Quantum Information at Colliders

Kazuki Sakurai

(University of Warsaw)

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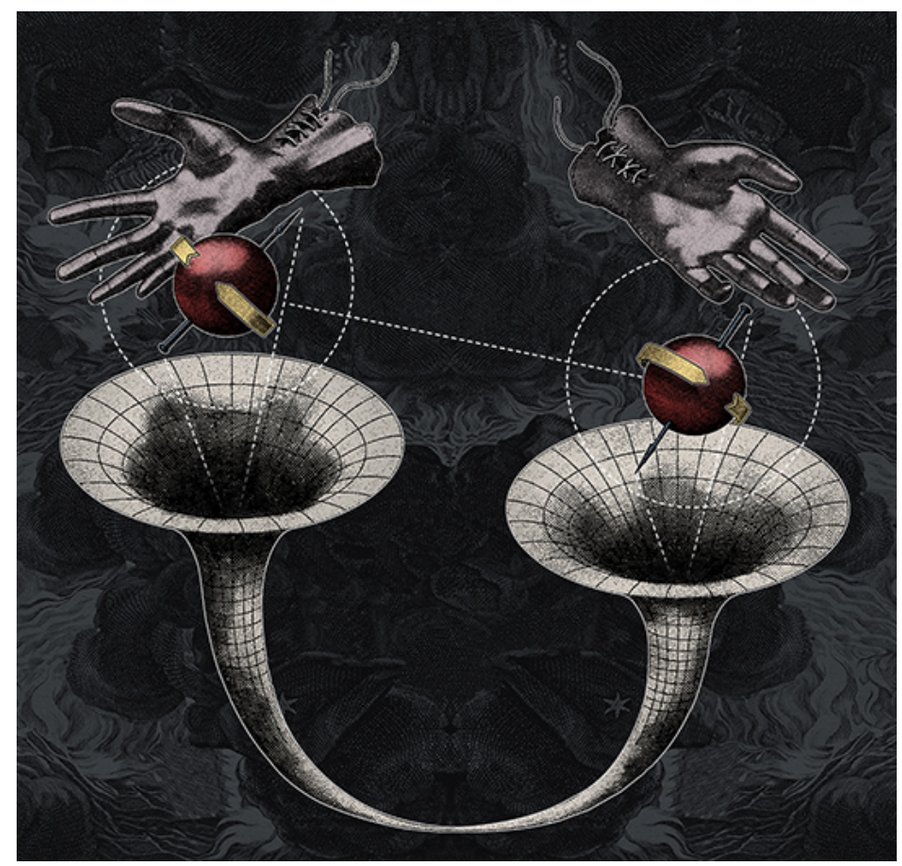
Entanglement and other quantum properties are crucial in:

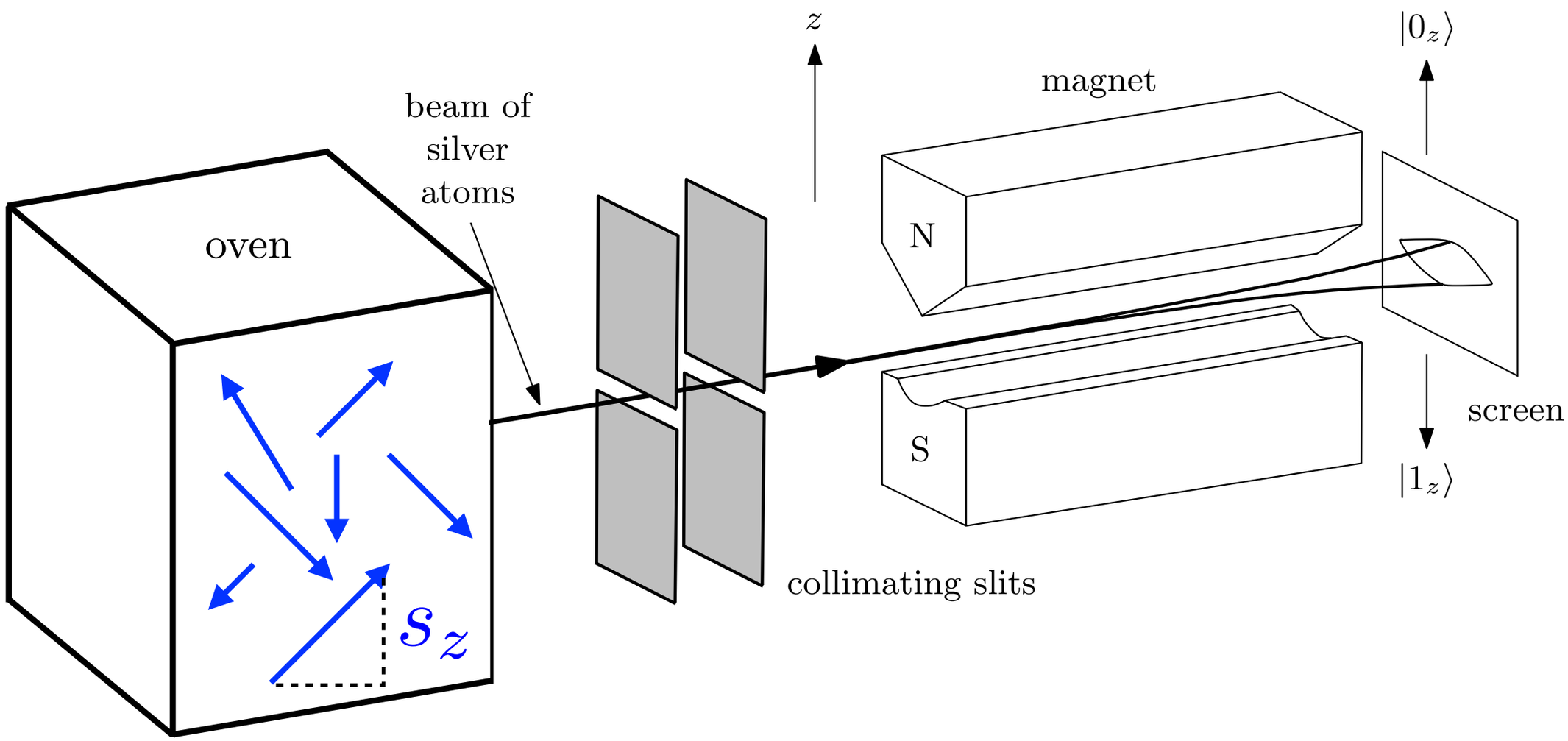
- developing **quantum technology/devices**

