Quark TMDs at Small-x

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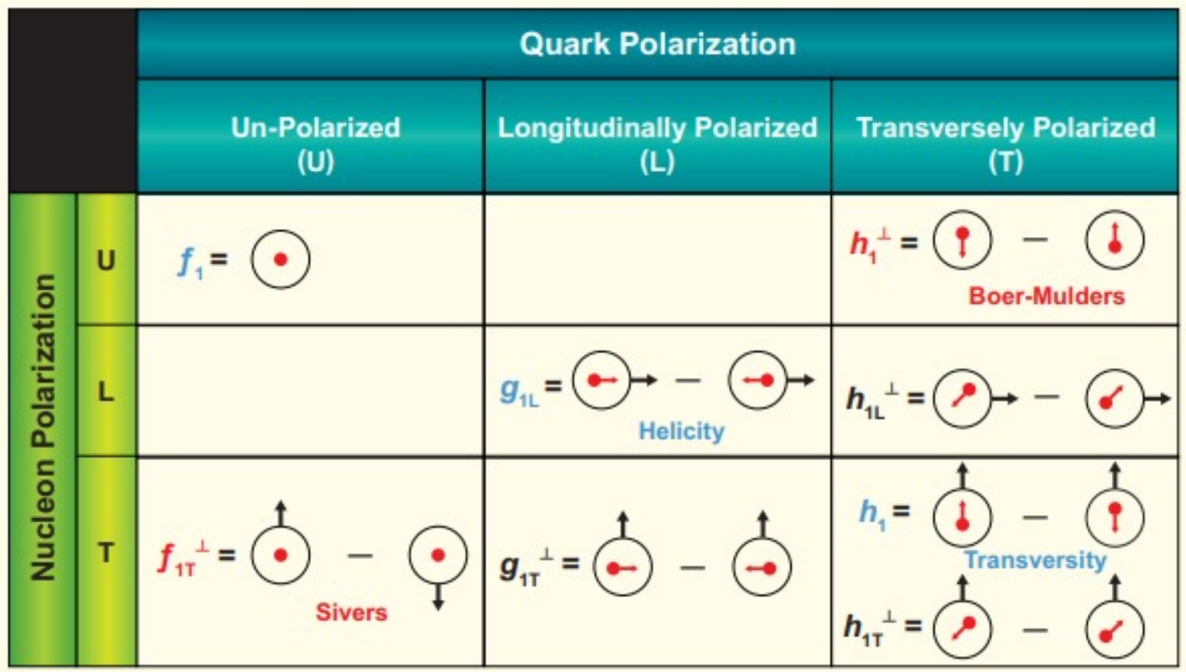
Based on work with Yuri V. Kovchegov, Daniel Adamiak, and Yossathorn Tawabutr

Outline

- TMD intro
- Light Cone Operator Treatment
  - From operator definitions to polarized dipole amplitudes
  - Sub-eikonal and sub-sub-eikonal operators
- Evolution and double logs
- Results for asymptotic scaling in BFKL regime

TMDs



- The leading-twist quark TMDs give various correlations between the transverse momentum and polarizations of the quarks within a hadron with the polarization of the parent hadron
- Their scale evolution in Q^2 is given by the CSS equations, but the small- x evolution is an ongoing effort

LCOT for TMDs

- Simplify
  - Rewrite operator definition in small- x limit using shockwave formalism
  - Expand to a given order in eikonality
  - Obtain expression for TMD in terms of 'polarized dipole amplitudes'
- Evolve
  - Calculate small- x gluon/quark emissions in dipole amplitude
  - Take (for example) large- N_c limit to obtain closed equations
- Solve
  - Solve integral equations analytically (if possible) or numerically
  - Plug evolved dipole amplitude back into TMD definition

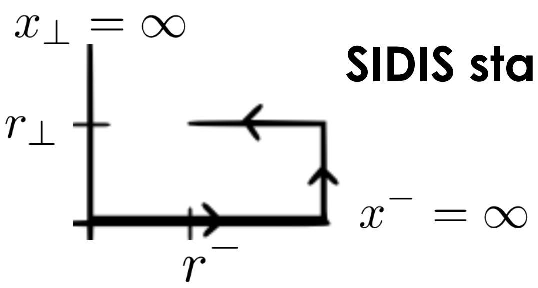
'Staple' Wilson Line becomes a dipole amplitude!

TMDs

- Quark TMDs are defined by the non-local operator product in the hadron state

$$\Phi^{[\Gamma]} = \int \frac{\mathrm{d}r^- \, \mathrm{d}^2 r_\perp}{2\,(2\pi)^3} e^{ik\cdot r} \langle P, S | \bar{\psi}(r) \mathcal{U}[r,0] \Gamma \psi(0) | P, S \rangle$$

$$\mathcal{U}[r,0] = \mathcal{P}\mathrm{exp}\left[ig \int_0^r \mathrm{d}x_\mu \, A^\mu(x) \right]$$

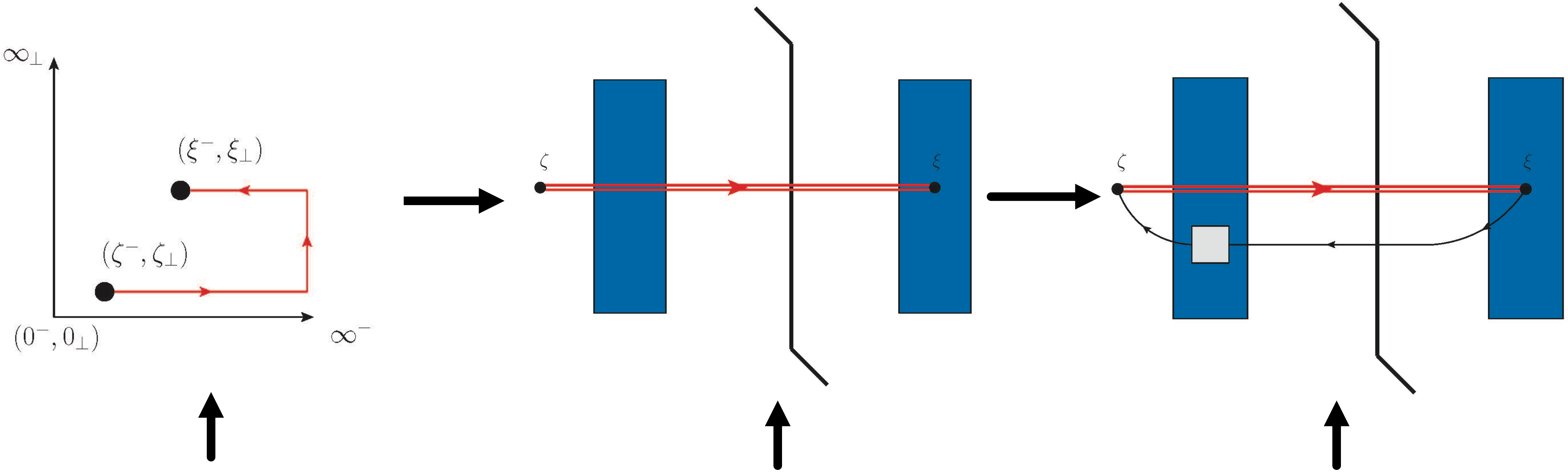


SIDIS staple

- Linear combinations of different TMDs come from different choices of the Dirac matrix Γ, for example the unintegrated quark density f_1^q and the Sivers function $f_{1T}^{\perp q}$ are given by the taking the matrix to be $\gamma^+/2$

$$f_1^q(x, k_T^2) - \frac{\underline{k} \times \underline{S}_P}{M_P} f_{1\,T}^{\perp\,q}(x, k_T^2) = \int \frac{\mathrm{d}r^- \, \mathrm{d}^2 r_\perp}{2\,(2\pi)^3} e^{ik\cdot r} \langle P, S | \bar{\psi}(r) \mathcal{U}[r,0] \frac{\gamma^+}{2} \psi(0) | P, S \rangle$$

