(Towards) Reconciling the kinematical constraint with the JIMWLK evolution equation

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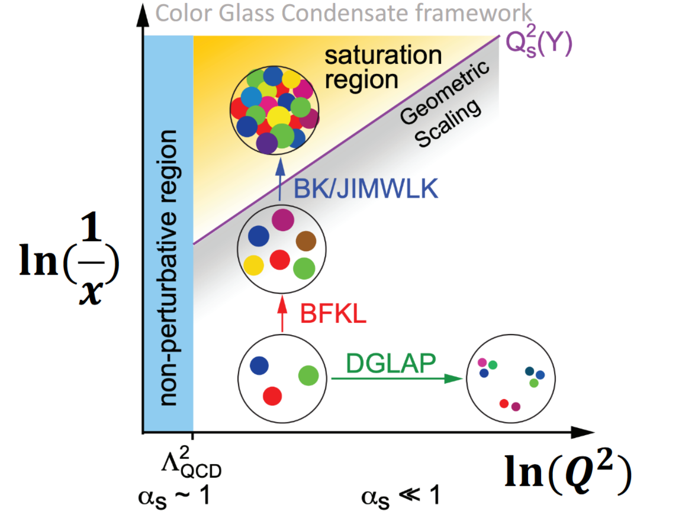
DIS2024, Grenoble, 8-13 April 2024

NATIONAL SCIENCE CENTRE POLAND This work is supported by NCN grant nr 2022/46/E/ST2/00346.

Evolution equations

ln Q^2 and ln $1/x$ evolution



Evolution equations

Color Glass Condensate framework

- effective description valid in the saturation regime, where dense and slow gluons (target) are described by classical fields traversed by a fast and energetic probe (projectile),

  [review by Gelis, Iancu, Jalilian-Marian, Venugopalan '10]

- basic degrees of freedom:
  - Wilson lines

$$U(\vec{x})$$

  - dipole correlation function

$$S(\vec{r}) = \left\langle \mathrm{tr}\left[U^\dagger(\vec{x})U(\vec{x}+\vec{r})\right]\right\rangle_{\vec{x}}.$$

- for forward and nearly back-to-back jets, one can apply both the TMD factorization and Color Glass Condensate (CGC) approaches to compute the di-jet cross-section

  [Marquet, Petreska, Roiesnel '16, Caucal, Salazar, Schenke, Stebel, Venugopalan '23]

Evolution equations

Evolution

- assuming a given distribution predict the distribution at larger Q^2
  - DGLAP equation
- assuming a given distribution predict the distribution at small x
  - BFKL (linear) equation
  - JIMWLK (non-linear) equation
  - BK (non-linear at leading color factor N) equation

Precision

- LO: fixed coupling constant, tree-level splitting and recombination amplitudes
- NLO: running coupling constant, NLO splitting and recombination amplitudes
- resummation: LO + all-order resummation of a particular class of contributions
  - kinematical constraint: resummation of contributions with $(\alpha_s \ln x)$

Kinematical constraint

BK with collinear improvement

- ordering of dipole lifetimes/sizes
- natural in the language of dipoles
- worked out and implemented for the BFKL and BK equations

[Motyka, Staśto '09]

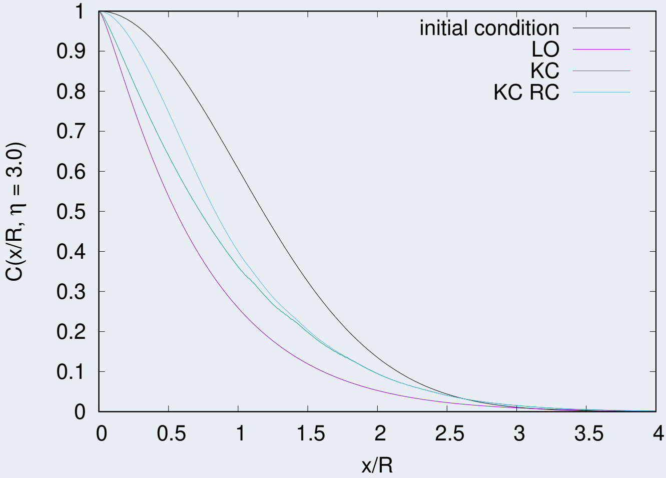
- evolution equation in the target rapidity η