- understanding **QFT** and quantum **gravity**

IBM Q system



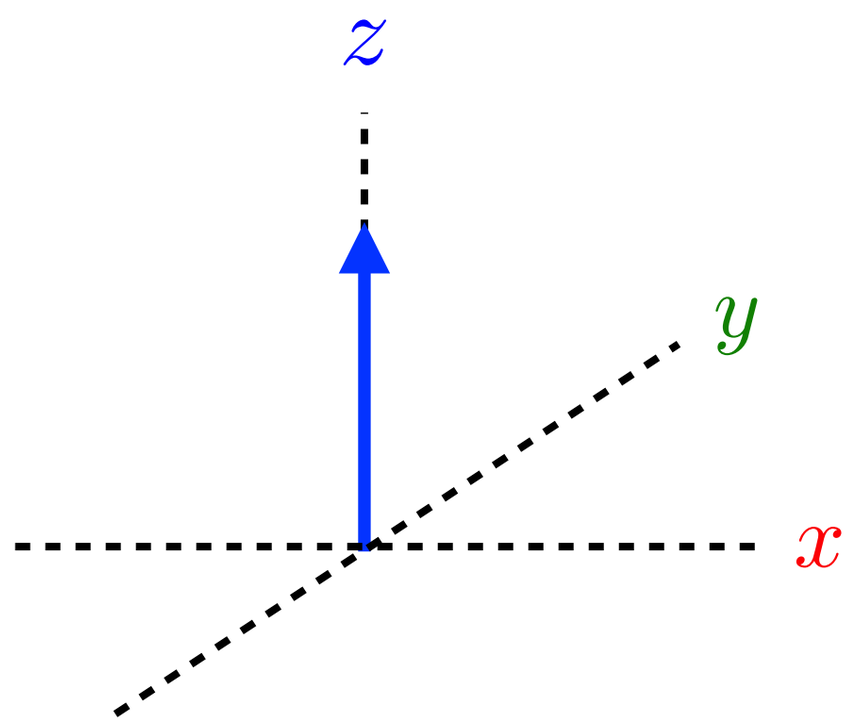
Spin



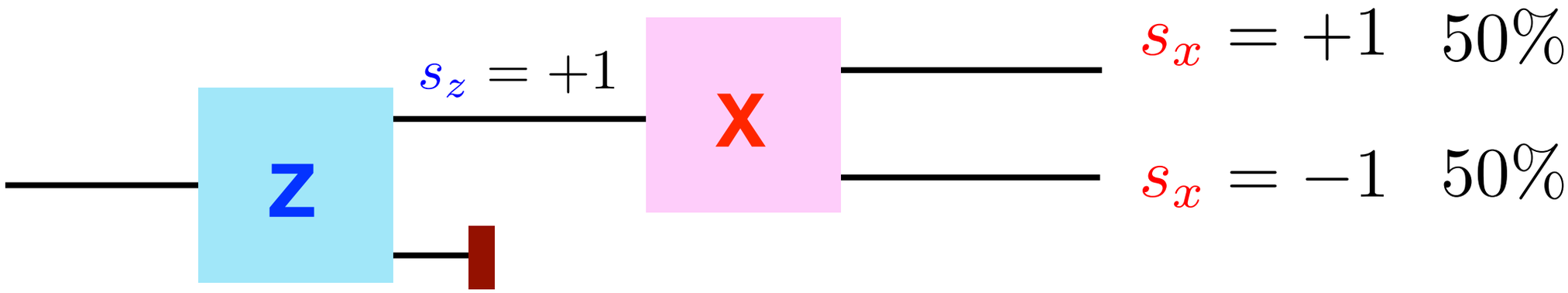
$$F_z \propto \partial_z(\vec{s}\cdot\vec{B}) \propto s_z B_z'$$

- **Z-component** of spin has only two possible values: **+1** and **-1**

- If spin is in the **Z**-direction, **X**, **Y**-component should be zero! (?)



- If spin is in the **Z**-direction, **X**-component should be zero! (?)