Simplify: Gauge link to dipole amplitude



$\langle P, S | \, \bar{\psi}(\xi) U[\xi, \zeta] \psi(\zeta) \, | P, S \rangle$

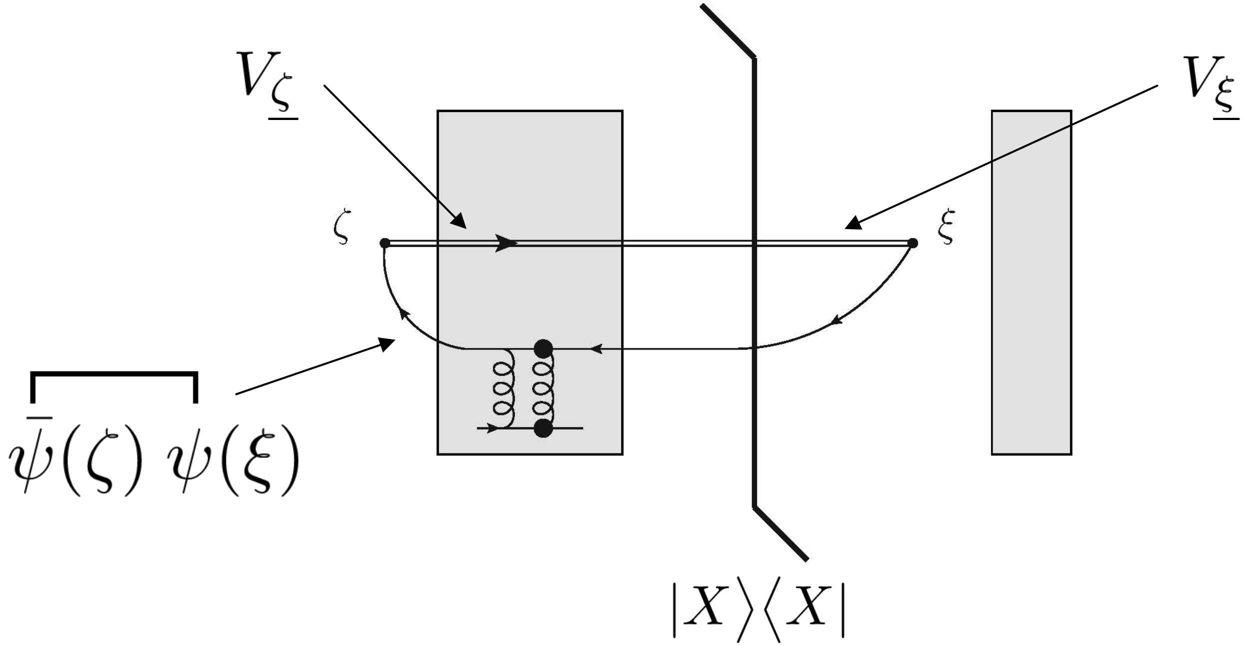
$\sum_X \langle P, S | \, \mathrm{T} \bar{\psi}(\xi) V_{\underline{\xi}}[\xi^-, \infty] \, | X \rangle \langle X | \, \mathrm{T} V_{\underline{\zeta}}[\infty, \zeta^-] \psi(\zeta) \, | P, S \rangle$

Leading order correction

Simplify: Shock wave picture

- By inserting a complete set of states, one can write the operator product as a sum over cut diagrams for the scattering of a quark on the shockwave of a target hadron

$$f_1^q(x,k_T^2) - \frac{\underline{k}\times\underline{S}_P}{M_P} f_{1T}^{\perp q}(x,k_T^2) = \frac{2p_1^+}{2(2\pi)^3}\sum_X \int \mathrm{d}\xi^- \, \mathrm{d}^2\xi_\perp \, \mathrm{d}\zeta^- \, \mathrm{d}^2\zeta_\perp \, e^{ik\cdot(\zeta-\xi)} \left[\frac{\gamma^+}{2}\right]_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) V_{\underline{\xi}}[\xi^-,\infty]|X\rangle\langle X| V_{\underline{\zeta}}[\infty,\zeta^-]\psi_\beta(\zeta)\right\rangle$$

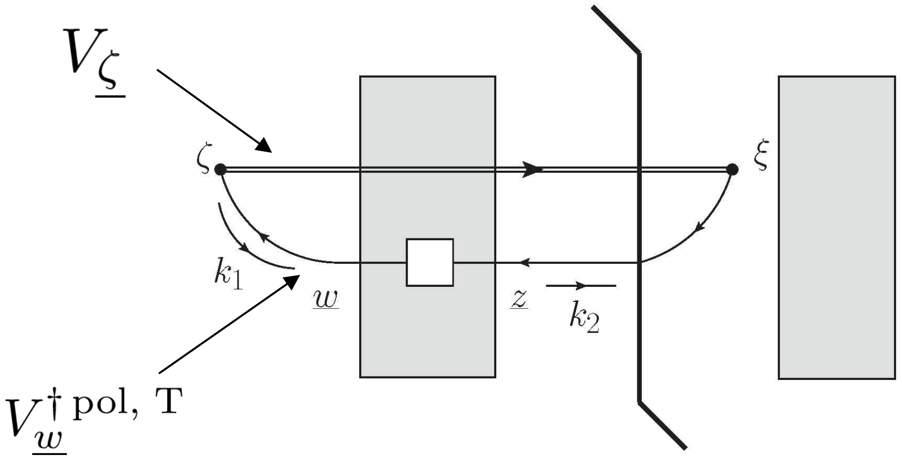


Simplify: Shock wave picture

- Writing the antiquark propagator as a polarized Wilson line lets us write the operator product in terms of a polarized dipole amplitude

$$f_1^q(x,k_T^2) - \frac{\underline{k}\times \underline{S}_P}{M_P} f_{1\,T}^{\perp\,q}(x,k_T^2) = -\frac{2p_1^+}{2(2\pi)^3}\int d^2\zeta_\perp\, d^2w_\perp\, \frac{d^2k_{1\perp}\, dk_1^-}{(2\pi)^3} e^{i(\underline{k}_1+\underline{k})\cdot(\underline{w}-\underline{\zeta})}$$