[Ducloué, Iancu, Soyez, Triantafyllopoulos '19]

$$\frac{\partial \bar{S}_{r=|x-y|}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int d^2z \frac{(x-y)^2}{(x-z)^2(z-y)^2} \theta\left(\eta - \delta_{xyz}\right) \times$$
$$\times \left[\bar{S}_{xz}(\eta - \delta_{xz,r})\bar{S}_{zy}(\eta - \delta_{zy,r}) - \bar{S}_{xy}(\eta)\right]$$

- rapidity shifts $\delta_{xz,r} = \max\{0, \ln \frac{r^2}{|x-z|^2}\}$
- $\delta_{xyz} = \max\{\delta_{xz,r}, \delta_{zy,r}\}$

BK with collinear improvement



JIMWLK evolution equation

Beyond the leading N order

- JIMWLK equation describes the non-linear small-x evolution
- it uses Wilson lines as fundamental degrees of freedom
- two-point correlation function $\langle U^\dagger(x) U(y) \rangle$ gives the dipole amplitude
- two-point correlation functions with derivatives provide a basis for small-x TMD structure functions

LO JIMWLK: Langevin formulation

$$U(\mathrm{x}, s+\delta s) = \exp\left(-\sqrt{\delta s} \sum_{\mathrm{y}} U(\mathrm{y}, s) (\mathrm{K}(\mathrm{x}-\mathrm{y}) \cdot \xi(\mathrm{y})) U^\dagger(\mathrm{y}, s) \right) \times$$
$$\times\, U(\mathrm{x}, s) \times \exp\left(\sqrt{\delta s} \sum_{\mathrm{y}} \mathrm{K}(\mathrm{x}-\mathrm{y}) \cdot \xi(\mathrm{y}) \right).$$

[Rummukainen, Weigert '04, Lappi, Mantysaari '14]

Saturation scale evolution speed

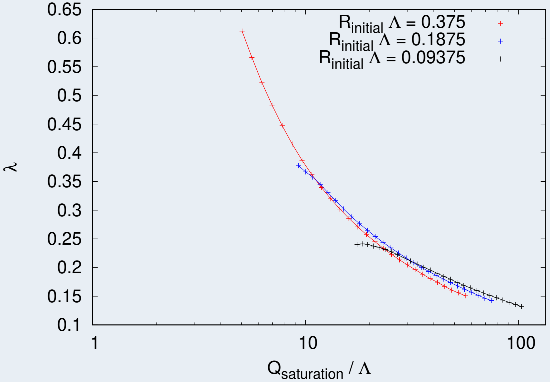


Figure: $R_{\text{initial}}\Lambda$ is the only parameter of the initial condition and of the evolution. Coinciding data from evolution for different values of $R_{\text{initial}}\Lambda$ corresponds to geometrical scaling.

JIMWLK in η with collinear improvement

Collinear improvement

All order resummation of corrections enhanced by kinematical constraints. Known from BFKL studies to be important to correctly describe phenomenology.
We build upon the proposal [Hatta, Iancu '16].

Proposal

$$
\begin{aligned}
&U(\mathrm{x}, R, \eta+\delta\varepsilon) = \\
&\exp\left(-\sqrt{\delta\varepsilon}\sum_{\mathrm{y}}\sqrt{\alpha_s}\,\theta(s-P^R_{\mathrm{xy}})\,U(\mathrm{y},\hat{R},s-\Delta^R_{xy})\left[\mathrm{K}_{\mathrm{xy}}\cdot\xi(\mathrm{y})\right]U^\dagger(\mathrm{y},\hat{R},s-\Delta^R_{xy})\right) \\
&\qquad\qquad\times U(\mathrm{x},R,s)\times \\
&\qquad\qquad\qquad \exp\left(\sqrt{\delta\varepsilon}\sum_{\mathrm{y}}\sqrt{\alpha_s}\,\theta(s-P^R_{\mathrm{xy}})\,\mathrm{K}_{\mathrm{xy}}\cdot\xi(\mathrm{y})\right),
\end{aligned}
$$

$$
P^R_{\mathrm{xy}} = \ln\frac{R^2}{(\mathrm{x}-\mathrm{y})^2},\ \Delta^R_{xy} = \theta\left(|x-y|-R\right)\rho^R_{\mathrm{xy}},\ \hat{R} = \max(|x-y|, R),\ s = \varepsilon\alpha_s.
$$