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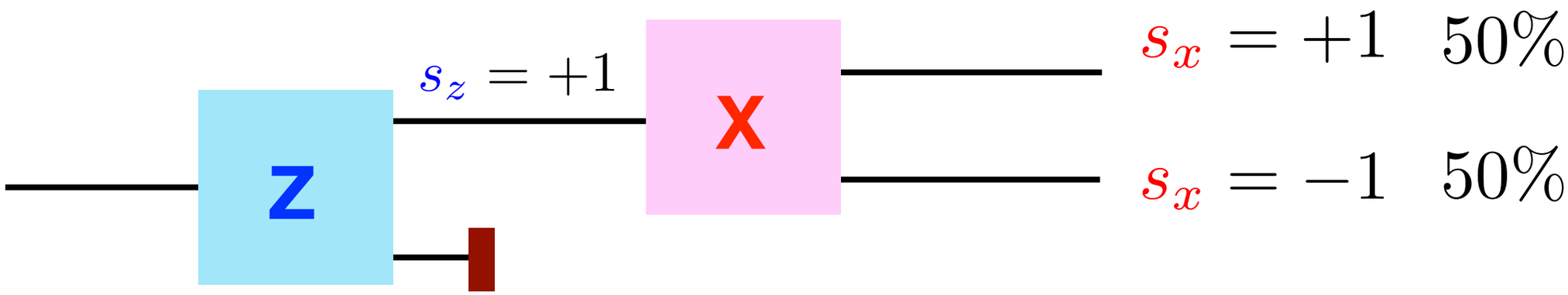

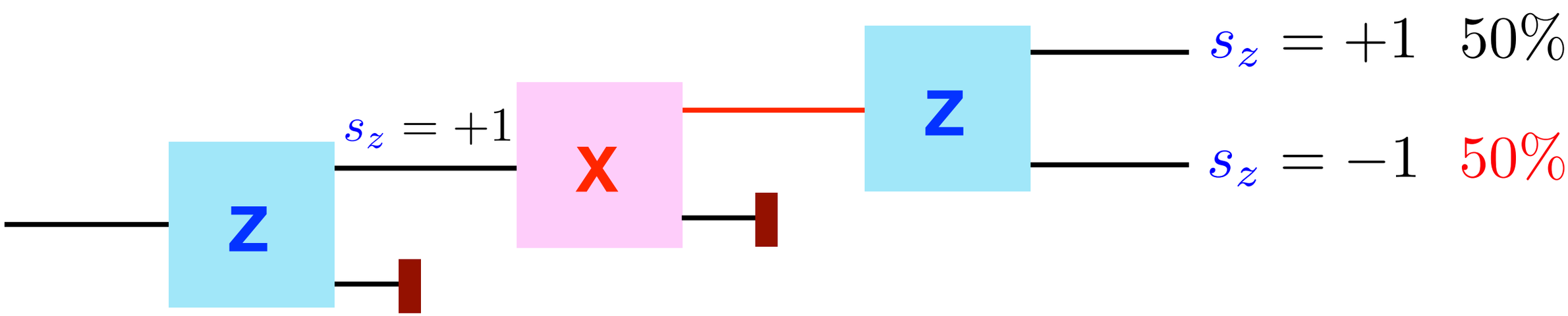

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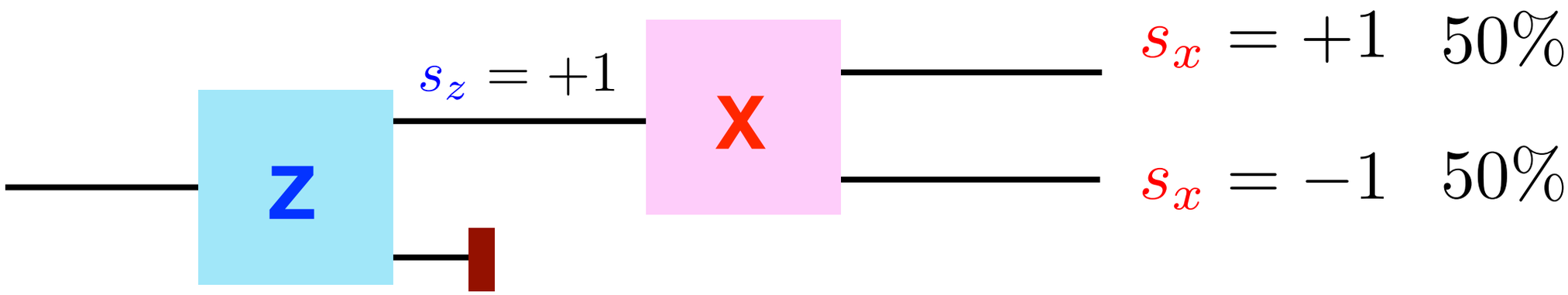
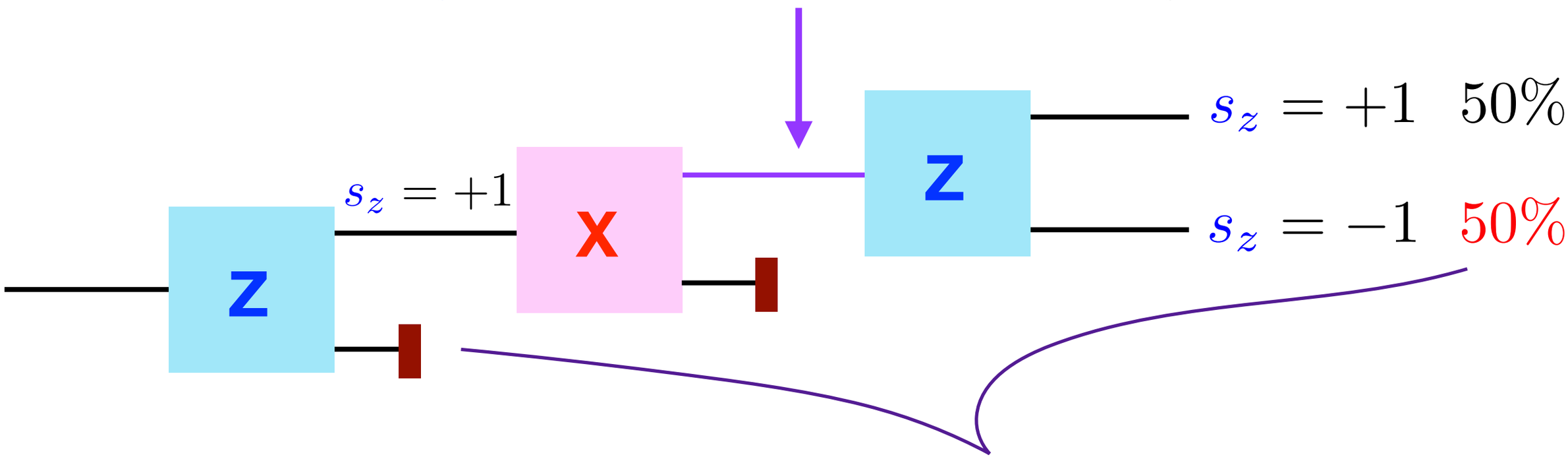