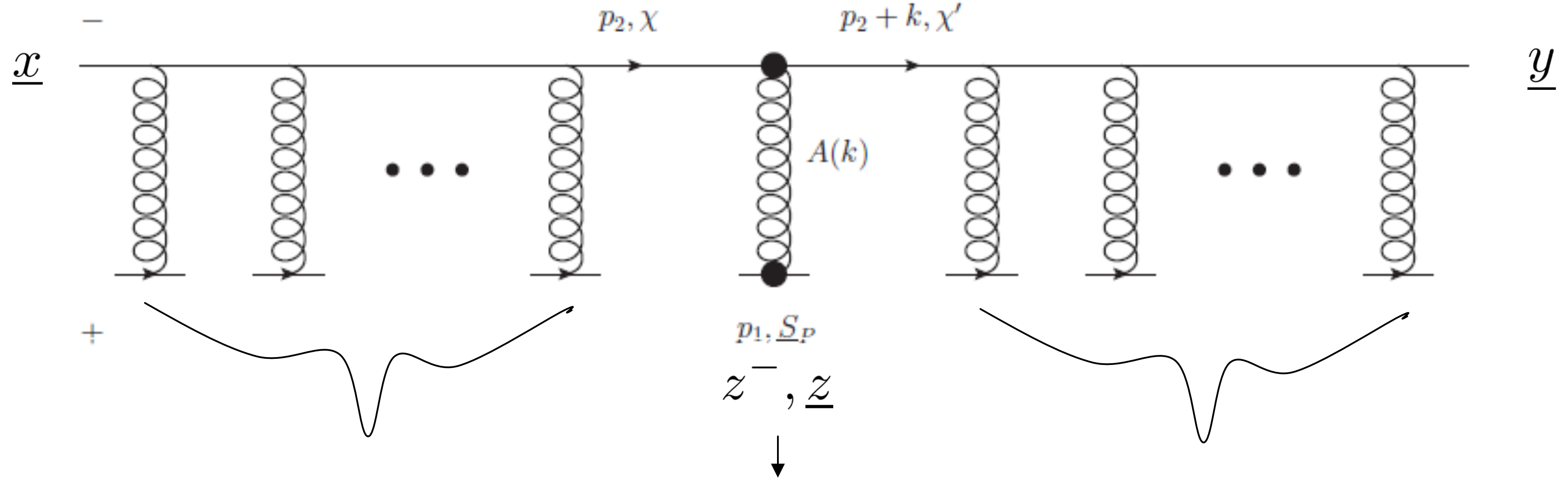
$$\theta(k_1^-)\,\frac{1}{(xp_1^+k_1^- + \underline{k}_1^2)(xp_1^+k_1^- + \underline{k}^2)} \sum_{\chi_1,\chi_2} \bar{v}_{\chi_2}(k_2)\frac{\gamma^+}{2}v_{\chi_1}(k_1)\left\langle \mathrm{T}\, V_{\underline{\zeta}}^{ij}\, \bar{v}_{\chi_1}(k_1) V_{\underline{w}}^{\dagger\,\mathrm{pol,\,T}ji} v_{\chi_2}(k_2)\right\rangle$$



$$\supset \mathrm{tr}\left[V_{\underline{\zeta}} V_{\underline{w}}^{\mathrm{pol,\,T}\dagger}\right]$$

Polarized Wilson Line

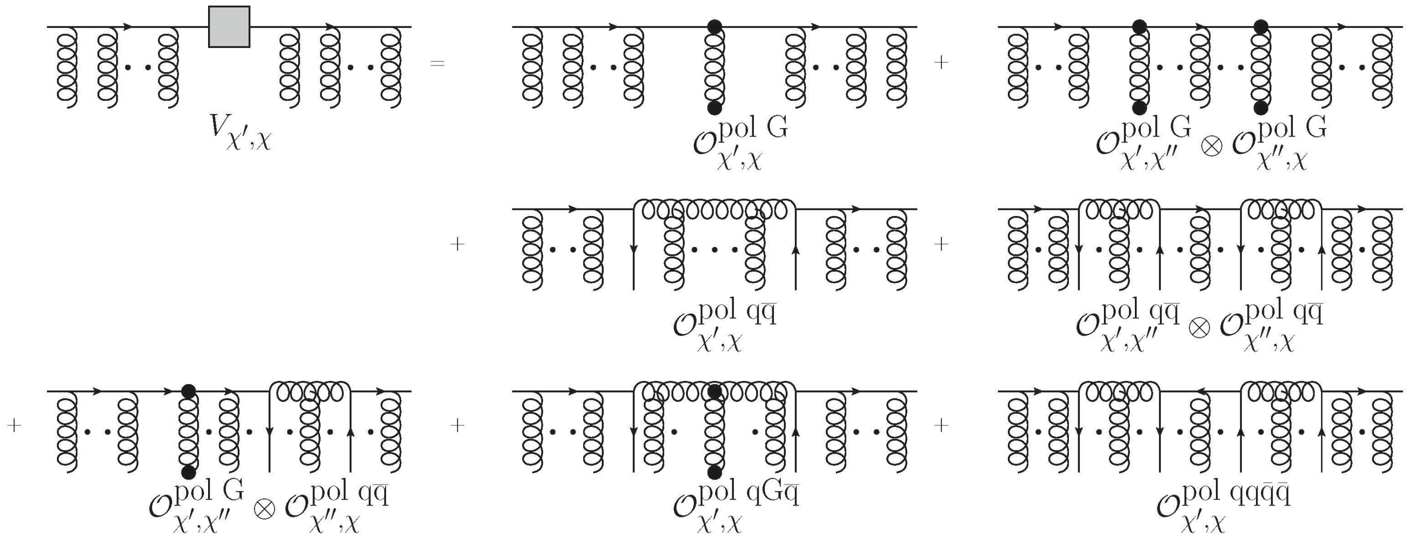
- Dependence on the spin of the quarks in the dipole requires the insertion of sub-eikonal operators in the Wilson lines
- Add all possible operator insertions integrated over x^- positions along Wilson lines



$$\int \mathrm{d}z^- \, \mathrm{d}^2 z_\perp \qquad V_{\underline{x}} \qquad \mathcal{O}^{\mathrm{pol\ G}}_{\chi,\chi'} \qquad V_{\underline{y}}$$

Polarized Wilson line

- It has been shown that to account for longitudinal (Kovchegov, Pitonyak and Sievert 2016) and transverse spin (Kovchegov and Sievert 2019) one needs to include corrections out to sub-sub-eikonal order
- We construct a general sub-sub-eikonal polarized Wilson line by adding all possible operator insertions



