Two-Loop Corrections to Top-Quark Pair Production

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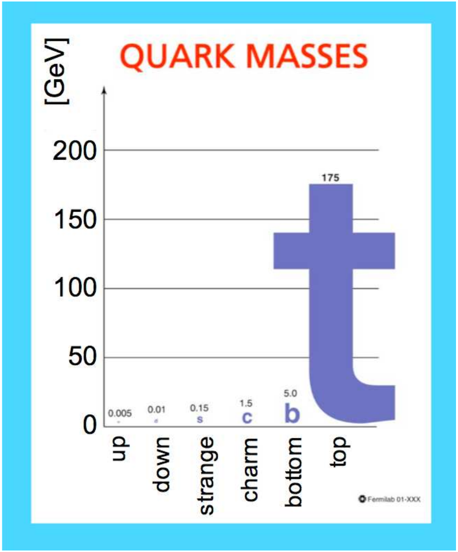
Johannes Gutenberg Universität Mainz

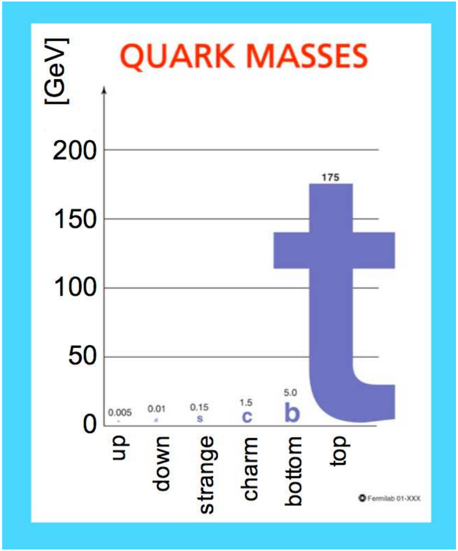
Grenoble, April 21, 2009

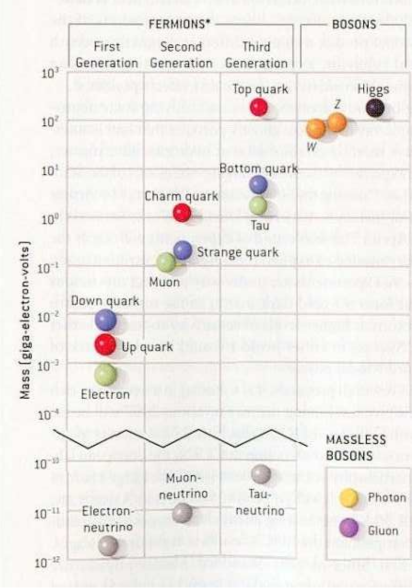
Outline

The Top Quark

A particle which
tends to stick out...




QUARK MASSES
[GeV]
200
150
100
50
0
0.005
0.01
0.15
1.5
5.0
175
c
b
t
up
down
strange
charm
bottom
top
Fermilab 01-XXX


FERMIONS*
BOSONS
First Generation
Second Generation
Third Generation
Mass (giga-electron-volts)
Top quark
Higgs
Z
W
Bottom quark
Charm quark
Tau
Strange quark
Muon
Down quark
Up quark
Electron
MASSLESS BOSONS
Muon-neutrino
Tau-neutrino
Electron-neutrino
Photon
Gluon

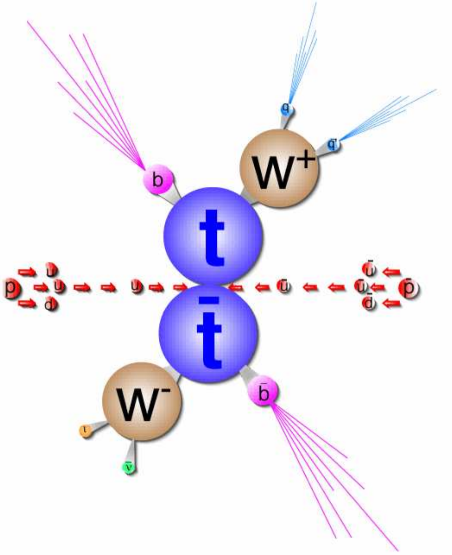
The Top Quark

- The top-quark is by far the heaviest fermion in the SM ($m_t \approx 172$ GeV)
- It decays very rapidly via EW interactions: $t \rightarrow bW$ ($\tau_t = 1/\Gamma_t \sim 5 \times 10^{-25} s$)
- The lifetime is one order of magnitude smaller than the hadronization scale ($\tau_{\text{had}} = 1/\lambda_{\text{QCD}} \approx 3 \times 10^{-24}$): the top quark decays before it can form hadronic bound states
- Because of its large mass, the top quark couples strongly to the electroweak symmetry breaking sector
- So far it was observed only at the Tevatron (few thousands top quarks produced)
- The mass of the top-quark could be measured with a percent accuracy
- Production cross-sections and couplings are know with larger uncertainties

Reviews

- M. Beneke *et al* *Top quark physics* hep-ph/0003033
- S. Dawson *The top quark, QCD, and new physics* hep-ph/0303191
- W. Wagner *Top quark physics in hadron collisions* hep-ph/0507207
- A. Quadt *Top quark physics at hadron colliders* EJPC (2006)
- W. Bernreuther *Top-quark physics at the LHC* 0805.1333

Top Quark Pairs



Top Quark Pairs

Key measurements at the Tevatron and the LHC include the top-quark pair production total cross section and differential distribution

$$p\bar{p}, pp \rightarrow t\bar{t}X$$

The top quarks decay almost exclusively in a W boson and a b jet. The observed processes are

$$\begin{aligned}
p\bar{p}, pp \rightarrow t\bar{t}X &\rightarrow l_1^+ + l_2^- + j_b + j_{\bar{b}} + p_T^{\text{miss}} + (n \geq 0) \quad \text{jets} \\
p\bar{p}, pp \rightarrow t\bar{t}X &\rightarrow l_1^\pm + j_b + j_{\bar{b}} + p_T^{\text{miss}} + (n \geq 2) \quad \text{jets} \\
p\bar{p}, pp \rightarrow t\bar{t}X &\rightarrow j_b + j_{\bar{b}} + (n \geq 4) \quad \text{jets}
\end{aligned}$$

All these channels were observed and analyzed at the Tevatron

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\end{aligned}$$

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Top Quark Pairs at the LHC

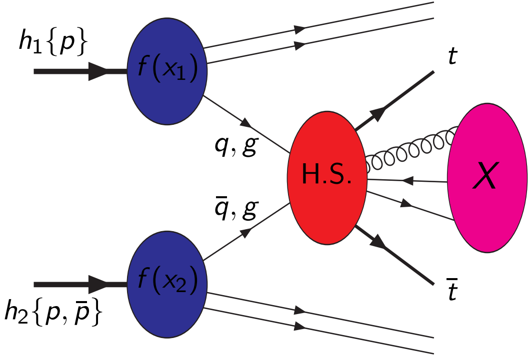
At the LHC, one expects to observe millions of top quarks already in the initial low luminosity phase ($L \sim 10\,\text{fb}^{-1}$)

With the large number of top quarks expected to be produced at the LHC, the study of its properties will become precision physics

- The total cross section is sensitive to $m_t \Longrightarrow$ mass measurement ($\Delta\sigma/\sigma \approx -5\Delta m_t/m_t$)
- LHC experiments will probe, in the $t\bar{t}$ channel, the existence of heavy resonances with masses up to several TeV
- Precise measurement of the total cross section at the LHC ($\sim 5\%$ uncertainty)
- The current theoretical predictions have an uncertainty of ($\sim 14\%$)

Top Quark Pair Hadroproduction

top-quark pair production is a hard scattering process which can be computed in perturbative QCD



$$\sigma^{t\bar{t}}_{h_1,h_2} = \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2 f_i^{h_1}(x_1, \mu_F) f_j^{h_2}(x_2, \mu_F) \hat{\sigma}_{ij}\left(\hat{s}, m_t, \alpha_s(\mu_R), \mu_F, \mu_R\right)$$

$$s = (p_{h_1} + p_{h_2})^2 \; , \; \hat{s} = x_1 x_2 s$$

Top Quark Pair Hadroproduction -II

$$\sigma_{h_1,h_2}^{t\bar{t}}(s_{\text{had}}, m_t^2) = \sum_{ij} \int_{4m_t^2}^{s_{\text{had}}} d\hat{s}\; L_{ij}\left(\hat{s}, s_{\text{had}}, \mu_f^2\right) \hat{\sigma}_{ij}(\hat{s}, m_t^2, \mu_f^2, \mu_r^2)$$

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where the luminosity L_{ij} is defined as

$$L_{ij}\left(\hat{s}, s_{\rm had}, \mu_f^2\right) = \frac{1}{s_{\rm had}} \int_{\hat{s}}^{s_{\rm had}} \frac{ds'}{s'} f_i^{h_1}\left(\frac{s'}{s_{\rm had}}, \mu_{\rm F}\right) f_j^{h_2}\left(\frac{\hat{s}}{s'}, \mu_{\rm F}\right)$$