Is there a ***definite*** s_z **value** of **this particle** before the s_z measurement?



previously blocked $s_z = -1$
reappears!

Is probabilistic outcomes in experiments **fundamental** or **due to our ignorance** about the physical state/system?

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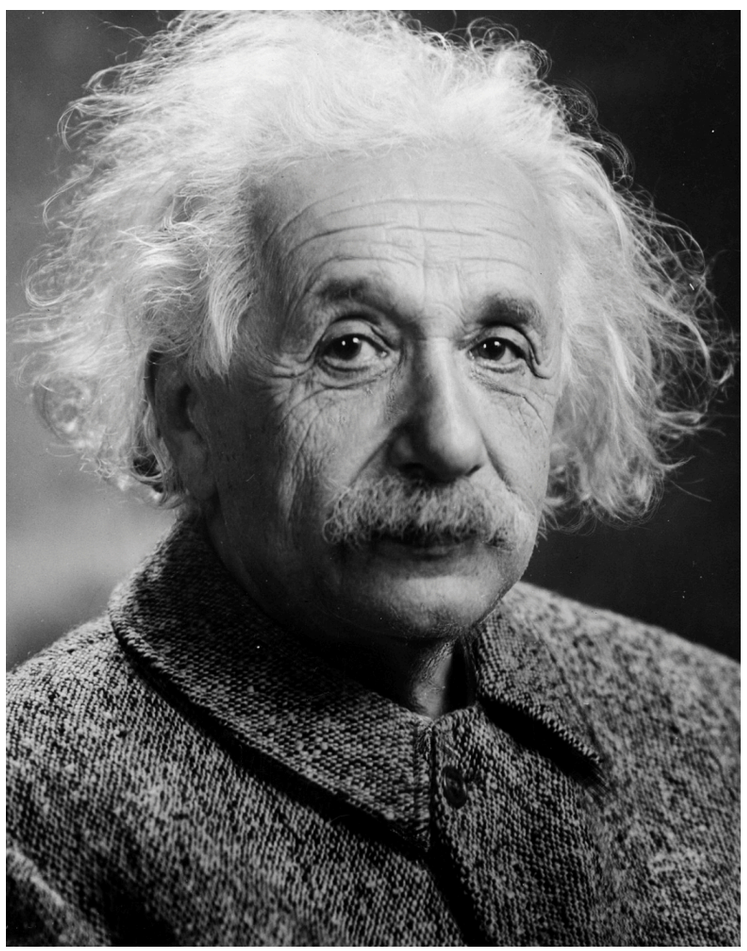
Can't be fundamental!

God does not play dice!



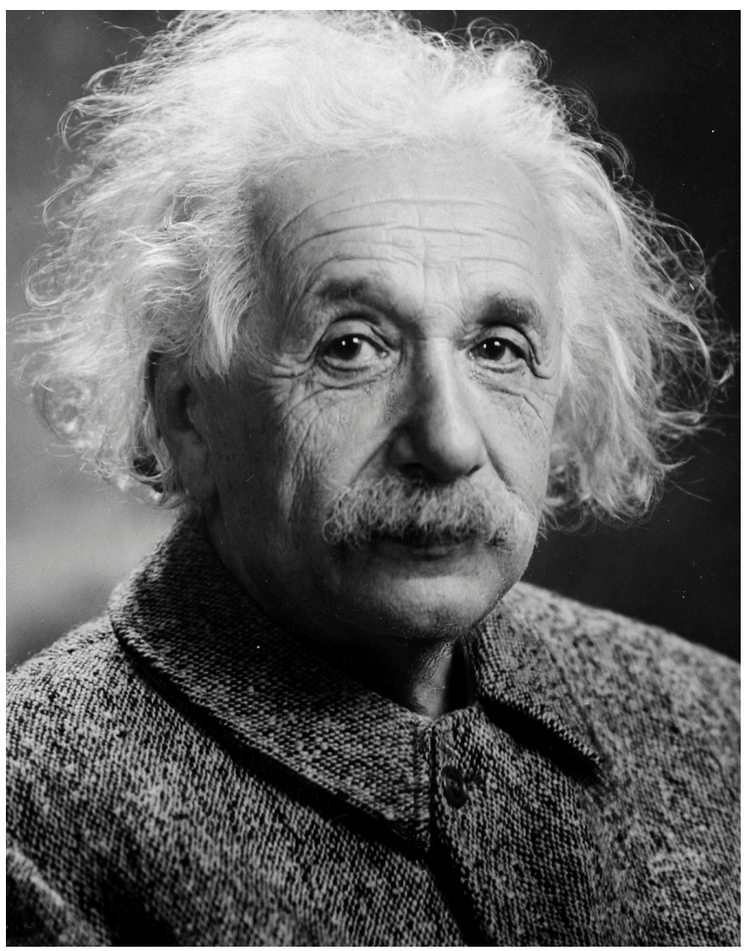
Albert Einstein

Is probabilistic outcomes in experiments **fundamental** or **due to our ignorance** about the physical state/system?