Simplify: Polarized Dipole Amplitudes

- For a given TMD, at a given order in the sub-eikonal expansion, we define polarized dipole amplitudes

$$
-\frac{\underline{k}\times\underline{S}_P}{M_P} f_{1T}^{\perp NS}(x,k_T^2)\Big|_{\text{sub-eikonal}} = \frac{16N_c}{(2\pi)^3}\int d^2x_{10}\,\frac{d^2k_{1\perp}}{(2\pi)^3}\,\frac{e^{i(\underline{k}+\underline{k}_1)\cdot\underline{x}_{10}}}{\underline{k}_1^2\,\underline{k}^2}\int\limits_{\frac{\Lambda^2}{s}}^{1}\frac{dz}{z}
$$

$$
\times\Big\{\underline{k}_1\cdot\underline{k}\,(k-k_1)^i\Big[\epsilon^{ij}\,S_P^j\,x_{10}^2\,F_{NS}^A(x_{10}^2,z)+x_{10}^i\,\underline{x}_{10}\times\underline{S}_P\,F_{NS}^B(x_{10}^2,z)+\epsilon^{ij}\,x_{10}^j\,\underline{x}_{10}\cdot\underline{S}_P\,F_{NS}^C(x_{10}^2,z)\Big]
$$

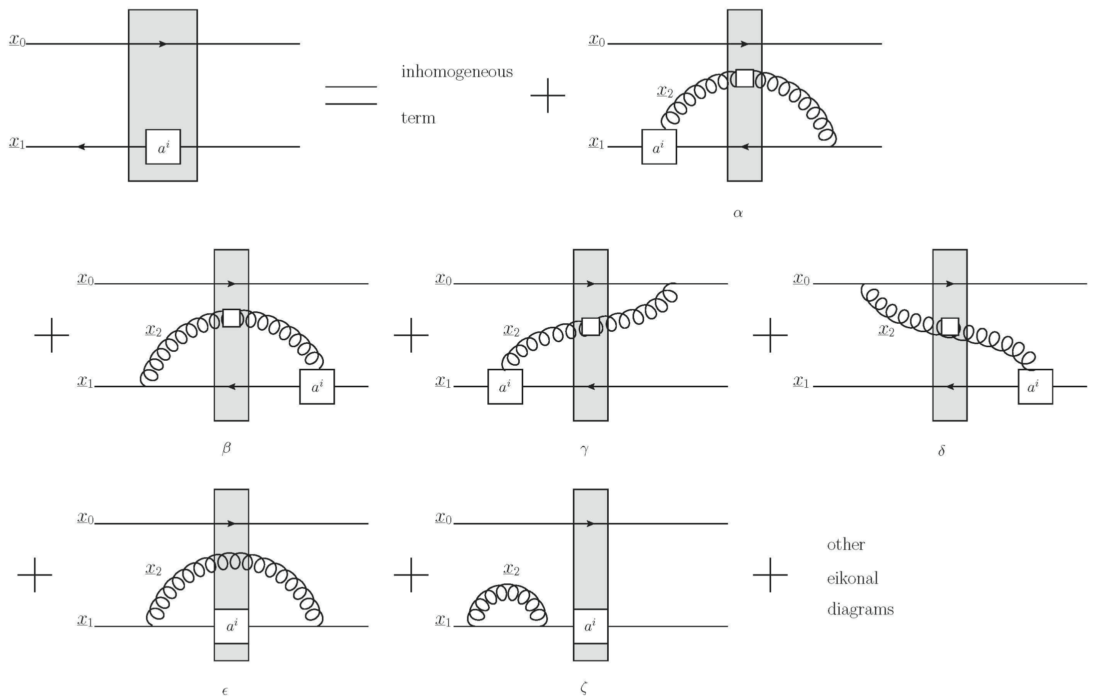
$$
+i\,\underline{k}_1\cdot\underline{k}\;\underline{x}_{10}\times\underline{S}_P\,F_{NS}^{[2]}(x_{10}^2,z)-i\,\underline{k}\times\underline{k}_1\,\underline{x}_{10}\cdot\underline{S}_P\,F_{NS}^{\text{mag}}(x_{10}^2,z)\Big\}
$$

$$
= \text{chromomagnetic interaction} + \text{covariant phase factor}
$$

$$
\mathcal{O}^{\text{pol}}\sim F^{12} \qquad\qquad \mathcal{O}^{\text{pol}}\sim \overleftarrow{\underline{D}}\cdot\overrightarrow{\underline{D}}
$$

Small-x Evolution

- Calculate gluon and quark emissions in the dipole amplitudes
- Sum over relevant diagrams/operators to extract evolution in large-N_c or large-N_c&N_f limit
- Obtain general operator level equations (in given approximation)



Polarized Wilson lines allow for logs as dipole size goes to zero → resum double logs $\alpha_s^n \ln^{2n}(1/x)$

Results for flavor non-singlet outside saturation region

Leading Twist Quark TMDs				
		Quark Polarization		
		U	L	T
Nucleon Polarization	U	$f_1^{\mathrm{NS}} \sim x^{-\sqrt{2\alpha_s C_F/\pi}}$		$h_1^{\perp \mathrm{NS}} \sim x$
	L		$g_1^{\mathrm{NS}} \sim x^{-\sqrt{\alpha_s N_c/\pi}}$	$h_{1L}^{\perp \mathrm{NS}} \sim x$
	T	$f_{1T}^{\perp \mathrm{NS}} \sim C_{\mathcal{O}} x^{-1} + C_1 x^{-3.4\sqrt{\alpha_s N_c/4\pi}}$	$g_{1T}^{\mathrm{NS}} \sim x^0$	$h_1^{\mathrm{NS}} \sim h_{1T}^{\perp \mathrm{NS}} \sim x^{1-2\sqrt{\alpha_s N_c/2\pi}}$

MGS 2024

- The diagonal TMDs all receive evolution contributions equal to that of the Reggeon (Kirschner and Lipatov 1983)
- Unpolarized and helicity TMDs also studied in InfraRed Evolution Equation (IREE) framework (Ermolaev, Manaenkov and Ryskin 1996, Bartels, Ermolaev and Ryskin 1996)
- Transversity TMD also studied previously (Kirschner, Mankiewicz, Schafer, and Szymanowski 1997)
- The off diagonal terms receive no evolution power correction in their leading terms!

Results for flavor singlet outside saturation region

Leading Twist Quark TMDs				
		Quark Polarization		
		U	L	T
Nucleon Polarization	U	$f_1^{\mathrm{S}} \sim x^{-\frac{4\alpha_s N_c}{\pi}\ln(2)}$		$h_1^{\perp \mathrm{S}} \sim x$
	L		$g_1^{\mathrm{S}} \sim x^{-3.66\sqrt{\alpha_s N_c/2\pi}}$	$h_{1L}^{\perp \mathrm{S}} \sim x$
	T	$f_{1T}^{\perp \mathrm{S}} \sim x^{-2.9\sqrt{\alpha_s N_c/4\pi}}$	$g_{1T}^{\mathrm{S}} \sim x^{-2.9\sqrt{\alpha_s N_c/4\pi}}$	$h_1^{\mathrm{S}} \sim h_{1T}^{\perp \mathrm{S}} \sim x^{1-2\sqrt{\frac{\alpha_s N_c}{2\pi}}}$

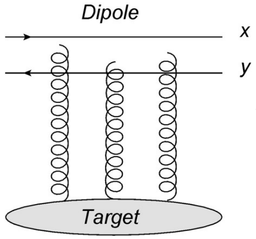
Adamiak, Tawabutr and MGS *in preparation*

- **Preliminary results**, but table is completed!
- Surprisingly, still have some off-diagonal TMDs with no linear evolution power corrections

Protection from evolution?