JIMWLK in η with collinear improvement

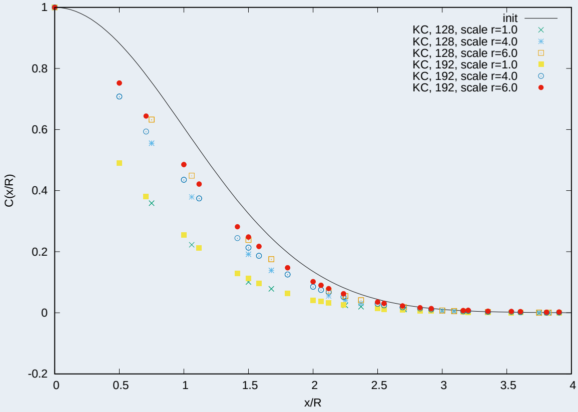


Figure: Preliminary results for the dipole amplitude with KC JIMWLK evolution equation at $\eta = 3.0$.

JIMWLK in η with collinear improvement

Reduction to the BK equation in η

In order to establish the dependence on η we expand $S(\mathrm{x},\mathrm{y}=\mathrm{x}+r,\eta+\varepsilon)$ in ε,

$$S(\mathrm{x},\mathrm{y}=\mathrm{x}+r,\eta+\varepsilon)=\frac{1}{N_c}\langle \mathrm{tr}\,U^{\dagger}(\mathrm{x},r,\eta+\varepsilon)U(\mathrm{x}+r,r,\eta+\varepsilon)\rangle.$$

All the terms yield:

$$\frac{\partial S(\mathrm{x},\mathrm{y},\eta)}{\partial\eta}=\frac{\bar{\alpha}_s}{2\pi}\int_{\mathrm{z}} S(\mathrm{x},\mathrm{y},\eta)$$

$$\Big\{-\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{yz}K^i_{yz}-\theta(n\varepsilon-\delta^r_{r_{xz}})K^i_{xz}K^i_{xz}+\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}\Big\}+$$

$$+\Big\{\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{yz}K^i_{yz}S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{yz},\delta_{yz},\eta)+$$

$$+\theta(n\varepsilon-\delta^r_{r_{xz}})K^i_{xz}K^i_{xz}S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{xz},\eta)+$$

$$-\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{yz},\delta_{yz},\eta)+$$

$$-\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{xz},\eta)\Big\}+$$

$$+\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}S_6(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{yz},\eta)$$

JIMWLK in η with collinear improvement

Recovering KC BK equation in η

$$S_6(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{yz},\eta) =$$
$$= \frac{1}{N_c^2}\mathrm{tr}\left[U_{n\varepsilon-\delta^r_{r_{xz}}}(\mathrm{z},r)U^\dagger_{n\varepsilon}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{y},r)U^\dagger_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)\right]\times$$
$$\times\,\mathrm{tr}\left[U^\dagger_{n\varepsilon-\delta^r_{r_{xz}}}(\mathrm{z},r)U_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)\right] =$$
$$= \frac{1}{N_c}\mathrm{tr}\left[U^\dagger_{n\varepsilon}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{y},r)\right] = S(\mathrm{x},\mathrm{y},\eta)$$

Assuming that $\delta_{xz} = \delta_{yz} = \delta$ we have and setting

$$S(\mathrm{x},\mathrm{y},\eta,\eta-\delta_{xz}) \equiv S(\mathrm{x},\mathrm{y},\eta-\delta_{xz})$$

in that case the final results reduces to

$$\frac{\partial S(\mathrm{x},\mathrm{y},\eta)}{\partial\eta} = \frac{\bar{\alpha}_s}{2\pi}\int_{\mathrm{z}}\mathscr{K}_{xyz}\theta(n\varepsilon-\delta)\left\{S(\mathrm{x},\mathrm{z},\eta-\delta)S(\mathrm{z},\mathrm{y},\eta-\delta)-S(\mathrm{x},\mathrm{y},\eta)\right\}$$