Top Quark Pair Hadroproduction -II

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Partonic Cross Section

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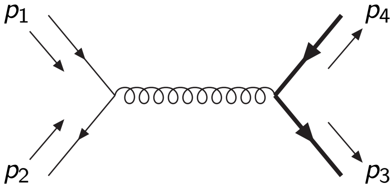
Tree Level QCD Partonic Processes

$$q(p_1) + \bar{q}(p_2) \longrightarrow t(p_3) + \bar{t}(p_4)$$



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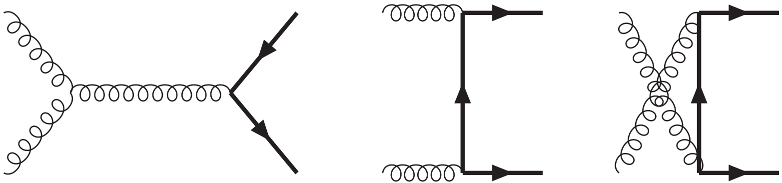

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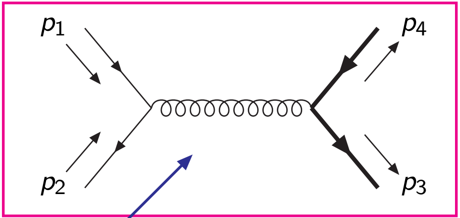

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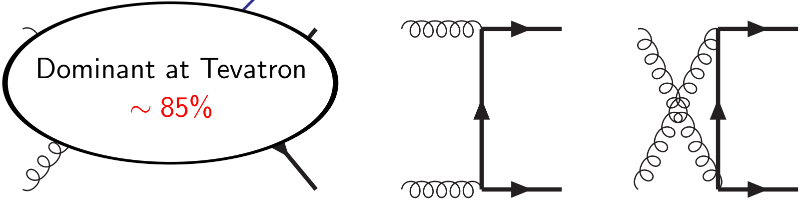

Dominant at Tevatron

$\sim$ 85%

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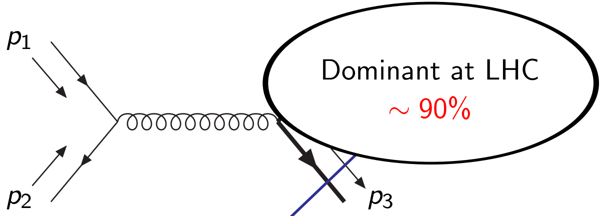

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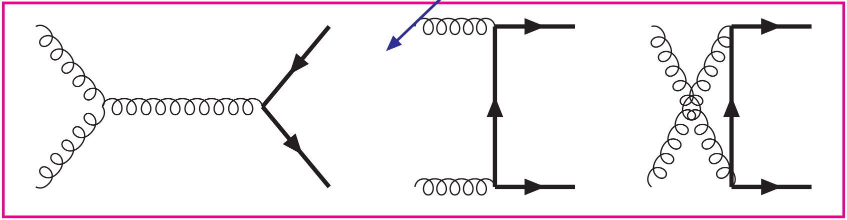

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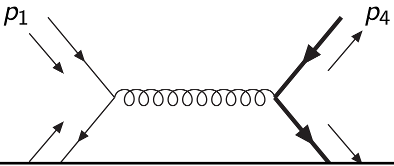
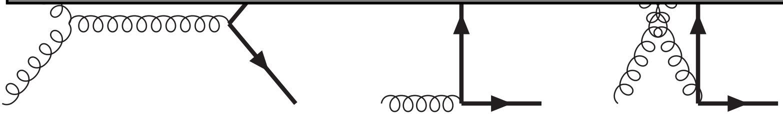


The NLO QCD corrections in both channels
(and to $qg \to t\bar{t}q$) are known since a long time

Nason, Dawson, Ellis ('88-'90) Beenakker, Kuijf, van Neerven, Smith ('89) Beenakker, van Neerven, Meng, Schuler ('91) Mangano, Nason, Ridolfi ('92) Frixione,Mangano, Nason, Ridolfi ('95)



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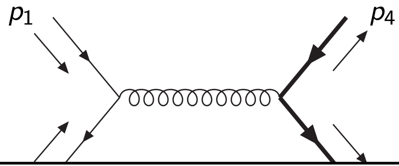
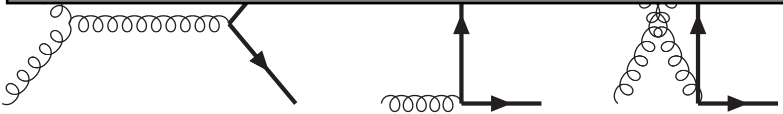
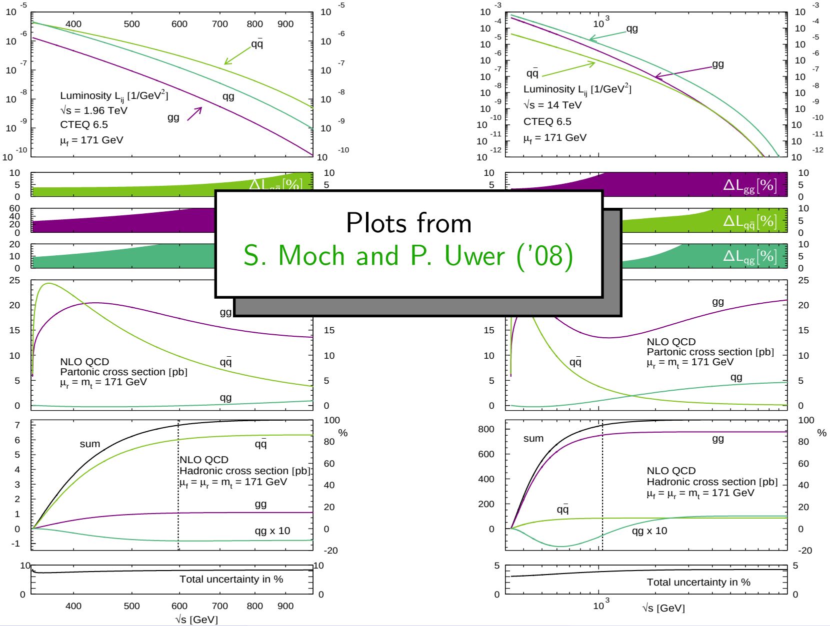


The mixed QCD-EW corrections in both channels are also known (they are smaller than current QCD uncertainties)

Beenakker *et al.* ('94) Bernreuther, Fuecker, and Si ('05-'08)
Kühn, Scharf, and Uwer ('05-'06)
Moretti, Nolten, and Ross ('06)



Luminosity and Partonic CS at NLO



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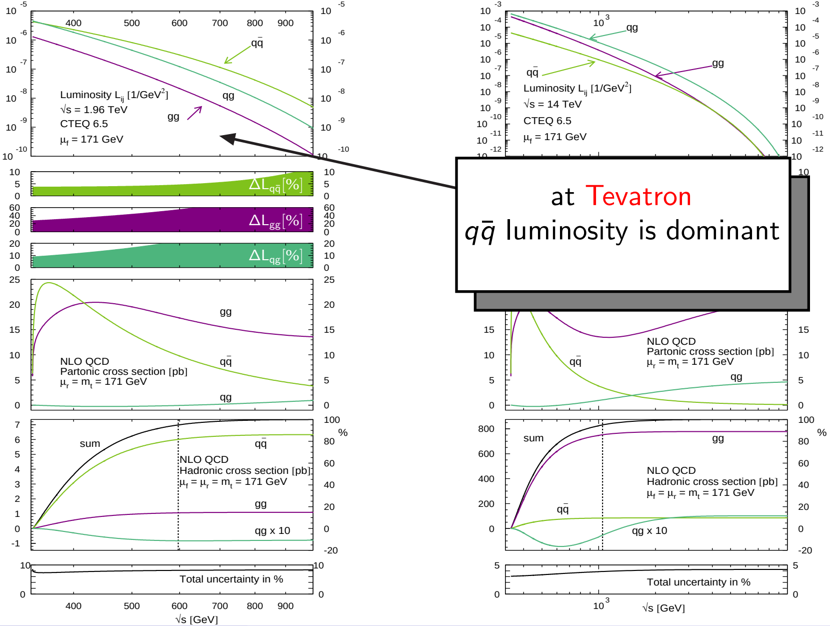
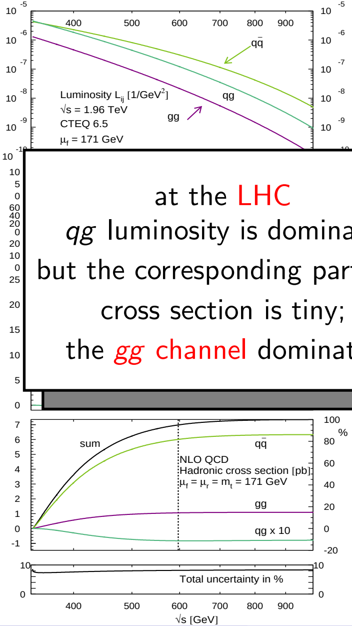