- One quite interesting feature is that multiple off-diagonal TMDs receive no corrections to their naïve power law
- Based on the known spin-dependent Odderon contribution to the Sivers function (Boer et al 2016), we had previously speculated that there could be protection for T-odd TMDs

$$\mathcal{O}(x,y) \sim$$



$$d^{abc} = \mathrm{tr}[t^a\{t^b, t^c\}]$$

$$\left\langle \mathrm{T}\,\mathrm{tr}[V_{\underline{\zeta}} V_{\underline{w}}^\dagger] + \bar{\mathrm{T}}\,\mathrm{tr}[V_{\underline{\zeta}} V_{\underline{w}}^\dagger] \right\rangle = 2N_c \left(\mathcal{S}_{\underline{\zeta w}} + i\,\mathcal{O}_{\underline{\zeta w}} \right)$$

$$\mathcal{O}(x,y) \xrightarrow[\text{small-}x]{} \left(\frac{1}{x}\right)^{1-g(\alpha_s N_c)} \sim \left(\frac{1}{x}\right)^{1-0}$$

Bartel, Lipatov and Vacca 2000

- New results show that this protection applies to other TMDs, perhaps some other symmetry?

Discussion

- We now know the small-x asymptotics of all 8 leading-twist quark TMDs
- Interesting patterns have emerged
- Many equations ready for phenomenological implementation
  - Significant progress made applying small-x for helicity TMDs by JAM (cf. JAM Adamiak et al 2023 and Josh's talk from Tuesday), also some work on the transversity TMD (JAM Cocuzza et al 2023)
- All N_c results require JIMWLK type evolution equation (cf. Cougoulic and Kovchegov 2019)
- Results can be derived for gluon TMDs as well
- Formalism can be extended to GPDs/GTMDs and potentially higher twist parton distributions
- Polarized Wilson lines can be used to calculate more general spin-dependent processes at high energy

Conclusions

- We have developed a general framework for including sub-eikonal effects into the small-x formalism
  - Similar formalisms for sub-eikonal corrections have been developed by various other authors, cf. Altinoluk et al 2016-2021, Chirilli 2019-2021
- We have obtained the small-x asymptotics for all eight leading-twist quark TMDs
- More improvements and applications of the formalism are in progress!

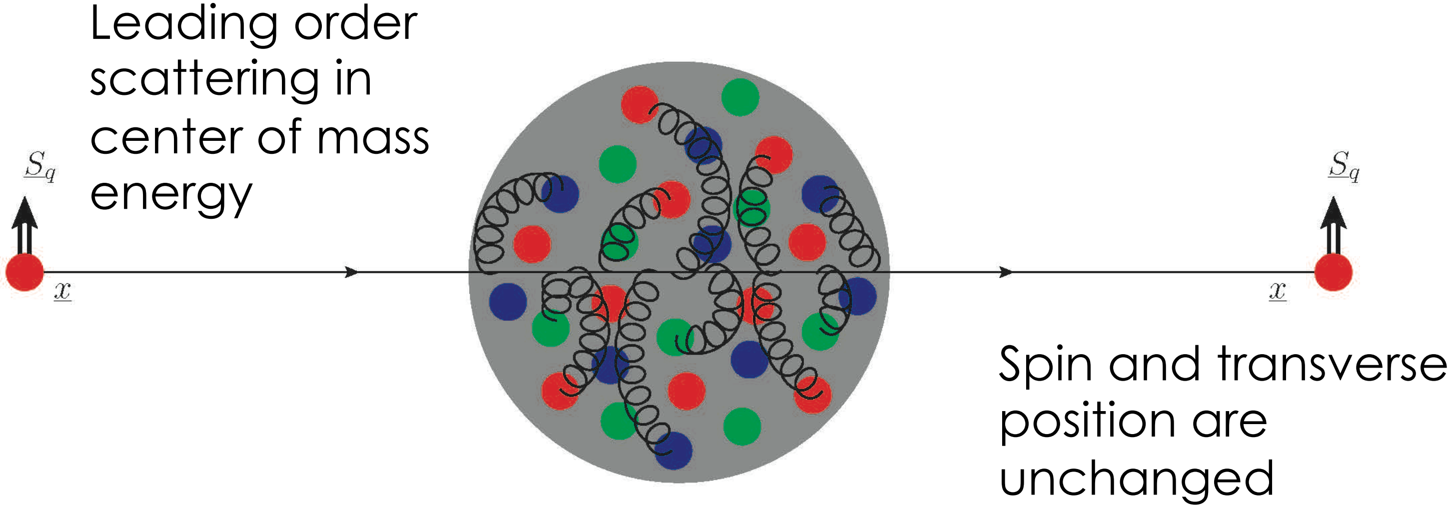
Backup Slides

Sub-eikonal power counting

- Eikonal distributions $q(x, k_T) \sim \frac{1}{x}$, no COM energy suppression
- Sub-eikonal distributions $q(x, k_T) \sim x^0, \frac{1}{s}$ energy suppression
- Sub-sub-eikonal distributions $q(x, k_T) \sim x, \frac{1}{s^2}$ energy suppression

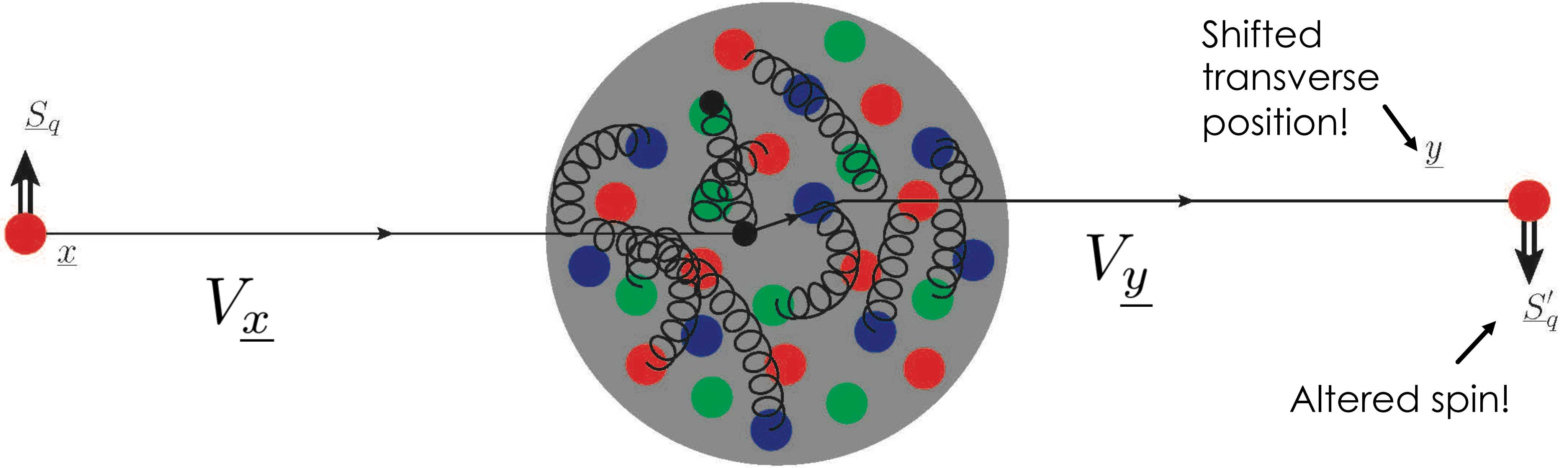
What about spin?