Can't be fundamental!

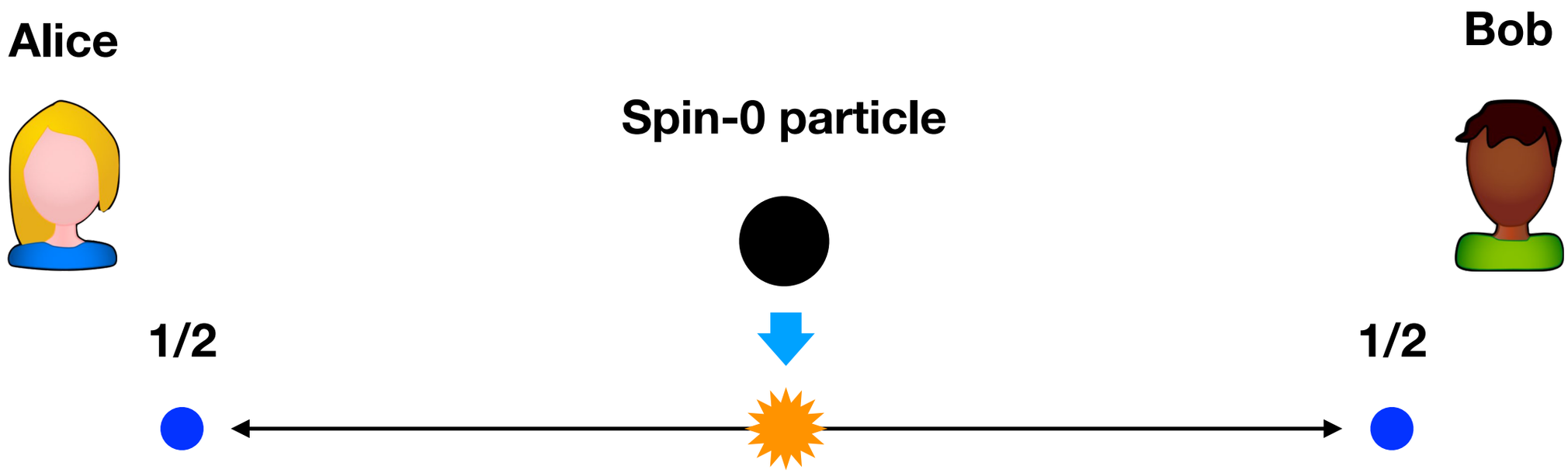
God does not play dice!

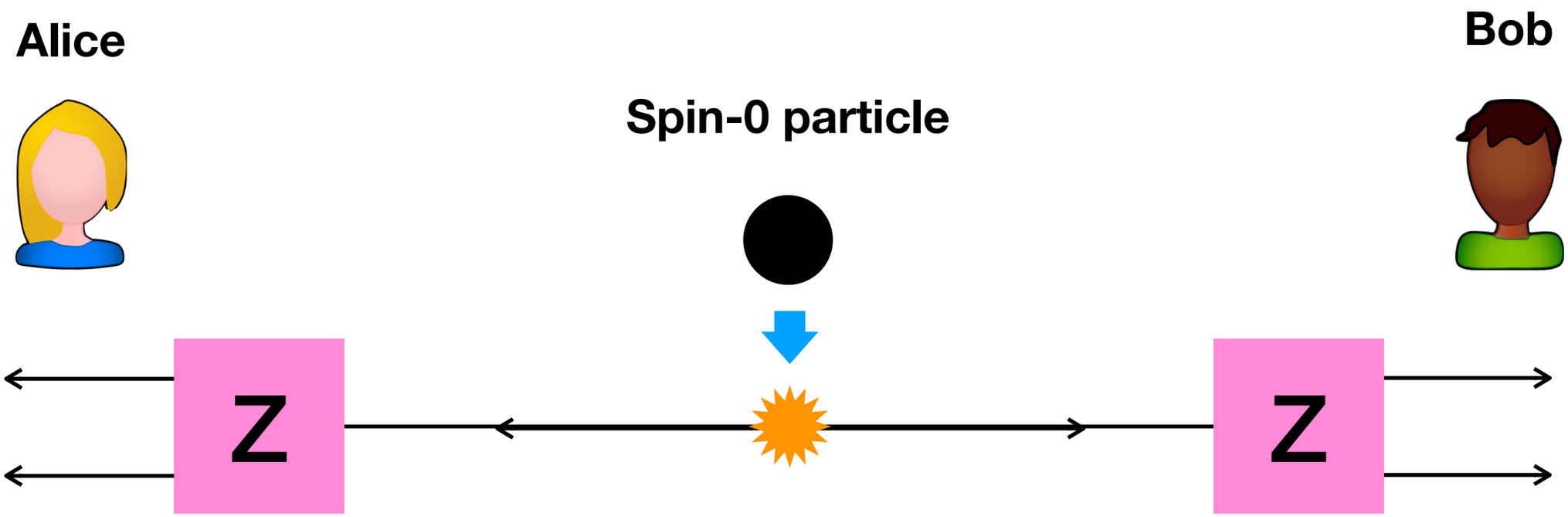
Can be tested experimentally :-)

