k \psi'(k+1)}{k+z+1} - \sum_{k=0}^{\infty} \frac{(-1)^k (\psi(k+1) - \psi(1))}{(k+z+1)^2}$

alternatively through $S_{-2,1}(z) = \sum_{k=1}^{z} \frac{(-1)^k}{k^2} \sum_{m=1}^{k} \frac{1}{m}$

# Bethe-Salpeter form of BFKL

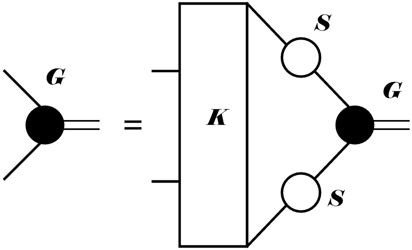
NLO BFKL eigenvalue has a general structure

$$E_{\nu n}^{NLO} \sim [f_0(z) + f_0(\bar{z})]\,[f_1(z) + f_1(\bar{z})] + f_2(z) + f_2(\bar{z}) \qquad (7)$$

has no holomorphic separability present at LO. This non-linearity suggests a different approach:

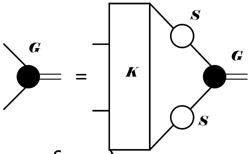
Bethe-Salpeter form of BFKL equation (Joubat, A.P. '20) can be schematically represented as follows

$$G = K \otimes S \otimes G \otimes S \qquad (8)$$



# BFKL eigenvalue and Bethe-Salpeter form of BFKL

$$G = K \otimes S \otimes G \otimes S \quad \Rightarrow$$




Regge pole decomposition (Bethe-Salpeter form)

$$1 = \frac{a}{\omega} \sum_{i=0}^{\infty} \omega^i \sum_{k=0}^{\infty} a^k f_{i,k}, \quad a = \frac{\alpha_s N_C}{2\pi} \tag{9}$$

each $f_{i,k}$ has holomorphic separability
Expand this to next-to-next-to-leading (NNLO) order

$$1 = \frac{a(f_{0,0} + af_{0,1} + a^2 f_{0,2})}{\omega} + a(f_{1,0} + af_{1,1}) + a\omega f_{2,0} \tag{10}$$

Solving it we get LO eigenvalue $\omega_0 = af_{0,0}$
NLO eigenvalue $\omega_1 = a^2 \left(f_{0,0} f_{1,0} + f_{0,1}\right)$
NNLO eigenvalue $\omega_2 = a^3 \left(f_{0,0}^2 f_{2,0} + f_{0,0} f_{1,1} + f_{0,0} f_{1,0}^2 + f_{1,0} f_{0,1} + f_{0,2}\right)$

- The BFKL eigenvalue is a function of the complex variable :

$$z = -\frac{1}{2} + i\nu + \frac{|n|}{2} \tag{11}$$

- $n \in \mathbb{Z}$ and called conformal spin,$\nu \in \mathbb{R}$ related to anomalous dimension.

For $n = 0$

$$\bar{z} = -1 - z \tag{12}$$

NLO BFKL eigenvalue NLO BFKL eigenvalue has a general structure

$$E_{\nu n}^{NLO} \sim [f_0(z) + f_0(\bar{z})]\,[f_1(z) + f_1(\bar{z})] + f_2(z) + f_2(\bar{z}) \tag{13}$$

can be written in separable form (Costa, Goncalves, Penedones '18)

$$F(z) + F(-1 - z). \tag{14}$$

This can de done in a systematic way using reflection identities (A.P. '19)

$$S_{\{a\}}(z)S_{\{b\}}(-1 - z) = S_{\{c\}}(z) + ... + S_{\{d\}}(-1 - z) + ... \tag{15}$$

# Reflection identities for Harmonic Sums

$$S_{\{a\}}(z)S_{\{b\}}(-1-z) = S_{\{c\}}(z) + \ldots + S_{\{d\}}(-1-z) + \ldots \quad (16)$$

Example, the simplest reflection identity

$$S_1(z)S_1(-1-z) = S_{1,1}(z) + S_{1,1}(-1-z) + \frac{\pi^2}{3}. \quad (17)$$

where (requires analytic continuation)

$$S_1(z) = \sum_{k=1}^{z} \frac{1}{k}, \quad S_{1,1}(z) = \sum_{k=1}^{z} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} = \frac{1}{2}\left(S_1(z)\right)^2 + \frac{1}{2} S_2(z) \quad (18)$$

This identity is well-known in the form

$$(\psi(1+z) - \psi(1))(\psi(-z) - \psi(1)) = \frac{1}{2}(\psi(1+z) - \psi(1))^2$$
$$-\frac{1}{2}\psi'(1+z) + \frac{1}{2}(\psi(-z) - \psi(1))^2 - \frac{1}{2}\psi'(-z) + \frac{\pi^2}{2} \quad (19)$$

only 2 reflection identities were known.

# Reflection identities for Harmonic Sums

$$S_{\{a\}}(z)S_{\{b\}}(-1-z) = S_{\{c\}}(z) + ... + S_{\{d\}}(-1-z) + ... \qquad (20)$$

only 2 reflection identities were known (discovered a century ago).
NNLO BFKL eigenvalue requires to know all identities up to weight 5.
All of them (more than 300) were calculated (Joubat, A.P. '20) using pole expansion around negative integers.
Example for weight 5

$$\begin{aligned} s_1\bar{s}_{2,2} = &-\zeta_2\bar{s}_{2,1} - \bar{s}_{3,2} + \bar{s}_{1,2,2} + \bar{s}_{2,1,2} - \frac{7}{10}\zeta_2^2\bar{s}_1 + 2\zeta_3\bar{s}_2 - \zeta_2 s_{2,1} \\ &- s_{4,1} + s_{2,2,1} + 3\zeta_3\zeta_2 - \frac{5\zeta_5}{2} + \frac{7}{10}\zeta_2^2 s_1 + \zeta_2 s_3 - \zeta_3 s_2 \end{aligned} \qquad (21)$$

where $s_{a,...} \equiv S_{a,...}(z)$, $\bar{s}_{a,...} \equiv S_{a,...}(\bar{z})$ and $\zeta_n$ is the Riemann zeta function $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$

# Summary

- Bethe-Salpeter form of BFKL equation seems to be more natural than the traditional Schrodinger form
- Restored holomorphic separability in Bethe-Salpeter approach to the BFKL equation is very useful for calculations
- Harmonic sums form a convenient formalism for analytic calculations due to plenty of their functional identities (shuffle, reflection etc.)
- Systematic labelling of harmonic sums and transcendental constants allows to apply algebraic approach, similar to Quantum Harmonic Oscillator, where the eigenstates were conveniently labelled.

Thank you for your attention