at Tevatron

$q\bar{q}$ luminosity is dominant

Luminosity and Partonic CS at NLO



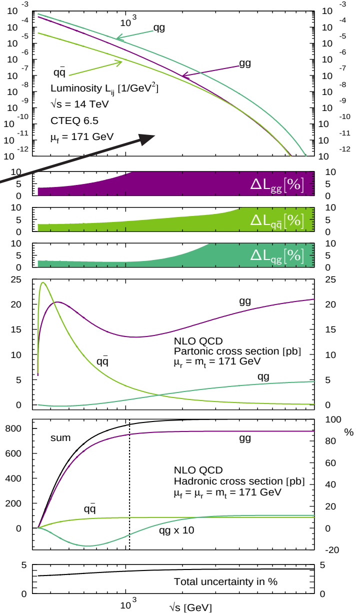
at the LHC

qg luminosity is dominant

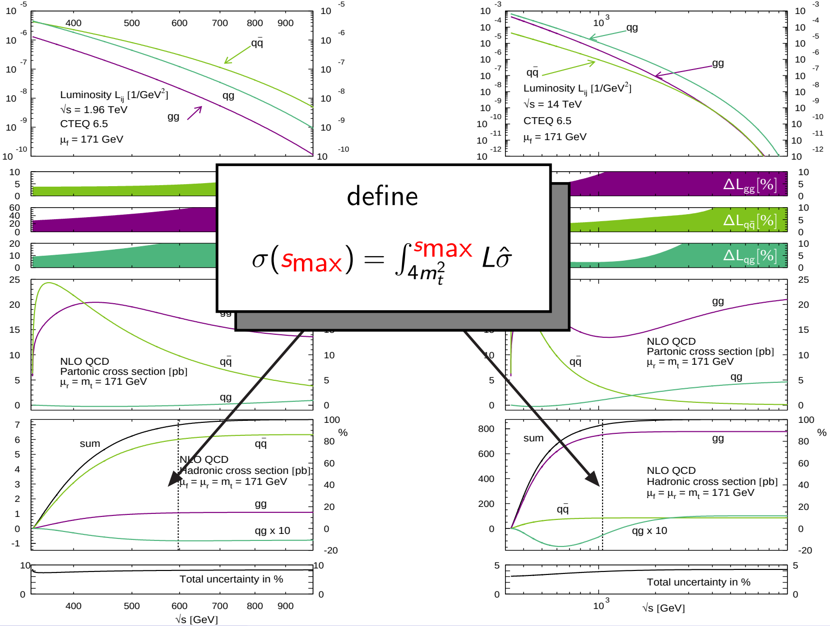
but the corresponding partonic

cross section is tiny;

the *gg* channel dominates



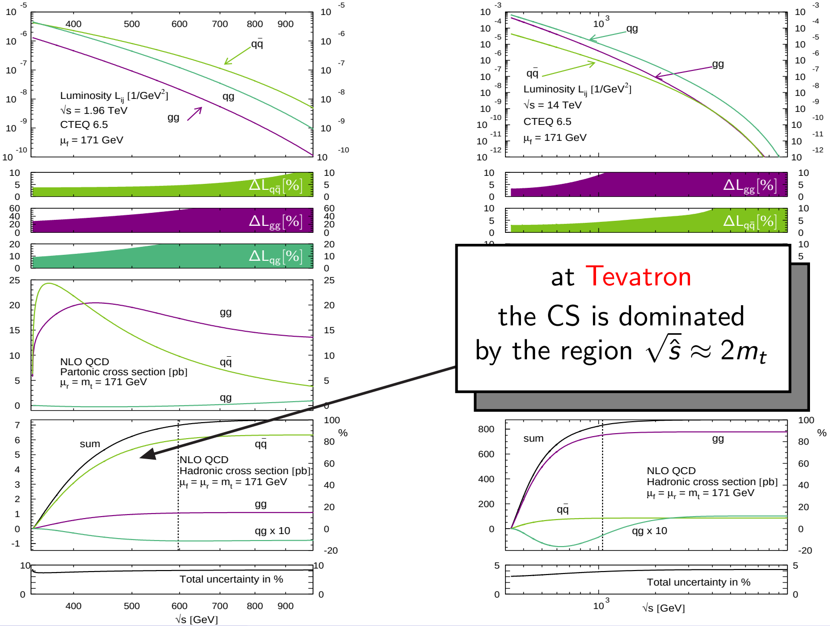
Luminosity and Partonic CS at NLO



define

$$\sigma(s_{max}) = \int_{4m_t^2}^{s_{max}} L\hat{\sigma}$$

Luminosity and Partonic CS at NLO



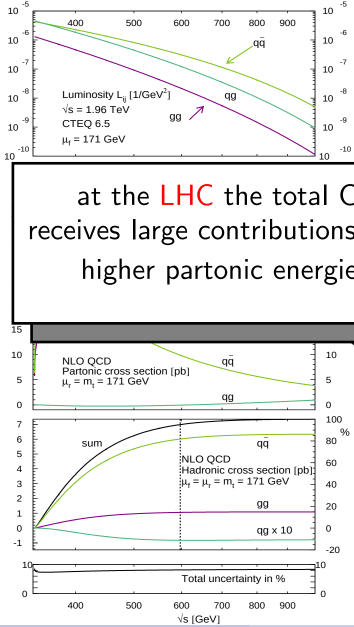
at Tevatron

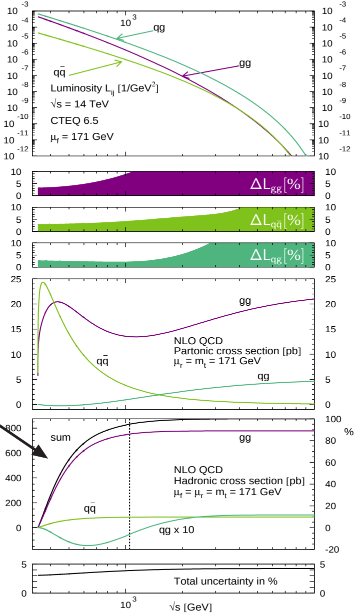
the CS is dominated

by the region $\sqrt{\hat{s}} \approx 2m_t$

Luminosity and Partonic CS at NLO

at the LHC the total CS
receives large contributions from
higher partonic energies

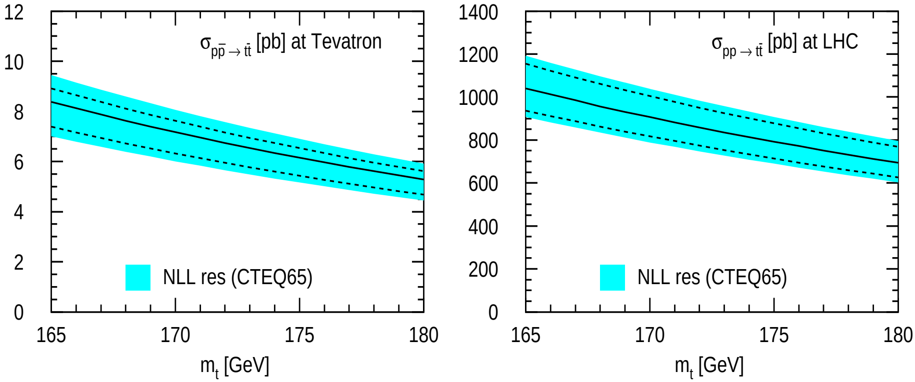




Uncertainty on NLO

The partonic cross sections involve terms like $\ln^2(1 - 4m^2/s)$ which become large threshold and must be resummed

Kidonakis and Sterman ('97), Bonciani *et al.* ('98), Kidonakis *et al.*('01), Kidonakis and Vogt ('03), Banfi and Laenen ('05)



S. Moch and P. Uwer ('08)

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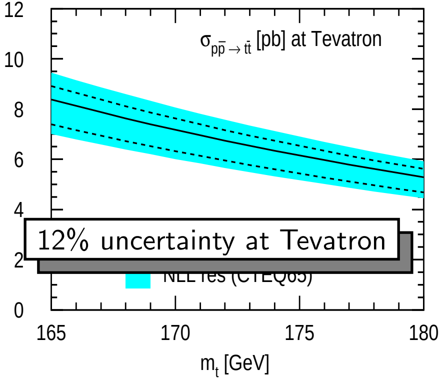
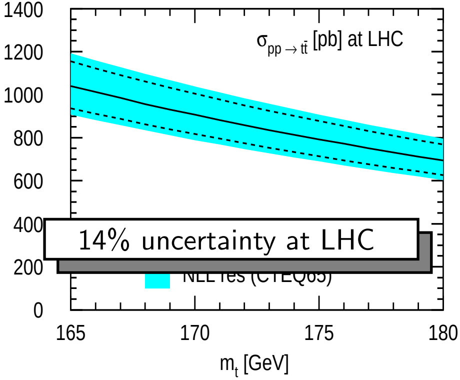