Recovering KC BK equation in η

In order to diagnoze the dynamics we investigate new correlation functions. The simplest is the correlation in η

$$C(\eta) = \frac{1}{VN_c}\langle \mathrm{tr}\; U^\dagger(x,0)U(x,\eta)\rangle_x,$$

$$C(r,\eta) = \frac{1}{VN_c}\langle \mathrm{tr}\; U^\dagger(x,r,0)U(x,r,\eta)\rangle_x.$$

and even more generally

$$W(x,y,\eta) = \frac{1}{N_c}\langle \mathrm{tr}\; U^\dagger(x,0)U(y,\eta)\rangle,$$

$$W(x,y,r,\eta) = \frac{1}{N_c}\langle \mathrm{tr}\; U^\dagger(x,r,0)U(y,r,\eta)\rangle.$$

JIMWLK in η with collinear improvement

BK-like equations for C, W and S

$$\frac{\partial W_{x,y}(\eta)}{\partial\eta} = \frac{\bar{\alpha}_s}{2\pi}\int_z \mathscr{K}_{xz}\Big(S_{x,z}(\eta)W_{z,y}(\eta) - W_{x,y}(\eta)\Big)$$

$$\frac{\partial C_x(\eta)}{\partial\eta} = \frac{\bar{\alpha}_s}{2\pi}\int_z \mathscr{K}_{xz}\Big(S_{x,z}(\eta)W_{z,x}(\eta) - C_x(\eta)\Big)$$

$$\frac{\partial S_{x,y}(\eta))}{\partial\eta} = \frac{\bar{\alpha}_s}{2\pi}\int_z \mathscr{M}_{xyz}\Big(S_{x,z}(\eta)S_{z,y}(\eta) - S_{x,y}(\eta)\Big)$$

Initial slope of C can be estimated analytically

$$\left.\frac{\partial C_x(\eta)}{\partial\eta}\right|_{\eta=0} = \frac{\bar{\alpha}_s}{2\pi}\int_z \mathscr{K}_{xz}\Big(S_{x,z}(0)S_{z,x}(0) - 1)\Big)$$

since $C(0) = 1$ and we can take $S_{x,z}(0) = \exp(-(|x-z|^2)/2R^2)$.

JIMWLK in η with collinear improvement

BK-like equations for C, W and S

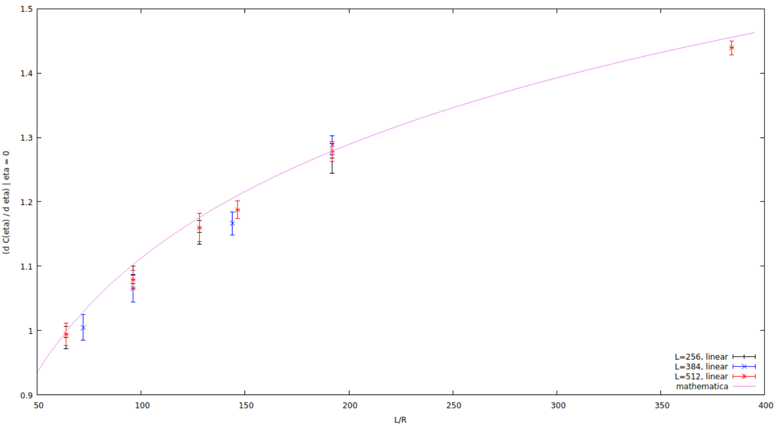


Figure: Comparison of the initial slope of $C(\eta)$ with the semi-analytic calculation in the continuum.

JIMWLK in η with collinear improvement

BK-like equations for C, W and S

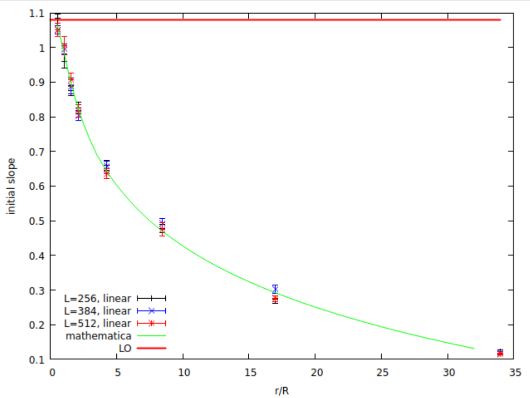


Figure: Comparison of the initial slope of $C(\eta)$ with the semi-analytic calculation in the continuum.