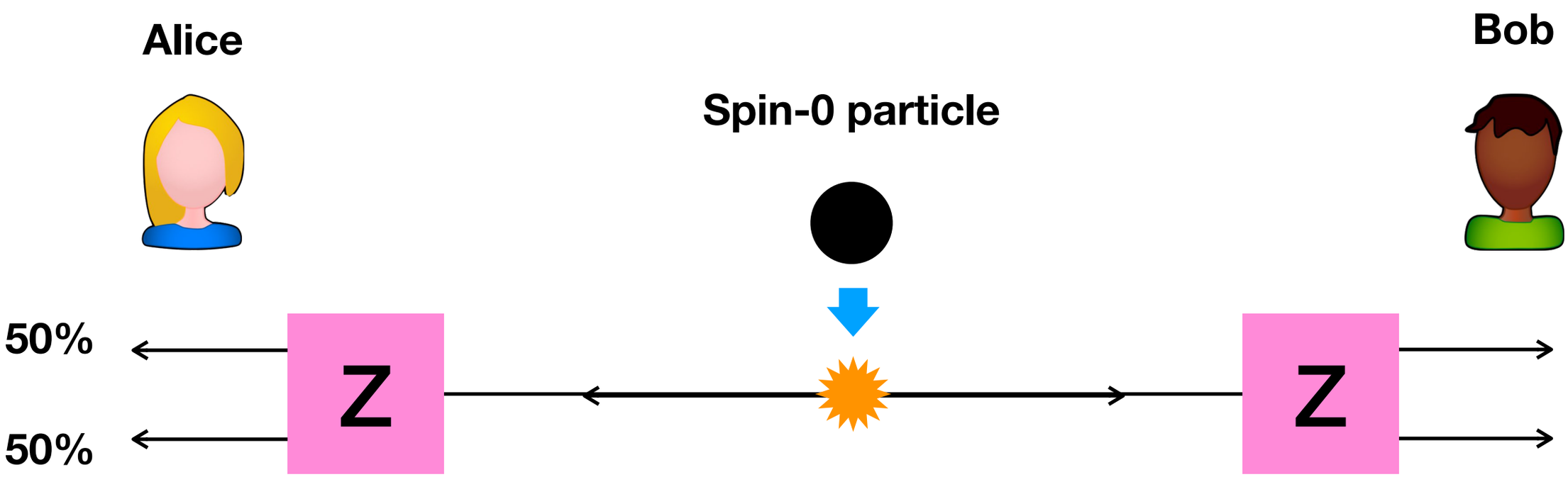


Albert Einstein

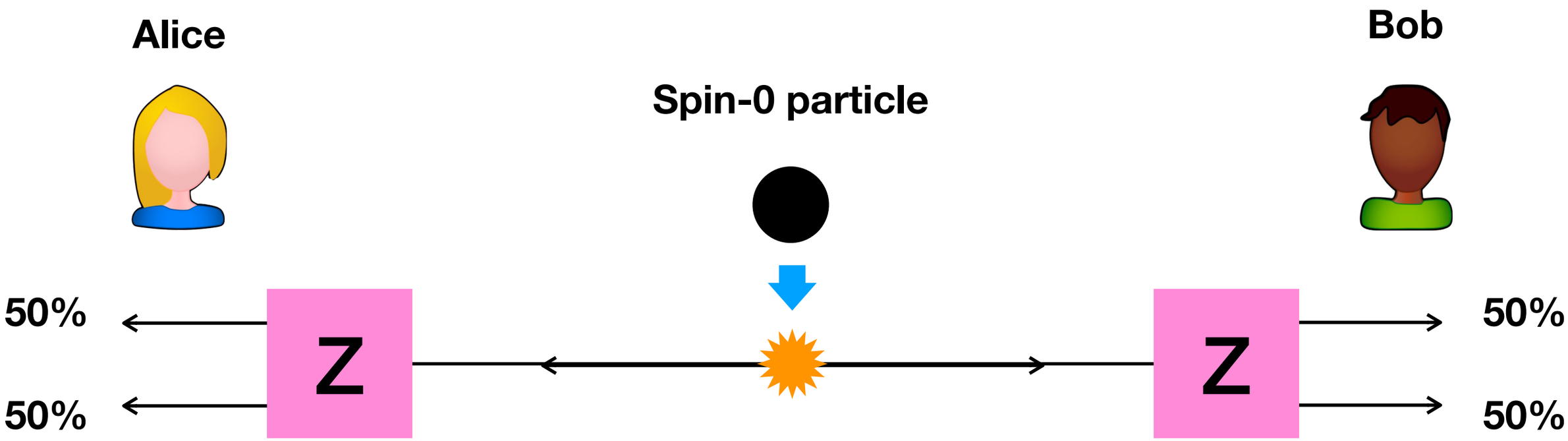
John Bell


Alice
Bob
Spin-0 particle
1/2
1/2

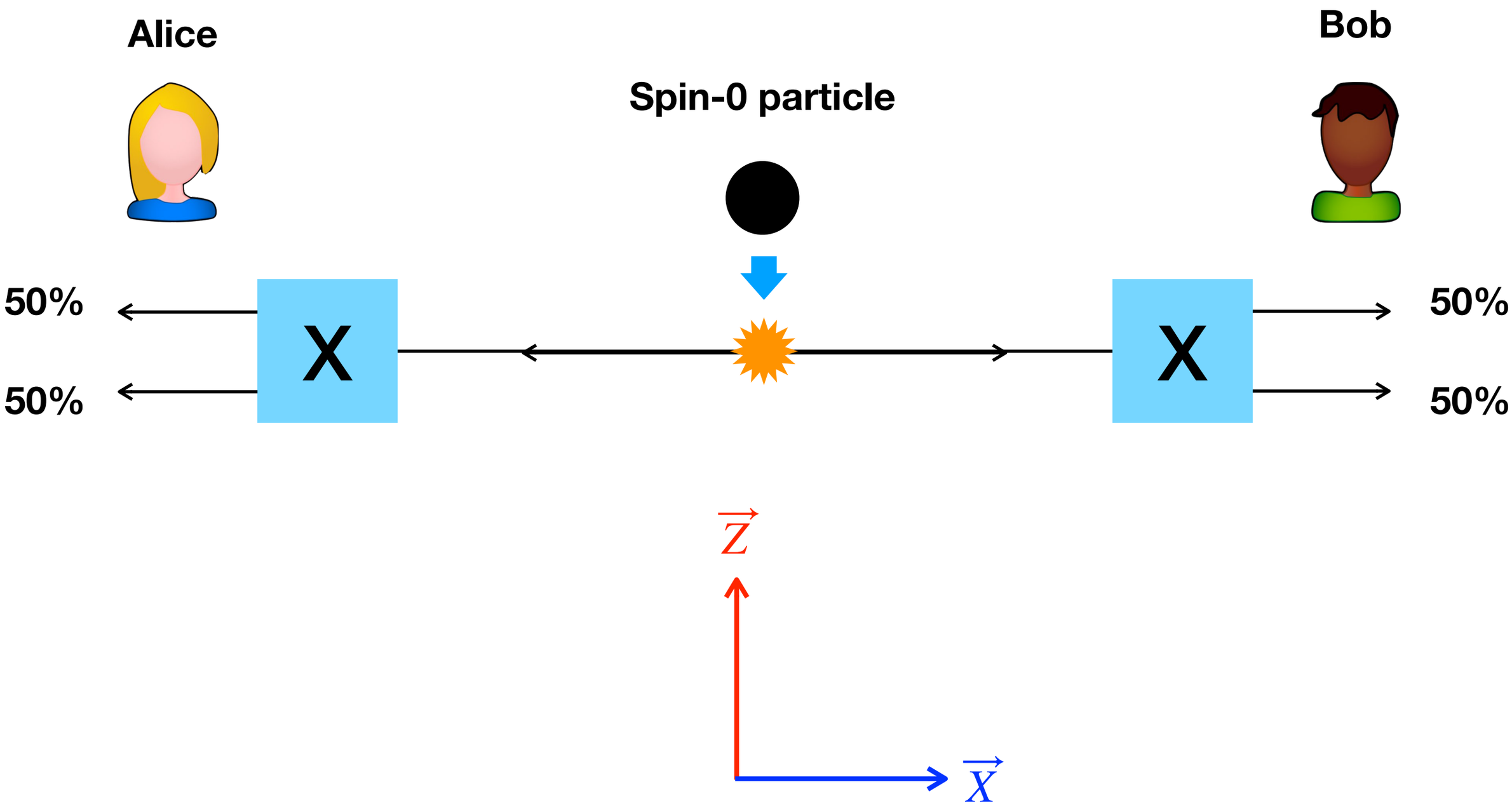

Alice
Bob
Spin-0 particle