12% uncertainty at Tevatron



14% uncertainty at LHC

S. Moch and P. Uwer ('08)

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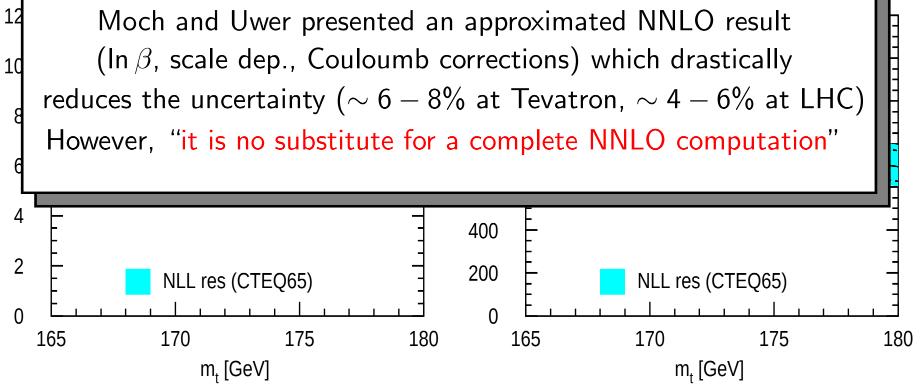
Moch and Uwer presented an approximated NNLO result ($\ln\beta$, scale dep., Couloumb corrections) which drastically reduces the uncertainty ($\sim 6-8\%$ at Tevatron, $\sim 4-6\%$ at LHC)

However, "it is no substitute for a complete NNLO computation"



S. Moch and P. Uwer ('08)

TWO-LOOP CORRECTIONS TO $q\bar{q} \to t\bar{t}$

The NNLO calculation of the top-quark pair hadroproduction requires three ingredient

- two-loop matrix elements for $q\bar{q} \to t\bar{t}$ and $gg \to t\bar{t}$
- one-loop matrix elements for the hadronic production of $t\bar{t} + 1$ parton
- tree-level matrix elements for the hadronic production of $t\bar{t} + 2$ partons

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Dittmaier, Uwer, Wenzierl ('07)

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Overall, in the $q\bar{q} \to t\bar{t}$ channel, `QGRAF` generates 218 two-loop diagrams (one massive flavor, one massless flavor):

- 31 non-vanishing diagrams with a massless quark loop
- 30 non-vanishing diagrams with a massive quark loop
- 64 non-vanishing planar "gluonic" diagrams

TWO-LOOP CORRECTIONS TO $q\bar{q} \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

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$$\boxed{\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}}$$

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

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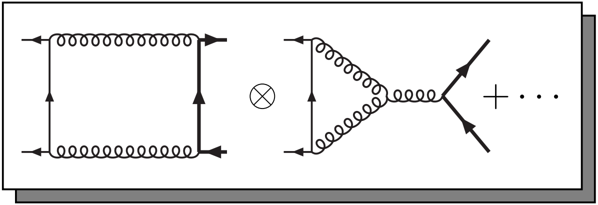
One-Loop × One-Loop

Körner, Merebashvili, Rogal ('05,'08)

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times0)} + \mathcal{A}_2^{(1\times1)}$$



Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

Two-Loop $\times$ Tree

$$\ldots^2\alpha_s^2 \left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

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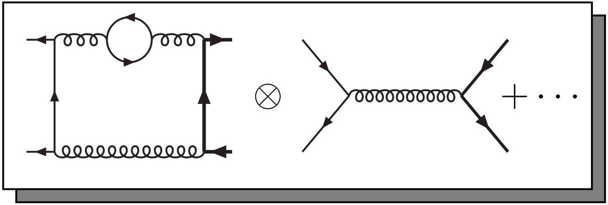

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

$$|\mathcal{M}|^2 (s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times 0)} \quad = \quad & N_c C_F \Big[N_c^2 A + B + \frac{C}{N_c^2} + N_l \left(N_c D_l + \frac{E_l}{N_c}\right) \\ & + N_h \left(N_c D_h + \frac{E_h}{N_c}\right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h \Big]\end{aligned}$$

10 different color coefficients

It is necessary to evaluate two-loop four-point functions depending on s,t, m_t^2

Massive from Massless $(s, |t|, |u| \gg m_t^2)$

It is necessary to evaluate two-loop four-point functions depending on $s, t,\ m_t^2$

start by considering the limit

$s, |t|, |u| \gg m_t^2$

MASSIVE FROM MASSLESS ($s, |t|, |u| \gg m_t^2$)

Is it possible to calculate graphs employing DIM REG to regulate both soft and collinear singularities and then translate a posteriori the collinear poles into collinear logs $\ln(m^2/s)$?

Massive from Massless $(s, |t|, |u| \gg m_t^2)$

Is it possible to calculate graphs employing DIM REG to regulate both soft and collinear singularities and then translate a posteriori the collinear poles into collinear logs $\ln(m^2/s)$?

For a generic QED/QCD process, with no closed fermion loops

$$\mathcal{M}^{(m\neq 0)} = \prod_{i\in\{\text{all legs}\}} Z_i^{\frac{1}{2}}(m,\varepsilon)\mathcal{M}^{(m=0)}$$

where Z is defined through the Dirac form factor

$$F^{(m\neq 0)}(Q^2) = Z(m,\varepsilon)\, F^{(m=0)}(Q^2) + \mathcal{O}(m^2/Q^2)$$

A. Mitov and S. Moch ('06)

Two-Loop Virtual Corrections in the $s \gg m_t^2$ Limit

M. Czakon, A. Mitov, S. Moch ('07)

Using the universal multiplicative relation between massless QCD amplitudes and massive amplitudes in the small mass limit, it was possible to calculate the two-loop virtual corrections to $q\bar{q} \to t\bar{t}$ starting from $q\bar{q} \to q'$(massless)$\bar{q}'$(massless) (C. Anastasiou *et al.* ('00))

The same result was also obtained by an approach based on the reduction to master integrals and the expansion of the master integrals in m^2/s through Mellin Barnes representations

Numerical Evaluation of the Two-Loop Corrections to $q\bar{q} \rightarrow t\bar{t}$

M. Czakon ('08)

Adding more terms in the expansion in powers of $m^2/s, m^2/|t|, m^2/|u|$ it is not sufficient to reach a permill accuracy in all the phase space (particularly near threshold)

The MIs can be evaluated by solving the corresponding differential equations numerically

- The evaluation of the full color structure with a 16 digit precision can require as much as 15 minutes per phase space point
- This does not allow for a direct implementation in a MC generator, but the functions are smooth enough to be interpolated starting from a grid of values

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Fermion Loop Corrections

7 color structures receive contributions from the diagrams involving closed fermion loops only

- N_l number of light (massless) quarks
- N_h number of heavy (massive) quarks

$$\mathcal{A}_2^{(2\times 0)} = N_c C_F \Big[N_c^2 A + B + \frac{C}{N_c^2} + N_l \left(N_c D_l + \frac{E_l}{N_c} \right) + N_h \left(N_c D_h + \frac{E_h}{N_c} \right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h \Big]$$

7 co ... sed

ferm

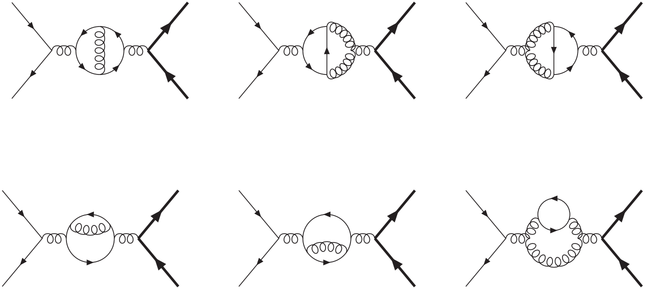
All the diagrams with a closed quark loop (massive or massless) were evaluated analytically

Bonciani, AF, Gehrmann, Maître, Studerus ('08)

Agreement with the numerical results by Czakon

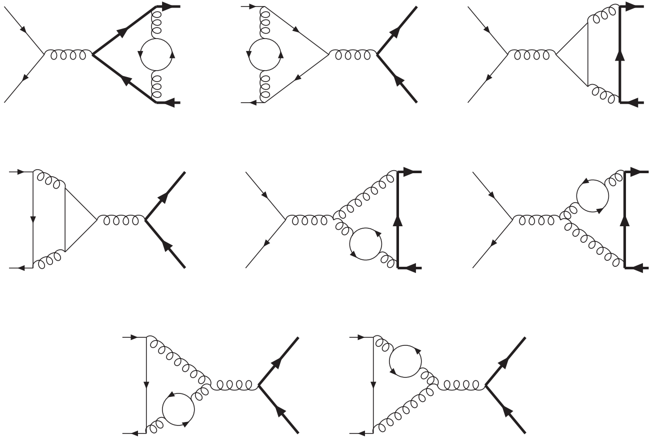
$$\mathcal{A}_2^{(2\times 0)} = N_c C_F \Big[N_c^2 A + B + \frac{C}{N_c^2} + N_l \left(N_c D_l + \frac{E_l}{N_c} \right) + N_h \left(N_c D_h + \frac{E_h}{N_c} \right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h \Big]$$

