Noncommutative geometry and Quantum gravity

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Noncommutative geometry and Quantum gravity : introduction

Quantum Gravity $\Longrightarrow$ encoded in **quantum space-time** $\Longrightarrow$ modelised with **Noncommutative geometry**

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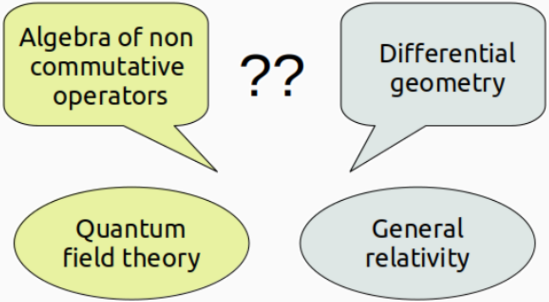

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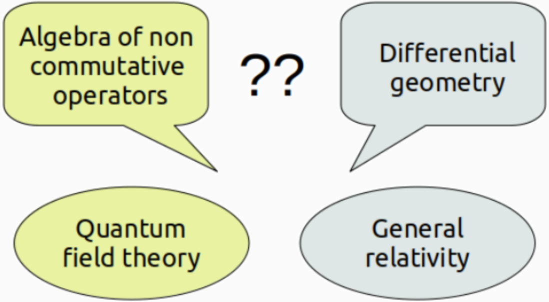

Common mathematical language $\Longrightarrow$ **Noncommutative geometry**

Noncommutativity of coordinates:

$$[\hat{x}_\mu, \hat{x}_\nu] \neq 0 \tag{1}$$

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Noncommutative spaces

- $\mathbb{R}^3_\kappa$: $[\hat{x}_i, \hat{x}_j] = i\kappa\epsilon_{ijk}x_k$, 3 dimensions, appear in **effective QG theory (Ponzano Regge)**
- $\mathcal{M}_\rho$: $[\hat{x}_0, \hat{x}_{1/2}] = \pm i\rho x_{2/1}, [\hat{x}_3, \hat{x}_\mu] = 0, [\hat{x}_1, \hat{x}_2] = 0$, 4 dimensions, **time operator has discret spectrum**

κ, ρ are **deformation parameters**

Summary

1. Unitary of the Ponzano Regge model. ($\mathbb{R}^3_\kappa$)
2. Gauge theories on ρ-Minkowski. ($\mathcal{M}_\rho$)

Unitarity of the Ponzano Regge model

Path integral formulation of 3D quantum gravity.[1]

$$Z = \int \mathcal{D}g\mathcal{D}\Psi \; e^{iS[\Psi,g]+iS_{\mathrm{GR}}[g]} \longrightarrow \int \mathcal{D}\Psi \; e^{iS_{\mathrm{eff}}[\Psi]}$$

The resulting QFT is **noncommutative** ($\mathbb{R}^3_\kappa$), **effective, and invariant under deformed Poincaré symmetries.**

Figure 1: Flat space-time

[2]

Figure 2: Particles insertions $\rightarrow$ space-time deformation, parametrized by an angle $\phi = \mathrm{m}\kappa$

[1] L.Freidel and E.R.Livine, *"Ponzao–regge model revisited: III. feynman diagrams and effective field theory"*, Classical and Quantum Gravity, 2006

[2] Matschull, Hans-Jürgen and Welling, Max,*"Quantum mechanics of a point particle in -dimensional gravity"*,Classical and Quantum Gravity, 1998,

Unitarity and optical theorem

Natural question: **Is this theory unitary?**

1. Effective theory point of view: No.
2. Noncommutative point of view: Yes.

Optical theorem

Figure 3: On the left side the imaginary part of the amplitude (off-shell) and on the right the amplitude decomposed over all possible intermediate states (on-shell).

$$2\,\mathrm{Im}(L_M(\vec{p})) = L_M^0(\vec{p}) \tag{2}$$

The 3D quantum gravity theory

The effective scalar field theory

$$S_{\text{eff}}[\Psi] = \int \frac{d^3x}{8\pi\kappa^3}[\frac{1}{2}\partial_\mu \Psi(x) \star \partial^\mu \Psi(x) - \frac{1}{2}M^2 \Psi(x) \star \Psi(x) + \frac{\lambda}{3!}\Psi \star \Psi(x) \star \Psi(x)] \quad (3)$$

with $\star$ the noncommutative product on $\mathbb{R}^3_\kappa$, $M = \frac{\sin(\phi\kappa)}{\kappa}$.