Z
Z



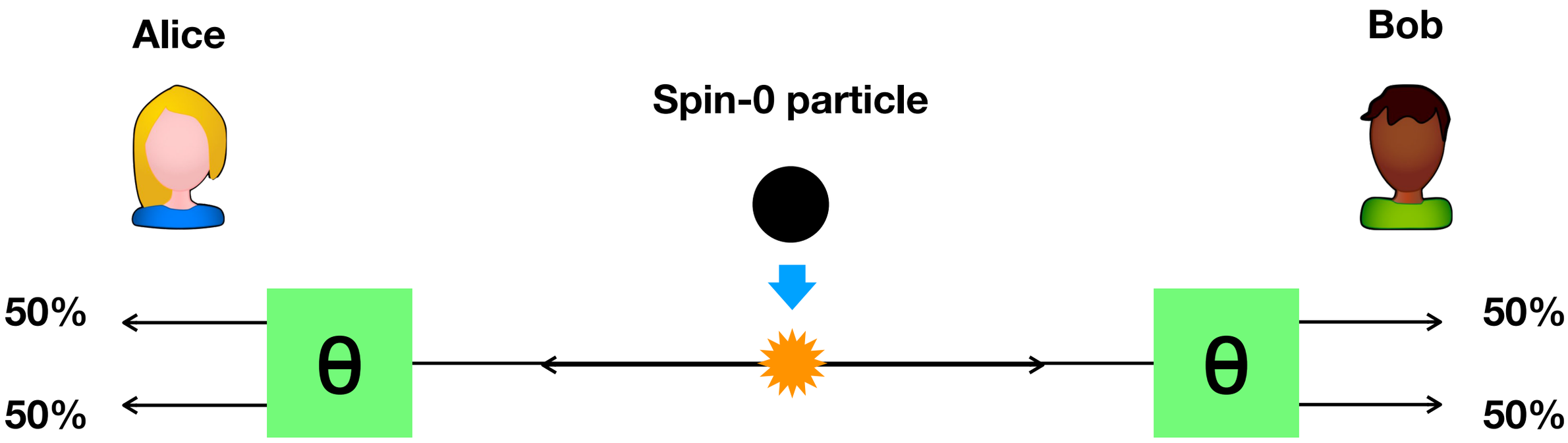
Alice	+	+	-	+	-	-	+	+	+	-	+	-

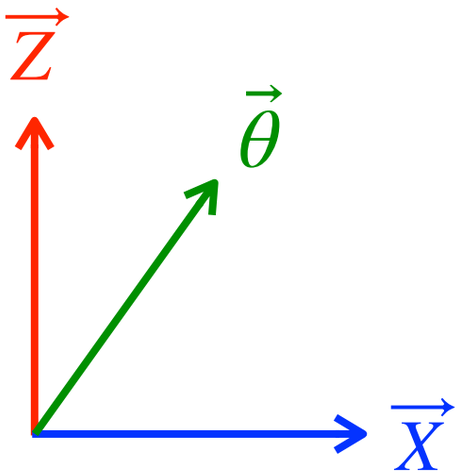


Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+

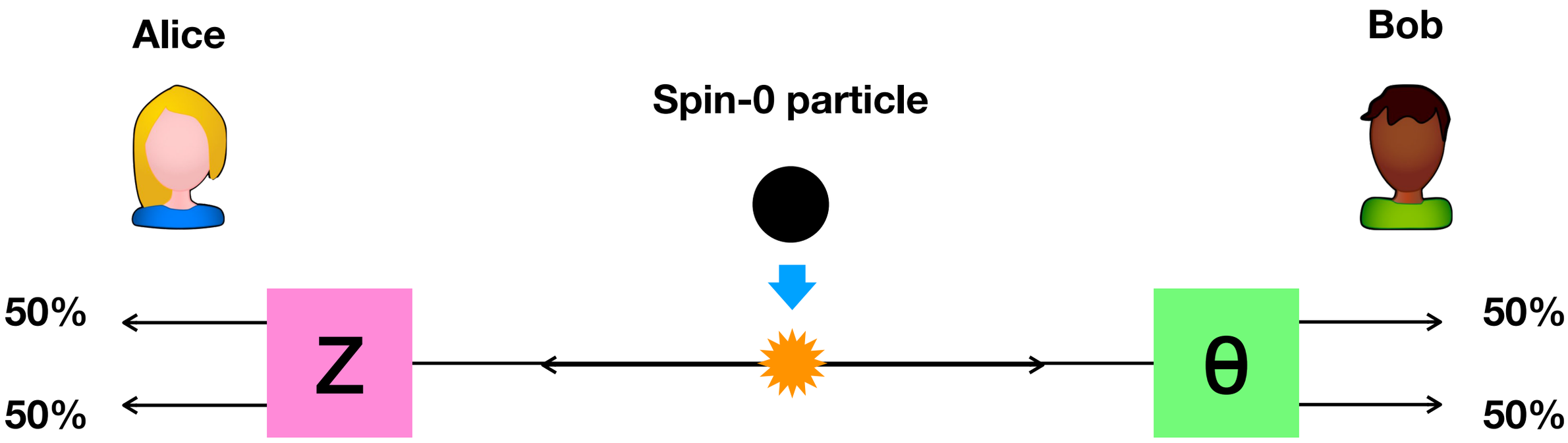