Fermionic Diagrams



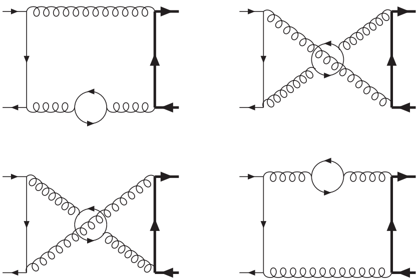
Self Energies

Fermionic Diagrams



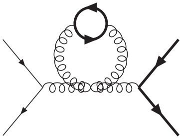
Vertices

Fermionic Diagrams

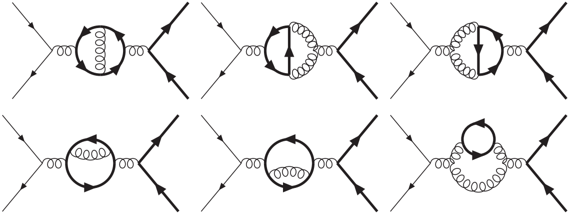


Boxes

Fermionic Diagrams



Tadpole



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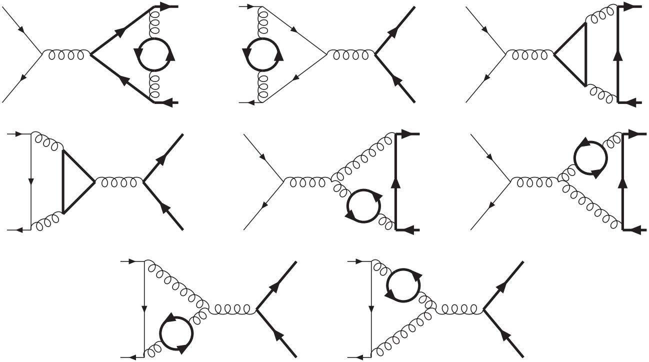

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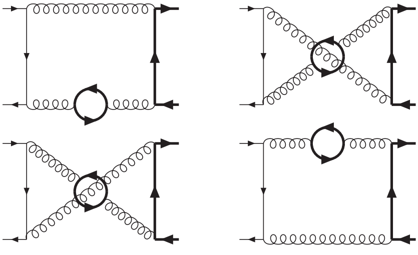
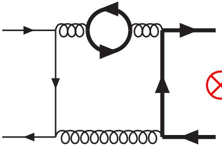


Boxes

Method: The General Strategy

After interfering a two-loop graph with the Born amplitude one obtains a linear combinations of scalar integrals



$\otimes$ Born Amplitude

$$\int \mathfrak{D}^d k_1 \mathfrak{D}^d k_2 \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}}$$

k_i	$\rightarrow$	integration momenta
p_i	$\rightarrow$	external momenta
S	$\rightarrow$	scalar products $k_i \cdot k_j$ or $k_i \cdot p_k$
$\mathcal{D}$	$\rightarrow$	propagators $[\sum c_i k_i + \sum d_j p_j]^2 \ (+m_t^2)$

Luckily, just a "small" number of these integrals are independent: the MIs

It is necessary to

- identify the MIs $\Longrightarrow$ Reduction through the Laporta Algorithm
- calculate the MIs $\Longrightarrow$ Differential Equation Method

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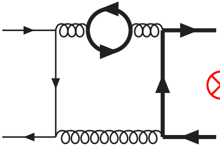

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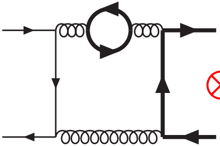

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The Laporta Algorithm

The set of denominators $\mathcal{D}_1, \cdots, \mathcal{D}_t$ defines a topology; for each topology

- The scalar integrals are related via Integration By Parts identities (10 identities per integral for a two-loop four-point function)

$$\int \mathfrak{D}^d k_1 \mathcal{D}^d k_2 \frac{\partial}{\partial k_i^{\mu}} \left[v^{\mu} \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right] = 0 \quad v^{\mu} = k_1, k_2, p_1, \cdots, p_3$$

- Building the IBPs for growing powers of the propagators and scalar products the number of equations grows faster that the number of unknown: one finds a system of equations which is apparently over-constrained
- Solving the system of IBPs (in a problem with a small number of scales) one finds that only a few of the scalar integrals above (if any) are independent: the MIs.

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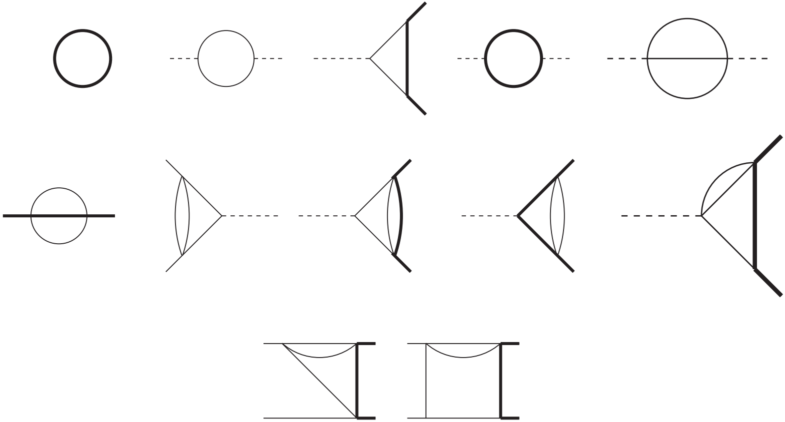
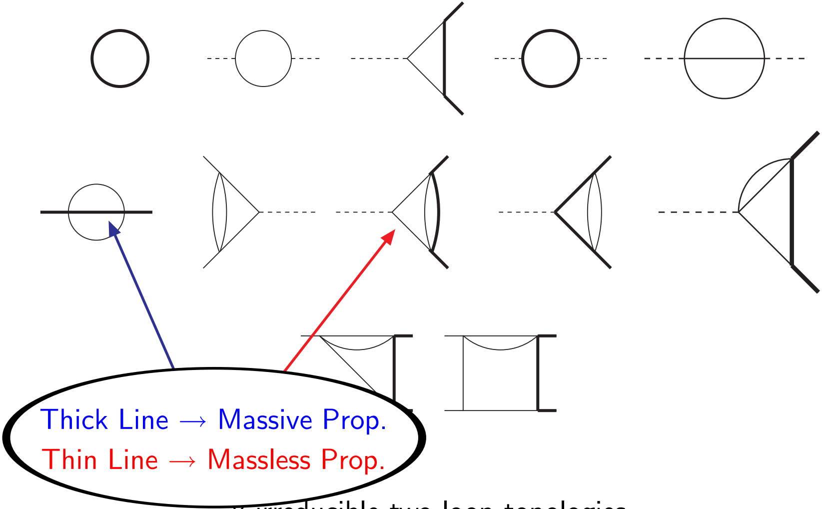


8 irreducible two-loop topologies

N_l Master Integrals



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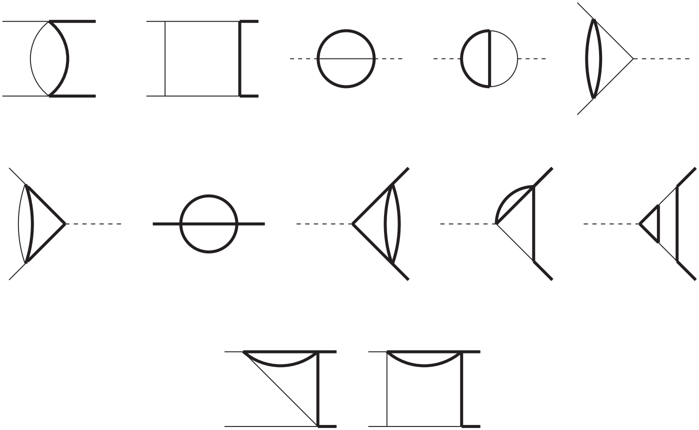
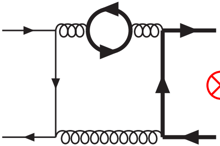
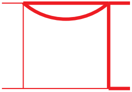
N_h Master Integrals



10 irreducible two-loop topologies

The two-loop box diagrams entering in the calculation of the heavy-fermion loop corrections are reducible, i.e. they can be rewritten in terms of integrals belonging to the subtopologies only:

 $\otimes$ Born Amplitude $\Longrightarrow \sum C$  $+$ Triangles

$+$ Bubbles $+$ Tadpoles

Calculation of the MIs: Differential Equation Method

For each Master Integral belonging to a given topology $F_l^{(q)} \to \{\mathcal{D}_1, \cdots, \mathcal{D}_q\}$

- Take the derivative of a given integrals with respect to the external momenta p_i