JIMWLK in η with collinear improvement

C as a function of η

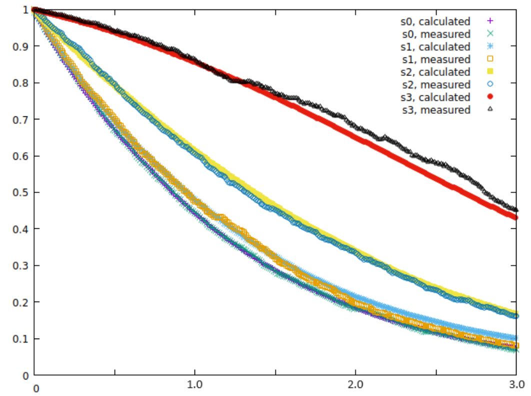


Figure: Comparison of $C(\eta)$ calculated using JIMWLK and BK-like equations

JIMWLK in η with collinear improvement

Regularize the divergence with a gluon mass

[Gardi, Kuokkanen, Rummukainen, Weigert '07].

The gluon mass modifies the elementary kernel

$$K^i_{xz} = \frac{(\mathrm{x}-\mathrm{z})^i}{|\mathrm{x}-\mathrm{z}|^2} \to \frac{(\mathrm{x}-\mathrm{z})^i}{|\mathrm{x}-\mathrm{z}|^2} e^{-m|\mathrm{x}-\mathrm{z}|}$$

Then

$$\mathscr{K}_{xz} = K^i_{xz} K^i_{xz} \to \frac{1}{|\mathrm{x}-\mathrm{z}|^2} e^{-2m|\mathrm{x}-\mathrm{z}|}$$

and

$$\mathscr{M}_{xyz} = K^i_{xz}K^i_{xz} + K^i_{yz}K^i_{yz} - 2K^i_{xz}K^i_{yz} = \frac{\left((y-z)e^{-m|\mathrm{x}-\mathrm{z}|} - (x+z)e^{-m|\mathrm{y}-\mathrm{z}|}\right)^2}{(x-z)^2(y-z)^2}$$

JIMWLK in η with collinear improvement

C as a function of η

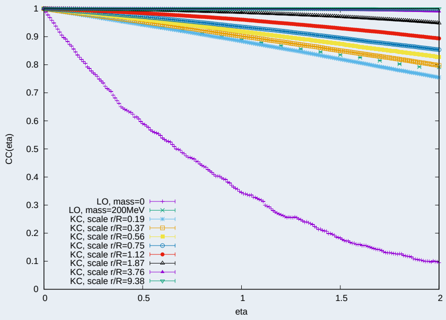


Figure: $C(\eta)$ calculated using JIMWLK with gluon mass of 200 MeV

JIMWLK in η with collinear improvement

S as a function of η

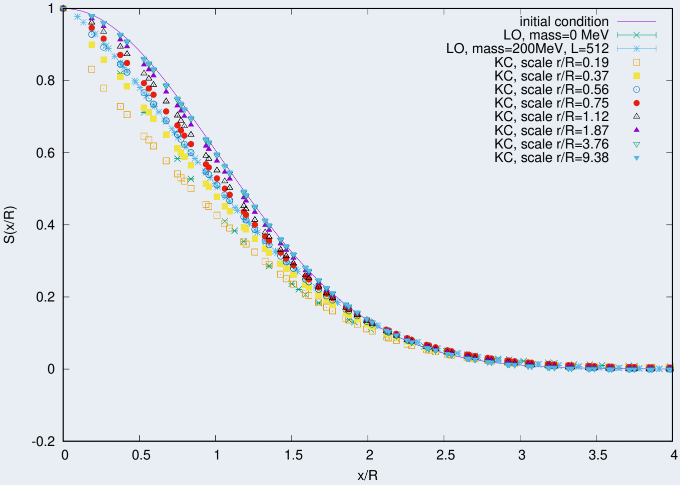


Figure: $S(\eta)$ calculated using JIMWLK with gluon mass of 200 MeV

Conclusions

Summary

- we have identified the origin of the instability of the numerical setup
- We have regularized it by introducing a gluon mass
- however, it seems that there remains some problem
- still more work is needed.