- The eikonal approximation only sees the projectile's color charge/representation



- Spin-dependence can only enter in the target background fields – eikonal Sivers function!

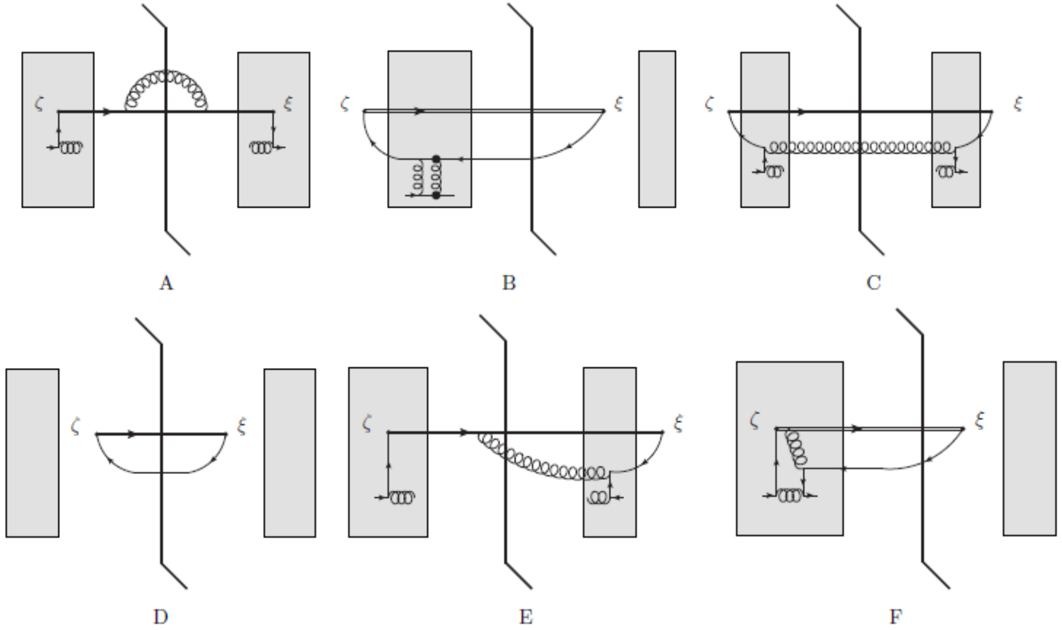
Polarized Wilson Line



Light Cone Operator Treatment (LCOT)

- These sub-eikonal corrections have been used to develop a framework for studying spin-dependent scattering at small-x over the course of several years
  - Initial calculations for helicity TMDs by Yuri Kovchegov, Daniel Pitonyak and Matthew Sievert in 2016
  - Many advancements and extension made since with major contributions by Daniel Adamiak, Jeremy Borden, Florian Cougoulic, Ming Li, Brandon Manley, MGS, Andrey Tarasov, Yossathorn Tawabutr
- Started by calculating cross sections, refined to calculate small-x TMDs starting directly from the operator definition
  - We call this new formalism the Light Cone Operator Treatment

Classes of leading small-x diagrams



- General analysis shows that class B gives leading contribution

Sub-sub-eikonal Polarized Wilson Line

$$V_{\underline{x},\underline{y};\chi',\chi} = V_{\underline{x}}\,\delta^2(\underline{x}-\underline{y})\,\delta_{\chi,\chi'} + \int\limits_{-\infty}^{\infty} \mathrm{d}z^-\, d^2z\; V_{\underline{x}}[\infty,z^-]\,\delta^2(\underline{x}-\underline{z})\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi}(z^-,\underline{z})\,V_{\underline{y}}[z^-,-\infty]\,\delta^2(\underline{y}-\underline{z})$$

$$+ \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^-\, d^2z_1 \int\limits_{z_1^-}^{\infty} \mathrm{d}z_2^-\, d^2z_2 \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_2^-]\,\delta^2(\underline{x}-\underline{z}_2)\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi''}(z_2^-,\underline{z}_2)\,V_{\underline{z}_1}[z_2^-,z_1^-]\,\delta^2(\underline{z}_2-\underline{z}_1)$$

$$\times\, \mathcal{O}^{\mathrm{pol\ G}}_{\chi'',\chi}(z_1^-,\underline{z}_1)\,V_{\underline{y}}[z_1^-,-\infty]\,\delta^2(\underline{y}-\underline{z}_1) + \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} dz_2^-\; V_{\underline{x}}[\infty,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi}(z_2^-,z_1^-;\underline{x},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty]$$

$$+ \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} \mathrm{d}z_2^- \int\limits_{z_2^-}^{\infty} \mathrm{d}z_3^- \int\limits_{z_3^-}^{\infty} \mathrm{d}z_4^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_4^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi''}(z_4^-,z_3^-;\underline{x},\underline{z})\,V_{\underline{z}}[z_3^-,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi'',\chi}(z_2^-,z_1^-;\underline{z},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty]$$

$$+ \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} \mathrm{d}z_2^- \int\limits_{z_2^-}^{\infty} \mathrm{d}z_3^- \int\limits_{z_3^-}^{\infty} \mathrm{d}z_4^-\, V_{\underline{x}}[\infty,z_4^-]\,\mathcal{O}^{\mathrm{pol\ qq\bar{q}\bar{q}}}_{\chi',\chi}(z_4^-,z_3^-,z_2^-,z_1^-;\underline{x})\,V_{\underline{y}}[z_1^-,-\infty]\delta^2(\underline{x}-\underline{y})$$

$$+ \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} dz_2^- d^2z_2 \int\limits_{z_2^-}^{\infty} dz_3^-\; V_{\underline{x}}[\infty,z_3^-]\,\delta^2(\underline{z}_2-\underline{x})\mathcal{O}^{\mathrm{pol}\ \ \mathrm{qG\bar{q}}}_{\chi',\chi}(z_1^-,z_2^-,z_3^-;\underline{x},\underline{z}_2)\delta^2(\underline{z}_2-\underline{y})V_{\underline{y}}[z_1^-,-\infty]$$

$$+ \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} \mathrm{d}z_2^- \int\limits_{z_2^-}^{\infty} \mathrm{d}z_3^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_3^-]\,\delta^2(\underline{x}-\underline{z})\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi''}(z_3^-;\underline{z})\,V_{\underline{z}}[z_3^-,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi'',\chi}(z_2^-,z_1^-;\underline{z},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty]$$

$$+ \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} \mathrm{d}z_2^- \int\limits_{z_2^-}^{\infty} \mathrm{d}z_3^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_3^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi''}(z_3^-,z_2^-;\underline{x},\underline{z})\,V_{\underline{z}}[z_2^-,z_1^-]\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi'',\chi}(z_1^-;\underline{z})\,V_{\underline{y}}[z_1^-,-\infty]\,\delta^2(\underline{y}-\underline{z})$$