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} F_l^{(q)} = p_j^\mu \int \mathfrak{D}^d k_1 \mathfrak{D}^d k_2 \frac{\partial}{\partial p_i^\mu} \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_q^{m_q}}$$

- The integral are regularized, therefore we can apply the derivative to the integrand in the r. h. s. and use the IBPs to rewrite it as a linear combination of the MIs
- Rewrite the diff. eq. in terms of derivatives with respect to s and t
- Fix somehow the initial condition(s) (ex. knowing the behavior of the integral at $s = 0$) and solve the system of DE(s)

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$$p_j^\mu \int \mathfrak{D}^d k_1 \mathfrak{D}^d k_2 \frac{\partial}{\partial p_i^\mu} \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_q^{m_q}} = \sum c_i F_i^{(q)} + \sum_{r \neq q} \sum_j k_j F_j^{(r)}$$

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$$\frac{\partial}{\partial s} F_l^{(q)}(s,t) = \sum_j c_j(s,t) F_j^{(q)}(s,t) + \sum_{r \neq q} \sum_l k_l(s,t) F_l^{(r)}(s,t)$$

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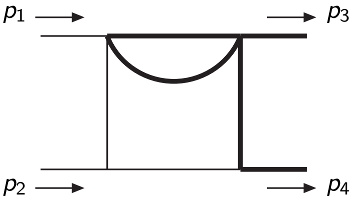
Five Denominator MIs

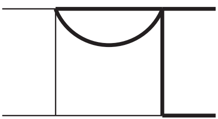
The most complicated irreducible topology in the calculation of the heavy fermion loop corrections is a five denominator box with two MIs
(thick lines indicate massive propagators, thin lines massless ones)

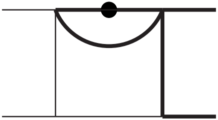


$$M(\alpha_1,\ldots,\alpha_9)=\int\frac{\mathfrak{D}^d k_1 \mathfrak{D}^d k_2\,(p_2\cdot k_1)^{\alpha_6}(p_1\cdot k_2)^{\alpha_7}(p_2\cdot k_2)^{\alpha_8}(p_3\cdot k_2)^{\alpha_9}}{P_0^{\alpha_1}(k_1+p_1)P_0^{\alpha_2}(k_1+p_1+p_2)P_m^{\alpha_3}(k_2)P_m^{\alpha_4}(k_1-k_2)P_m^{\alpha_5}(k_1+p_3)}$$

$$P_0(q)=q^2 \qquad\qquad P_m(q)=q^2+m^2$$

Five Denominator MIs-II

$M_1 =$ 

$M_2 =$ 

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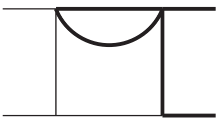
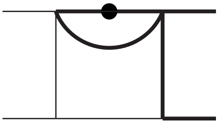
$M_1 =$  $M_2 =$ 

- the two MIs satisfy two independent first order differential equations

$$\frac{dM_i(s,t)}{dt} = C_i(s,t)M_i(s,t) + \Omega_i(s,t)$$

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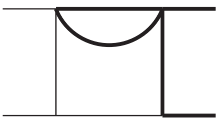
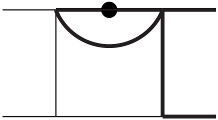
$M_1 =$ 

$M_2 =$ 

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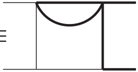
$$\frac{dM_i(s,t)}{dt} = C_i(s,t)M_i(s,t) + \Omega_i(s,t)$$

- One of the two needed initial conditions can be fixed by imposing the regularity of the integrals in $t = 0$
  The second integration constant can be fixed by calculating the integral in $t = 0$ with MB techniques

By solving the differential equation(s) it is possible to obtain analytic expressions for the MIs ($s = -m^2(1-x)^2/x\,,\ \ y = -t/m^2$)

$M_1(d,m,t,s) \quad \equiv$  $= \dfrac{A^{(-3)}}{(d-4)^3} + \dfrac{A^{(-2)}}{(d-4)^2} + \dfrac{A^{(-1)}}{(d-4)} + A^{(0)}$

$$
\begin{aligned}
A_{-3} &= \frac{1}{32(y+1)} \\
A_{-2} &= \frac{1}{32(x-1)(y+1)}\Big[-2G(-1;y)(x-1)+2(x-1)-(1+x)G(0;x)\Big] \\
A_{-1} &= \frac{1}{32(x-1)(y+1)}\Big[3\zeta(2)x+4x+6(x+1)G(-1,0;x)+(-5x-1)G(0,0;x) \\
&\quad -4(x-1)G(-1;y)+G(0;x)(2(x+1)G(-1;y)-2(x+1)) \\
&\quad +4(x-1)G(-1,-1;y)-2(x-1)G(0,-1;y)+3\zeta(2)-4\Big] \\
A_{0} &= \frac{1}{32(x-1)(y+1)}\Big[-36(x+1)G(-1,-1,0;x)+18(x+1)G(-1,0,0;x) \\
&\quad +6(5x+1)G(0,-1,0;x)+(-5x-1)G(0,0,0;x)+\cdots
\end{aligned}
$$

Analytic Expression for the MIs

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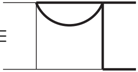
$M_1(d, m, t, s) \equiv$  $= \dfrac{A^{(-3)}}{(d-4)^3} + \dfrac{A^{(-2)}}{(d-4)^2} + \dfrac{A^{(-1)}}{(d-4)} + A^{(0)}$

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$$A_{-1} = \frac{1}{32(x-1)(y+1)}\Big[3\zeta(2)x + 4x + 6(x+1)G(-1,0;x) + (-5x-1)G(0,0;x)$$

A_0

The analytic results depend on ln, Li_2, Li_3, $S_{1,2}$

They are conveniently expressed in terms of HPLs and 2dHPLs

(Remiddi, Vermaseren, Gehrmann)

Harmonic Polylogarithms (HPLs)

E. Remiddi, J. Vermaseren (1999)

E. Remiddi, T. Gehrmann (2001)

Functions of the variable x and a set of indices $\vec{a}$ with weight w; each index can assume values $1, 0, -1$

$$H(\mathbf{a}; x)$$

Definitions: $w = 1$

$$\begin{aligned} H(1;x) &= \int_0^x \frac{dt}{1-t} = -\ln(1-x) \\ H(0;x) &= \ln x \\ H(-1;x) &= \int_0^x \frac{dt}{1+t} = \ln(1+x) \end{aligned}$$

$$\frac{d}{dx} H(a;x) = f(a;x) \quad f(1;x) = \frac{1}{1-x} \quad f(0;x) = \frac{1}{x} \quad f(-1;x) = \frac{1}{1+x}$$

Definitions: $w > 1$

$$\text{if } \vec{a} = 0, 0, \ldots, 0 \ (w \text{ times}) \quad H(\vec{0}_w; x) = \frac{1}{w!} \ln^w x$$

$$\text{else} \quad H(i, \vec{a}; x) = \int_0^x dt f(i; t) H(\vec{a}; t)$$

consequences: $\frac{d}{dx} H(i, \vec{a}; x) = f(i; x) H(\vec{a}; x) \quad H(\vec{a} \notin \vec{0}; 0) = 0$

a few examples @ $w = 2$

$$H(0, 1; x) = \int_0^x dt f(0; t) H(1; t) = -\int_0^x dt \frac{1}{t} \ln(1 - t) = \mathrm{Li}_2(x)$$

$$H(1, 0; x) = \int_0^x dt f(1; t) H(0; t) = \int_0^x dt \frac{1}{1 - t} \ln t$$

$$= -\ln x \ln(1 - x) + \mathrm{Li}_2(x)$$

HPLs as a Generalization of the Nielsen's PolyLogs

The HPLs include the Nielsen's PolyLogs

$$S_{n,p(x)} = \frac{(-1)^{n+p-1}}{(n+p)!p!}\int_0^1 \frac{dt}{t}\ln^{n-1} t \ln^p(1-xt) \quad \mathrm{Li}_n(x) = S_{n-1,1}(x)$$

$$\begin{aligned} \mathrm{Li}_n(x) &= H(\vec{0}_{n-1}, 1; x) \\ S_{n,p}(x) &= H(\vec{0}_n, \vec{1}_p; x) \end{aligned}$$

but the HPLs are a larger set of functions: from $w = 4$ one finds things as

$$H(-1,0,0,1;x) = \int_0^x \frac{dt}{1+t}\mathrm{Li}_3(x) \notin \sum \text{Nielsen's PolyLogs}$$

The HPLs Algebra

- Shuffle Algebra:

$$H(\vec{p};x)H(\vec{q};x) = \sum_{\vec{r}=\vec{p}\uplus\vec{q}} H(\vec{r};x)$$

some examples

$$\begin{aligned} H(a;x)H(b;x) &= H(a,b;x) + H(b,a;x) \\ H(a;x)H(b,c;x) &= H(a,b,c;x) + H(b,a,c;x) + H(b,c,a;x) \end{aligned}$$