Expression of $x \in \mathbb{R}^3_\kappa$, such that $[x_i, x_j] = i\epsilon_{ijk}\kappa x_k$, with element of Lie groups G: [3]

- In Euclidian case **G= SO(3) or SU(2)**
- In Lorentzian case: **G= SO(2,1) or SU(1,1)**

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[3]
$$S_{\text{eff}}[g] = \int \frac{dg}{2}(P^2(g) - \frac{1}{2}M^2)\tilde{\Psi}(g)\tilde{\Psi}(g^{-1}) + \frac{\lambda}{3!}\int dg dh dk \delta(ghk)\tilde{\Psi}(g)\tilde{\Psi}(h)\tilde{\Psi}(k) \quad (4)$$

Commutative case ($\kappa \to 0$)

In the Euclidian and Lorentzian case we have for the off-shell one loop:

$$\mathrm{Im}(L_m(\vec{p})) = \begin{cases} 0 & \text{if} \quad |p| < 2m \\ \frac{i}{16|\vec{p}|} & \text{if} \quad |p| > 2m \end{cases}$$

For the on-shell one-loop we have different results in the Eulidiean and Lorentzian case:

$$L_m^{0,\mathrm{E}}(\vec{p}) = \begin{cases} \frac{i}{8|p|} & \text{if} \quad |p| < 2m \\ 0 & \text{if} \quad |p| > 2m \end{cases} \qquad L_m^{0,\mathrm{L}}(\vec{p}) = \begin{cases} 0 & \text{if} \quad |p| < 2m \\ \frac{i}{8|p|} & \text{if} \quad |p| > 2m \end{cases} \tag{5}$$

In the Lorentzian case we have the true optical theorem whereas in the Euclidean case we have an inversion on the mass momentum condition.

Results

Is the optical theorem verified ?

.	Euclidean	Lorentzian timelike	Lorentzian spacelike
Commutative	Yes, with inversion on mass-momentum condition	Yes	Yes
SO(3), SO(2,1)	Yes, with inversion on mass-momentum condition but **only for small mass**	Yes but **only for small mass**	?
SU(2), SU(1,1)	Yes, with inversion on mass-momentum condition and **we must add spin representation** $j = -\frac{1}{2}$	Yes and **we must add spin representation** $j = -\frac{1}{2}$	?

Conclusion and perspectives

- Unitarity in noncommutative geometry: have only been fully investigated in **Moyal** space [4].
- For **Lorentzian timelike** case in **SO(2,1)** we recover the results of [5]
- $SU(2)$ (Euclidan) $SU(1,1)$ (Lorentzian) are good candidates to express the Ponzano Regge model.
- **Next step** : verify optical theorem for **spacelike** elements in the Lorentzian case.

[4] J.Gomis and T.Mehen, "*Space–time noncommutative field theories and unitarity*", Nuclear Physics B, 2000

[5] Sasai, Yuya and Sasakura, Naoki, "*The Cutkosky rule of three dimensional noncommutative field theory in Lie algebraic noncommutative spacetime*", Journal of High Energy Physics, 2009

Gauge theories on ρ-Minkowski

ρ-Minkowski space (ρ deformation parameter, $[\rho] = m$)

$$[x_0, x_1] = i\rho x_2, \quad [x_0, x_2] = -i\rho x_1, \quad [x_1, x_2] = [x_3, x_\mu] = 0 \tag{6}$$

We define an **algebra of functions** $(\mathcal{C}(\mathcal{M}_\rho), \star)$ equipped with an $\star$-product and an involution $\dagger$. [6]

$$(f \star g)(x_0, \vec{x}, x_3) = \int \frac{dp_0}{2\pi}\, dy_0\; e^{-ip_0 y_0} f(x_0 + y_0, \vec{x}, x_3) g(x_0, R(-\rho p_0)\vec{x}, x_3), \tag{7}$$

$$f^\dagger(x_0, \vec{x}, x_3) = \int \frac{dp_0}{2\pi}\, dy_0\; e^{-ip_0 y_0} \overline{f}(x_0 + y_0, R(-\rho p_0)\vec{x}, x_3), \tag{8}$$

This algebra has for **symetry group the Hopf [7] algebra ρ-Poincaré**, generated by deformed translations, rotations and boosts, (P_μ, M_μ, N_μ).

[6] K.Hersent and J-C. Wallet, *"Field theories on ρ -deformed Minkowski space-time"*,Journal of High Energy Physics,2023

[7] Hopf algebra := bialgebra (algebra + co algebra with compatible structure) + antipode .

Motivations

$\mathcal{M}_\rho$: Promising quantum space with several interesting properties:

- At **commutative limit**, $\rho \to 0$, we recover Minkowski space-time and Poincaré algebra.
- Possible emergence in **QNM of black-holes** [8] [9]
- The **time operator has discret spectrum** [10]

[8] M. Dimitrijević Cirić and N.Konjik and A.Samsarov, *"Noncommutative scalar quasinormal modes of the Reissner–Nordström black hole"*,Classical and Quantum Gravity,2018

[9] M. Dimitrijević Cirić and N.Konjik and A.Samsarov, *"Noncommutative scalar field in the nonextremal Reissner-Nordström background: Quasinormal mode spectrum"*,Physical Review D,2020

[10] F.Lizzi, P.Vitale, *"Time discretization from noncommutativity"*,Physics Letters B,2021

Gauge theories in noncommutative geometry [11], [12]