Sub-sub-eikonal Operators

$$\mathcal{O}^{\text{pol G}}_{\chi',\chi}(x^-,\underline{x}) = -i\,\delta_{\chi,\chi'}\left[\overleftarrow{D}^i\,\frac{1}{2(P_2^- + iD^-)}\vec{D}^i + \frac{m^2}{2(P_2^- + iD^-)}\right]$$
$$+\frac{ig}{2}\left\{\delta_{\chi,-\chi'}\left[F^{12}\,\frac{1}{P_2^- + iD^-} - \frac{i}{(P_2^-)^2}\epsilon^{ij}\,\overleftarrow{D}^i\,F^{-j}\right] + \chi\,\delta_{\chi,\chi'}\frac{m}{(P_2^-)^2}\,\epsilon^{ij}S^i F^{-j} + \chi\,\delta_{\chi,-\chi'}\frac{im}{(P_2^-)^2}\,S^i\,F^{-i}\right\}$$

Sub-sub-eikonal Operators

$$\mathcal{O}^{\text{pol q}\bar{\text{q}}}_{\chi',\chi}(z_2^-, z_1^-; \underline{z}_2, \underline{z}_1) = -\frac{g^2\, p_1^+}{8\, s}\, t^b\, \psi_\beta(z_2^-, \underline{z}_2) \left[\delta^{b'b''} - \frac{i\, p_1^+\, \mathscr{D}_{z_2}^{b'b''-}}{2s}\right] U_{\underline{z}_2}^{b''a''}[z_2^-, z_1^-]\, \delta^2(\underline{z}_2 - \underline{z}_1)$$

$$\times \left[\delta^{a''a'} - \frac{i\, p_1^+\, \mathscr{D}_{z_1}^{a''a'-}}{2s}\right] \Bigg\{ \delta_{\chi,\chi'} \Bigg[\gamma^+\, \delta^{a'a}\, \delta^{bb'} - \frac{2mp_1^+}{s}\, \delta^{a'a}\, \delta^{bb'}$$

$$+ \frac{p_1^+}{s} \bigg((\gamma^1 - i\gamma^5\gamma^2)\, [i\underline{S}\cdot \overleftarrow{\underline{\mathscr{D}}}_{z_2}^{bb'} + \underline{S} \times \overleftarrow{\underline{\mathscr{D}}}_{z_2}^{bb'}]\, \delta^{a'a} - (\gamma^1 + i\gamma^5\gamma^2)\, [i\underline{S}\cdot \underline{\mathscr{D}}_{z_1}^{a'a} + \underline{S} \times \underline{\mathscr{D}}_{z_1}^{a'a}] \bigg)\, \delta^{bb'} \bigg) \Bigg]$$

$$+ \delta_{\chi,-\chi'} \Bigg[\gamma^+\, \gamma^5\, \delta^{a'a}\, \delta^{bb'} - \frac{2mp_1^+}{s}\, i\, \gamma^1\, \gamma^2\, \delta^{a'a}\, \delta^{bb'}$$

$$+ \frac{p_1^+}{s} \bigg((i\gamma^2 - \gamma^5\gamma^1)\, [i\underline{S}\cdot \overleftarrow{\underline{\mathscr{D}}}_{z_2}^{bb'} + \underline{S} \times \overleftarrow{\underline{\mathscr{D}}}_{z_2}^{bb'}]\, \delta^{a'a} + (i\gamma^2 + \gamma^5\gamma^1)\, [i\underline{S}\cdot \underline{\mathscr{D}}_{z_1}^{a'a} + \underline{S} \times \underline{\mathscr{D}}_{z_1}^{a'a}] \bigg)\, \delta^{bb'} \bigg) \Bigg]$$

$$+ \chi\delta_{\chi,\chi'}\, \frac{p_1^+}{s}\, \delta^{a'a}\, \delta^{bb'} \bigg[\Big[i\gamma^5 \underline{S}\cdot \overleftarrow{\underline{D}}_{z_2} - \underline{S} \times \overleftarrow{\underline{D}}_{z_2} \Big] (1 - i\gamma^5\gamma^1\gamma^2) + \Big[i\gamma^5 \underline{S}\cdot \underline{D}_{z_1} - \underline{S} \times \underline{D}_{z_1} \Big] (1 + i\gamma^5\gamma^1\gamma^2) \bigg]$$

$$+ \chi\delta_{\chi,-\chi'}\, \frac{p_1^+}{s}\, \delta^{a'a}\, \delta^{bb'} \bigg[\Big[i\underline{S}\cdot \overleftarrow{\underline{D}}_{z_2} - \gamma^5 \underline{S} \times \overleftarrow{\underline{D}}_{z_2} \Big] (1 - i\gamma^5\gamma^1\gamma^2) + \Big[i\underline{S}\cdot \underline{D}_{z_1} - \gamma^5 \underline{S} \times \underline{D}_{z_1} \Big] (1 + i\gamma^5\gamma^1\gamma^2) \bigg] \Bigg\}_{\alpha\beta}$$

$$\times \bar{\psi}_\alpha(z_1^-, \underline{z}_1)\, t^a + \mathcal{O}\left(\frac{1}{s^3}\right)$$

Sub-sub-eikonal Operators

$$\mathcal{O}^{\text{pol qG}\bar{\text{q}}}_{\chi',\chi} = \frac{ig^2(p_1^+)^2}{16s^2} t^b\, \psi_\beta(z_3^-, \underline{x})\, U_{\underline{x}}^{b'b}[z_3^-, z_2^-] \Big\{ \delta_{\chi,\chi'} \Big[\gamma^+ \overleftarrow{\underline{\mathscr{D}}}^{bc}_{z_2} \cdot \underline{\mathscr{D}}^{ca}_{z_2} + \gamma^+\gamma^5\, 2g\, \big(\mathcal{F}^{12}(z_2)\big)^{ba} \Big] + \delta_{\chi,-\chi'} \Big[\gamma^+\gamma^5\, \overleftarrow{\underline{\mathscr{D}}}^{bc}_{z_2} \cdot \underline{\mathscr{D}}^{ca}_{z_2} + \gamma^+\, 2g\, \big(\mathcal{F}^{12}(z_2)\big)^{ba} \Big] \Big\}_{\alpha\beta}$$
$$\times\, U_{\underline{y}}^{aa'}[z_2^-, z_1^-]\, \bar{\psi}_\alpha(z_1^-, \underline{y})\, t^{a'}$$

$$\mathcal{O}^{\text{pol qq}\bar{\text{q}}\bar{\text{q}}}_{\chi',\chi} = -\frac{g^4(p_1^+)^2}{64s^2} t^d \psi_\delta(z_4^-, \underline{x}) U_{\underline{x}}^{dc}[z_4^-, z_3^-] \bar{\psi}_\beta(z_2^-, \underline{x}) t^b V_{\underline{x}}^\dagger[z_3^-, z_2^-]$$
$$\times \Big\{ \delta_{\chi,\chi'} \Big[\big(\gamma^+\big)_{\alpha\delta} \big(\gamma^+\big)_{\beta\gamma} - \big(\gamma^+\gamma^5\big)_{\alpha\delta} \big(\gamma^+\gamma^5\big)_{\beta\gamma} \Big] + \delta_{\chi,-\chi'} \Big[\big(\gamma^+\big)_{\alpha\delta} \big(\gamma^+\gamma^5\big)_{\beta\gamma} - \big(\gamma^+\gamma^5\big)_{\alpha\delta} \big(\gamma^+\big)_{\beta\gamma} \Big] \Big\}$$
$$\times\, t^c \psi_\gamma(z_3^-, \underline{x}) U_{\underline{x}}^{ba}[z_2^-, z_1^-] \bar{\psi}_\alpha(z_1^-, \underline{x}) t^a$$