- Product Ids:

$$\begin{aligned} H(m_1,\ldots,m_q;x) &= H(m_1;x)H(m_2,\ldots,m_q;x) \\ &- H(m_2,m_1;x)H(m_3,\ldots,m_q;x) \\ &+ \cdots + (-1)^{q+1}H(m_q,\ldots,m_1;x) \end{aligned}$$

2-dimensional Harmonic Polylogarithms (2dHPLs)

E. Remiddi, T. Gehrmann (2000)

As for the HPLs, they are obtained by repeated integration over a new set of factors depending on a second variable.

$$f(-y;x) = \frac{1}{x+y} \quad f(-1/y;x) = \frac{1}{x+1/y}$$

$$G(i,\vec{a};x) = \int_0^x dt f(i;t) G(\vec{a};t)$$

a few examples:

$$G(-y;x) = \int_0^x \frac{dz}{z+y} = \ln\left(1+\frac{x}{y}\right) \quad G(-1/y;x) = \int_0^x \frac{dz}{z+1/y} = \ln\left(1+xy\right)$$

$$G(-y,0;x) = \ln x \ln\left(1+\frac{x}{y}\right) + \mathrm{Li}_2\left(-\frac{x}{y}\right)$$

2-dimensional Harmonic Polylogarithms (2dHPL)-II

The 2dHPLs share the properties of the HPLs

The analytic properties of both HPLs & 2dHPLs are know
Codes for their numerical evaluation are available

E. Remiddi, T. Gehrmann (2001-2002)

Up to $w = 3$ (our case) the 2dHPLs can be expressed in terms of $\ln$, Li_2, Li_3, $S_{1,2}$

$$
\begin{aligned}
G(-1/y;x) &= \ln(1+xy)\,, \\
G(-1/y,0;x) &= \ln(x)\ln(xy+1)+\mathrm{Li}_2(-xy)\,, \\
G(-y,1;x) &= \frac{1}{2}\ln^2(y+1)-\ln(1-x)\ln(y+1)-\ln(y)\ln(y+1) \\
&\quad +\ln(1-x)\ln(x+y)-\mathrm{Li}_2(-y)+\mathrm{Li}_2\left(\frac{1-x}{y+1}\right)-\frac{\pi^2}{6}\,, \\
G(-y,1,0;x) &= -\frac{1}{3}\ln^3(1-x)-\ln(x)\ln^2(1-x)+\cdots
\end{aligned}
$$

HPLs as Multiple Sums

S. Weinzierl and J. Vollinga ('04)

We defined (2-d)HPLs in terms of iterated integrations

$$G(z_1, \cdots, z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \cdots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$$

but they can be written also in terms of multiple sums

$$G(\underbrace{0, \cdots, 0}_{m_1 - 1}, z_1, \cdots, z_{k-1}, \underbrace{0, \cdots, 0}_{m_k - 1}, z_k; y) \equiv G_{m_1, \cdots, m_k}(z_1, \cdots, z_k; y)$$

$$\begin{aligned} G_{m_1, \cdots, m_k}(z_1, \cdots, z_k; y) &= \sum_{j_1=1}^{\infty} \cdots \sum_{j_1=1}^{\infty} \frac{1}{(j_1 + \cdots + j_k)^{m_1}} \left(\frac{y}{z_1}\right)^{j_1} \times \\ &\times \frac{1}{(j_2 + \cdots + j_k)^{m_2}} \left(\frac{y}{z_1}\right)^{j_2} \cdots \frac{1}{j_k^{m_k}} \left(\frac{y}{z_1}\right)^{j_k} \end{aligned}$$

Results

All results are written in analytic form:

```
[F1] = + 50/81
       - 80/27*[1-x]^-6
       + 80/9*[1-x]^-5
       - 620/81*[1-x]^-4
       + 40/81*[1-x]^-3
       - 20/81*[1-x]^-2
       + 40/27*[1-x]^-1
       - 160/27*y*[1-x]^-6
       + 160/9*y*[1-x]^-5
       - 1480/81*y*[1-x]^-4
       + 560/81*y*[1-x]^-3
       + 20/27*y*[1-x]^-2
       - 100/81*y*[1-x]^-1
       - 80/27*y^2*[1-x]^-6
       + 80/9*y^2*[1-x]^-5
       - 620/81*y^2*[1-x]^-4
       + 40/81*y^2*[1-x]^-3
       + 100/81*y^2*[1-x]^-2
       - 80/27*G(0,x)*[1-x]^-7
       + 232/27*G(0,x)*[1-x]^-6
       - 176/27*G(0,x)*[1-x]^-5
       - 8/9*G(0,x)*[1-x]^-4
       + 8/27*G(0,x)*[1-x]^-3
       + 4/3*G(0,x)*[1-x]^-2
       + 8/9*G(0,x)*[1-x]^-1
       - 160/27*G(0,x)*y*[1-x]^-7
       + 464/27*G(0,x)*y*[1-x]^-6
       - 16*G(0,x)*y*[1-x]^-5
       - 224/27*G(1,x)*y*[1-x]^-3
       - 8/9*G(1,x)*y*[1-x]^-2
       + 40/27*G(1,x)*y*[1-x]^-1
       + 32/9*G(1,x)*y^2*[1-x]^-6
       - 32/3*G(1,x)*y^2*[1-x]^-5
       + 248/27*G(1,x)*y^2*[1-x]^-4
       - 16/27*G(1,x)*y^2*[1-x]^-3
       - 40/27*G(1,x)*y^2*[1-x]^-2
       + ...
       + 32/9*G(1,0,x)*[1-x]^-7
       - 112/9*G(1,0,x)*[1-x]^-6
       + 128/9*G(1,0,x)*[1-x]^-5
       - 40/9*G(1,0,x)*[1-x]^-4
       - 16/9*G(1,0,x)*[1-x]^-2
       + 64/9*G(1,0,x)*y*[1-x]^-7
       - 224/9*G(1,0,x)*y*[1-x]^-6
       + 32*G(1,0,x)*y*[1-x]^-5
       - 160/9*G(1,0,x)*y*[1-x]^-4
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       - 16/9*G(1,0,x)*y^2*[1-x]^-3
       + 8/9*G(1,0,x)*y^2*[1-x]^-2
;
```

Results-II

The color factors can be easily evaluated numerically (either with `Mathematica` or with `Fortran` code):

```
################################################

 Fl  =          4.93507679E+01
 Flh =          9.87015354E+01
 Fh  =          4.93507675E+01

 El  =   1/ep^3*(        2.05000001E-01)
       + 1/ep^2*(        1.02365896E+00)
       + 1/ep  *(       -1.81803776E+02)
       +        (        2.67510489E+03)

 Dl  =   1/ep^3*(       -2.05000001E-01)
       + 1/ep^2*(       -4.51882816E-01)
       + 1/ep  *(        2.07220644E+02)
       +        (       -4.46537433E+03)

 Eh  =   1/ep^2*(        5.78057996E+00)
       + 1/ep  *(       -2.10928847E+02)
       +        (        2.77869548E+03)

 Dh  =   1/ep^2*(       -5.78057996E+00)
       + 1/ep  *(        2.40508604E+02)
       +        (       -4.58561923E+03)

################################################
```

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       +        (       -4.58561923E+03)