Classical geometry	Noncomutative geometry
$\mathbb{A} = (\mathcal{C}^\infty(\mathcal{M}), .)$ associative algebra	$\mathbb{A} = (\mathcal{C}(\mathcal{M}_\rho), \star)$ associative algebra
$\mathrm{Der}(\mathbb{A}) = \{P_\mu : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathcal{C}^\infty(\mathcal{M})\vert$ Linear and Leibniz $\} = \Gamma(TM)$	$\mathrm{Der}(\mathbb{A}) = \{P_\mu : \mathbb{A} \rightarrow \mathbb{A}\vert$ Linear and Leibniz $\}$
$\mathcal{E}$: vector bundle $\quad \forall P_\mu \in \mathrm{Der}(\mathbb{A})$ $\nabla_{P_\mu} : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E})$	$\mathbb{E}$: module $\quad \forall P_\mu \in \mathrm{Der}(\mathbb{A})$ $\nabla_{P_\mu} : \mathbb{E} \longrightarrow \mathbb{E}$

In the following we consider the module $\mathbb{E} := \mathcal{M}_\rho$ and the right action $\triangleleft$ of the algebra on the module by

$$\forall m \in \mathbb{E}, \forall a \in \mathbb{A}, \quad (m \triangleleft a) := m \star a \tag{9}$$

[11] K.Hersent, M.Philippe, J-C.Wallet,"*Gauge theories on quantum spaces*",Phys.Rept.,2023

[12] M.Dubois-Voilette,"*Lectures on graded differential algebras and noncommutative geometry*",Math. Phys. Stud.,2001

Derivations

The twisted derivations of ρ-Minkowski are:

$$\mathfrak{D} = \left\{ P_\mu : \mathcal{M}_\rho \to \mathcal{M}_\rho,\ \mu = 0, 3, \pm,\ \{P_0, P_3\}_{\mathbb{I}} \oplus \{P_+\}_{\mathcal{E}_+} \oplus \{P_-\}_{\mathcal{E}_-} \right\} \tag{10}$$

where P_0, P_3, $P_\pm = P_1 \pm iP_2$ generates the Hopf subalgebra $\mathcal{T}_\rho \subset \mathcal{P}_\rho$

They obey to the twisted Leibniz rules:[13]

$$P_\mu(a \star b) = P_\mu(a) \star b + \mathcal{E}_\mu(a) \star P_\mu(b) \tag{11}$$

With the twist $\mathcal{E}_\mu = (\mathbb{I}, \mathcal{E}, \mathcal{E}^{-1}, \mathbb{I}), \quad \mathcal{E} \in \mathrm{Aut}(\mathcal{M}_\rho)$

[13] J.T.Hartwig, D.Larsson and S.D. Silvestrov, *"Deformations of Lie algebras using σ -derivations"*,Journal of Algebra,2006

Connection

We define a **twisted** connection as an application, $\nabla_\mu : \mathbb{E} \to \mathbb{E}, \quad \forall P_\mu \in \mathfrak{D}_I I = 0, \pm$ satisfying for any $m \in \mathbb{E}$, $f \in \mathcal{M}_\rho$, $z \in \mathrm{Z}(\mathcal{M}_\rho)$[14]:

$$\begin{cases} \nabla_{P_\mu + P'_\mu}(m) = \nabla_{P_\mu}(m) + \nabla_{P'_\mu}(m) \\ \nabla_{z.P_\mu}(m) = \nabla_{P_\mu}(m) \star z, \\ \nabla_{P_\mu}(m \star f) = \nabla_{P_\mu}(m) \triangleleft f + \mathcal{E}_\mu(m) \star P_\mu(f) \end{cases} \tag{12}$$

Explicite form of the connection

$$\nabla_\mu(f) = A_\mu \star f + P_\mu(f), \quad A_\mu = \nabla_\mu(\mathbb{I}) \tag{13}$$

[14](where $\mathrm{Z}(\mathcal{M}_\rho)$ is the center of ρ-Minkowski)

Curvature and gauge invariance

Curvature

$$F(P_\mu, P_\nu) := F_{\mu\nu} : \mathbb{E} \to \mathbb{E}, \qquad F_{\mu\nu} := \mathcal{E}_\mu^{-1}\nabla_\mu \mathcal{E}_\nu^{-1}\nabla_\nu - \mathcal{E}_\nu^{-1}\nabla_\nu \mathcal{E}_\mu^{-1}\nabla_\mu \tag{14}$$

Twisted gauge conditions:

$$A_\mu^g = \mathcal{E}_\mu(g^\dagger) \star A_\mu \star g + \mathcal{E}_\mu(g^\dagger) \star P_\mu(g), \qquad F_{\mu\nu}^g = \mathcal{E}_\nu \mathcal{E}_\mu u^\dagger \star F_{\mu\nu} \star u \tag{15}$$

Gauge groupe :

$$\mathcal{U}_\rho(1) = \{g \in \mathbb{E} | g^\dagger \star g = g \star g^\dagger = 1\} \tag{16}$$

Action

We define the following action for gauge theory in ρ-Minkowski:

$$S_\rho = h(F, F) = \int d^4x \; F^\dagger_{\mu\nu} \star F_{\mu\nu} \tag{17}$$

This action has the following properties:

- At the limite $\rho \to 0$ we **recover the standard Abelian gauge theory**
- **Invariant** under **gauge** symetries.
- **Invariant** under ρ-**Poincaré** symetries

The End

Thank you for your attention