Non-singlet Sivers function Polarized Dipole Amplitudes

$$F^{NS\,i}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c}\,\text{Re}\left\langle\!\!\left\langle \text{T tr}\left[V_{\underline{\zeta}}\,V^{i\,\dagger}_{\underline{w}}\right] - \text{T tr}\left[V^{i}_{\underline{w}}\,V^{\dagger}_{\underline{\zeta}}\right]\right\rangle\!\!\right\rangle,$$

$$F^{NS\,[2]}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c}\,\text{Im}\left\langle\!\!\left\langle \text{T tr}\left[V_{\underline{\zeta}}\,V^{[2]\,\dagger}_{\underline{w};\underline{k},\underline{k}_1}\right] - \text{T tr}\left[V^{[2]}_{\underline{w};\underline{k},\underline{k}_1}\,V^{\dagger}_{\underline{\zeta}}\right]\right\rangle\!\!\right\rangle,$$

$$F^{NS\,\text{mag}}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c}\,\text{Re}\left\langle\!\!\left\langle \text{T tr}\left[V_{\underline{\zeta}}\,V^{\text{mag}\,\dagger}_{\underline{w}}\right] - \text{T tr}\left[V^{\text{mag}}_{\underline{w}}\,V^{\dagger}_{\underline{\zeta}}\right]\right\rangle\!\!\right\rangle$$

$$V^{\text{i}}_{\underline{x}} = -\frac{p_1^+}{8\,s}\int_{-\infty}^{\infty} dz^-\;\; V_{\underline{x}}[\infty, z^-]\,(\overleftarrow{D}^i_z - \vec{D}^i_z)\,V_{\underline{x}}[z^-, -\infty]$$

$$V^{[2]}_{\underline{x}} = \frac{i\,p_1^+}{8\,s}\int_{-\infty}^{\infty} dz^-\;\; V_{\underline{x}}[\infty, z^-]\,[(\overleftarrow{D}^i_z - \vec{D}^i_z)^2 - (\underline{k}_1 - \underline{k})^2]\,V_{\underline{x}}[z^-, -\infty]$$

$$-\frac{g^2\,p_1^+}{4\,s}\int_{-\infty}^{\infty} dz_1^-\int_{z_1^-}^{\infty} dz_2^-\;\; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-, \underline{x})\,U^{ba}_{\underline{x}}[z_2^-, z_1^-]\left[\frac{\gamma^+}{2}\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-, \underline{x})\,t^a\,V_{\underline{x}}[z_1^-, -\infty]$$

$$V^{\text{mag}}_{\underline{x}} = \frac{igp_1^+}{2s}\int_{-\infty}^{\infty} dz^-\;\; V_{\underline{x}}[\infty, z^-]F^{12}(z^-, \underline{x})V_{\underline{x}}[z^-, -\infty]$$

$$-\frac{g^2\,p_1^+}{4\,s}\int_{-\infty}^{\infty} dz_1^-\int_{z_1^-}^{\infty} dz_2^-\;\; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-, \underline{x})\,U^{ba}_{\underline{x}}[z_2^-, z_1^-]\left[\frac{\gamma^+\gamma^5}{2}\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-, \underline{x})\,t^a\,V_{\underline{x}}[z_1^-, -\infty]$$

Here $\left\langle\!\!\left\langle \ldots \right\rangle\!\!\right\rangle = zs\left\langle \ldots \right\rangle$ with z the internal longitudinal momentum fraction and s the center of mass energy squared

Non-singlet Sivers function large-N_c linearized DLA equations

$$F_A^{NS}(x_{10}^2, z) = F_A^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z' s}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[6\, F_A^{NS}(x_{21}^2, z') - F_B^{NS}(x_{21}^2, z') + F_C^{NS}(x_{21}^2, z')\right]$$

$$F_B^{NS}(x_{10}^2, z) = F_B^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z' s}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[-2\, F_A^{NS}(x_{21}^2, z') + 5\, F_B^{NS}(x_{21}^2, z') - F_C^{NS}(x_{21}^2, z')\right],$$

$$F_C^{NS}(x_{10}^2, z) = F_C^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z' s}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[2\, F^{NS\,\mathrm{mag}}(x_{21}^2, z') + 6\, F_C^{NS}(x_{21}^2, z')\right],$$

$$F^{NS\,\mathrm{mag}}(x_{10}^2, z) = F^{NS\,\mathrm{mag}\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[\Gamma^{NS\,\mathrm{mag}}(x_{10}^2, x_{21}^2, z') + 2\, \Gamma_A^{NS}(x_{10}^2, x_{21}^2, z') - \Gamma_B^{NS}(x_{10}^2, x_{21}^2, z') + 3\, \Gamma_C^{NS}(x_{10}^2, x_{21}^2, z')\Big]$$

Dipole evolution equations

$$\Gamma_A^{NS}(x_{10}^2, x_{21}^2, z') = F_A^{NS\,(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z'' s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \times \left[6\, F_A^{NS}(x_{32}^2, z'') - F_B^{NS}(x_{32}^2, z'') + F_C^{NS}(x_{32}^2, z'')\right]$$

$$\Gamma_B^{NS}(x_{10}^2, x_{21}^2, z') = F_B^{NS\,(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z'' s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \times \left[-2\, F_A^{NS}(x_{23}^2, z'') + 5\, F_B^{NS}(x_{32}^2, z'') - F_C^{NS}(x_{32}^2, z'')\right],$$

$$\Gamma_C^{NS}(x_{10}^2, x_{21}^2, z') = F_C^{NS\,(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z'' s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \times \left[2\, F^{NS\,\mathrm{mag}}(x_{32}^2, z'') + 6\, F_C^{NS}(x_{32}^2, z'')\right],$$

$$\Gamma^{NS\,\mathrm{mag}}(x_{10}^2, x_{21}^2, z') = F^{NS\,\mathrm{mag}\,(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \times \Big[\Gamma^{NS\,\mathrm{mag}}(x_{10}^2, x_{32}^2, z'') + 2\, \Gamma_A^{NS}(x_{10}^2, x_{32}^2, z'') - \Gamma_B^{NS}(x_{10}^2, x_{32}^2, z'') + 3\, \Gamma_C^{NS}(x_{10}^2, x_{32}^2, z'')\Big]$$

'Neighbor' dipole evolution equations