################################################
```

- by expanding the results for $s \gg m_t^2$ we find analytic agreement with `Czakon, Mitov, and Moch ('07)`
- we we find numerical agreement with `Czakon ('08)`
- we obtained analytic expression in the $\beta = \sqrt{1 - 4m^2/s} \to 0$ limit

PLANAR DIAGRAMS IN $q\bar{q} \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned} \mathcal{A}_2^{(2\times 0)} \quad = \quad & N_c C_F \Big[N_c^2 A + B + \frac{C}{N_c^2} + N_l \left(N_c D_l + \frac{E_l}{N_c} \right) \\ & + N_h \left(N_c D_h + \frac{E_h}{N_c} \right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h \Big] \end{aligned}$$

Planar Diagrams in $q\bar{q} \to t\bar{t}$

The coefficient of N_c^2 can also be evaluated analytically

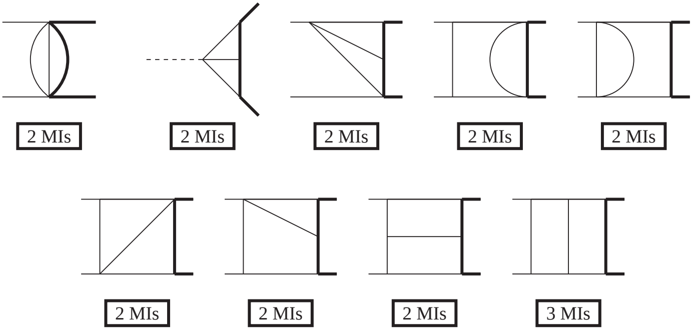
it involves only planar diagrams

Bonciani, AF, Gehrmann, Studerus (in progress)

$$\mathcal{A}_2^{(2\times 0)} = N_c C_F \Big[N_c^2 A + B + \frac{C}{N_c^2} + N_l \left(N_c D_l + \frac{E_l}{N_c} \right) + N_h \left(N_c D_h + \frac{E_h}{N_c} \right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h \Big]$$

MIs for the planar diagrams

With respect to the quark loop case, there are 9 new irreducible topologies



C. Studerus ('09)

New Issues and Technical Problems

- It is necessary to introduce and study 2dHPLs depending on new weight functions

$$\frac{1}{t - \frac{1}{2}\left(1 \pm \sqrt{3}\right)}, \qquad \frac{1}{t - \left(1 - \frac{1}{x} - x\right)}$$

- For the calculation of all the planar diagrams we need HPLs and 2dHPLs up to weight 4 (we needed only weight 3 in the calculation of the quark loop diagrams)
- It is not easy to switch the variables in the weights and the variables in the arguments

$$G(1 - 1/x - x, \cdots ; y) \Longrightarrow \sum G(\cdots, g(y), \cdots ; x)$$

- In some cases to fix the integration constants requires the direct evaluation of the integral for special values of s and t with techniques based on Mellin-Barnes representations

Complete Analytic NLO calculation

Recently, the NLO total cross section was evaluated analytically
M. Czakon, A. Mitov ('08)

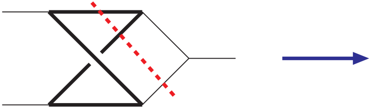
Laporta algorithm and differential equation method are employed also for the phase space integrals (by exploiting the optical theorem); cut propagators are treated by using

$$\delta\left(q^2+m^2\right)=\frac{1}{2\pi i}\left(\frac{1}{q^2+m^2-i\delta}-\frac{1}{q^2+m^2+i\delta}\right)$$

Anastasiou Melnikov ('02)

The $gg \to t\bar{t}X$ cross section cannot be completely written in terms of HPLs and their generalizations.

Integrals over elliptic functions



$$\begin{aligned} K(k) &= \int_0^1 dz \frac{1}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} \\ E(k) &= \int_0^1 dz \frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}} \end{aligned}$$

Two-Loop Corrections to $gg \to t\bar{t}$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

One-Loop × One-Loop

Anastasiou, Aybat ('08)
Körner, Kniehl, Merebashvili, Rogal ('08)

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times 0)} \quad = \quad & (N^2-1)\Bigg(N^3 A + N B + \frac{1}{N} C + \frac{1}{N^3} D + N^2 N_l E_l + N^2 N_h E_h \\ & + N_l F_l + N_h F_h + \frac{N_l}{N^2} G_l + \frac{N_h}{N^2} G_h + N N_l^2 H_l + N N_h^2 H_h \\ & + N N_l N_h H_{lh} + \frac{N_l^2}{N} I_l + \frac{N_h^2}{N} I_h + \frac{N_l N_h}{N} I_{lh}\Bigg)\end{aligned}$$

16 color structures

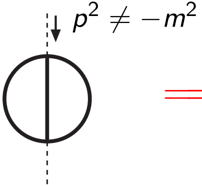
- $\mathcal{A}_2^{(2\times 0)}$ is known only in the limit $s \gg m_t^2$

  Czakon, Mitov, Moch ('07)

- The diagrams involving massless quark loops can be calculated analytically in the usual way

  Bonciani, AF, Gehrmann, Studerus (in progress)

- The remaining part of the virtual corrections involve many MI which cannot be expressed in terms of HPLs only $\Longrightarrow$ Numerical approach? New ideas?



$\Longrightarrow$

Elliptic Functions

$$K(z) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

Laporta Remiddi ('04)

Summary & Conclusions

- The top-quark pair production cross section may eventually be measurable with a 5% uncertainty at the LHC; This requires the complete computation of the NNLO QCD corrections to the top-quark pair production cross section
- a crucial ingredient of the NNLO program is the calculation of the two-loop corrections in both the $q\bar{q}$ and gg channels
- In the $q\bar{q}$ there is already a complete numerical results, and the analytical calculation of all the quark-loop diagrams; the analytical calculation of all the planar diagrams is at an advanced stage
- Much less is known in the gg channel; the light-quark loop diagrams can be evaluated analytically, other contributions cannot be expressed in terms of HPLs and might require a numerical approach

Summary & Conclusions

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The moral of the story is always the same:
there is a lot of work to do . . .

Backup Slides

The Lorentz Invariance Identities (LIs)

T. Gehrmann, E. Remiddi ('99)

A scalar integral is invariant under Lorentz transformation of the external momenta:

$$p^\mu \to p^\mu + \delta p^\mu = p^\mu + \delta\epsilon^\mu_\nu p^\nu \quad \delta\epsilon^\mu_\nu = -\delta\epsilon^\nu_\mu$$
$$I(p_1, p_2, p_3) = I(p_1 + \delta p_1, p_2 + \delta p_2, p_3 + \delta p_3)$$

implying the following 3 identities for a 4-point functions

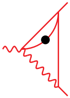
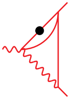
$$(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu) \sum_{n=1}^{3} \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) = 0$$
$$(p_2^\mu p_3^\nu - p_2^\nu p_3^\mu) \sum_{n=1}^{3} \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) = 0$$
$$(p_1^\mu p_3^\nu - p_1^\nu p_3^\mu) \sum_{n=1}^{3} \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) = 0$$

The General Symmetry Relations Identities

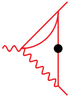
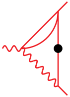
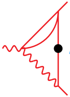
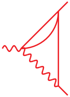
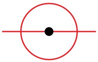
Further identities arise when a Feynman graph has symmetries

It is in those cases possible to perform a transformation of the loop momenta that does not change the value of the integral but allows to express the integrand as a combination of different integrands

Some relations are immediately seen:

$0 =$  $-$ 

others are more involved

$0 =$  $k_1 \cdot k_2 +$  $p_1 \cdot k_2 +$  $p_2 \cdot k_1 + \frac{s}{2}$  $+ \frac{1}{2}$ 

Equations and Unknowns

In two-loop $2 \rightarrow 2$ processes there are two integration momenta and three external momenta $\longrightarrow$ 9 possible scalar products

Consider the integrals $I_{t,r,s}$ where

- t $\rightarrow$ # of propagators
- 9-t $\rightarrow$ # of irreducible scalar products
- r $\rightarrow$ sum of the powers of all propagators
- s $\rightarrow$ sum of the powers of all irreducible scalar products

The number of integrals belonging to the $I_{t,r,s}$ set is

$$N(I_{t,r,s}) = \binom{r-1}{r-t}\binom{8-t+s}{s}$$

It is possible to build $(N_{\mathrm{IBP}} + N_{\mathrm{LI}})N(I_{t,r,s})$ identities

Euler Method

First Order Differential Equation

$$\frac{d}{dx} f(x) + C(x) f(x) = \Omega(x)$$

1. find the solution of the homogeneous equation

$$\frac{d}{dx} h(x) + C(x) h(x) = 0$$

2. build the general solution of the non-homogeneous equation

$$f(x) = h(x) \left[k + \int dx \frac{\Omega(x)}{h(x)} \right]$$

3. fix the integration constant k

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Euler Method-II

Second Order Differential Equation

$$\frac{d^2}{dx^2} f(x) + A(x)\frac{d}{dx} f(x) + B(x) f(x) = \Omega(x)$$

1. find the two solution of the homogeneous equation

$$\frac{d^2}{dx^2} h_{1,2}(x) + A(x)\frac{d}{dx} h_{1,2}(x) + B(x) h_{1,2}(x) = 0$$

2. build the Wronskian

$$W(x) = h_1(x)\frac{d}{dx} h_2(x) - h_2(x)\frac{d}{dx} h_1(x)$$

3. build the solution

$$f(x) = h_1(x)\left(k_1 - \int_0^x \frac{dw}{W(w)} h_2(w)\Omega(w)\right) + h_2(x)\left(k_2 + \int_0^x \frac{dw}{W(w)} h_1(w)\Omega(w)\right)$$

4. fix the integration constants k_1 and k_2

Euler Method-II

Second Order Differential Equation

$$\frac{d^2}{dx^2} f(x) + A(x)\frac{d}{dx} f(x) + B(x) f(x) = \Omega(x)$$

1. find the two solution of the homogeneous equation

$$\frac{d^2}{dx^2} h_{1,2}(x) + A(x)\frac{d}{dx} h_{1,2}(x) + B(x) h_{1,2}(x) = 0$$

2. build the Wronskian

$$W(x) = h_1(x)\frac{d}{dx}h_2(x) - h_2(x)\frac{d}{dx}h_1(x)$$

3. build the solution

$$f(x) = h_1(x)\left(k_1 - \int_0^x \frac{dw}{W(w)} h_2(w)\Omega(w)\right) + h_2(x)\left(k_2 + \int_0^x \frac{dw}{W(w)} h_1(w)\Omega(w)\right)$$

4. fix the integration constants k_1 and k_2

Euler Method-II

Second Order Differential Equation

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