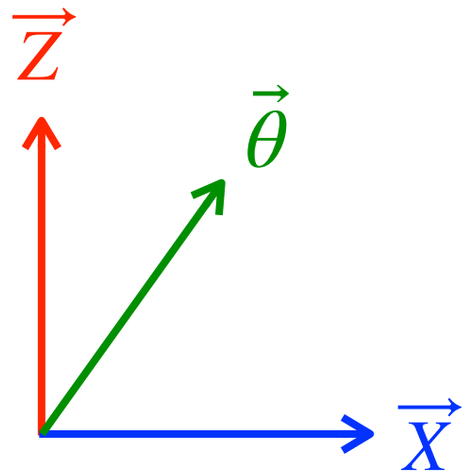
Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+





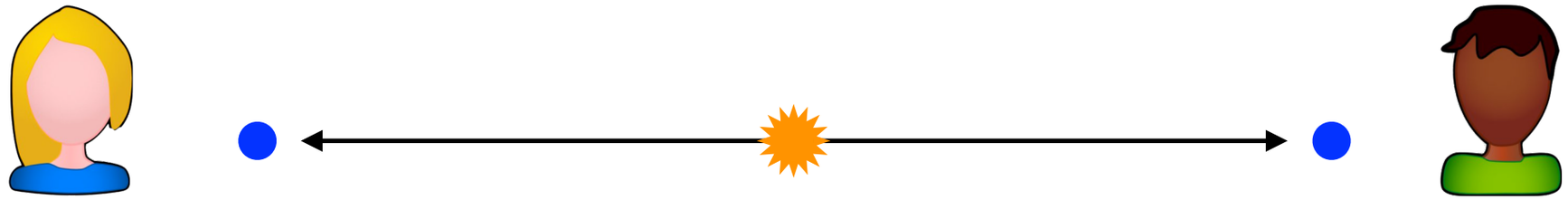
Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+



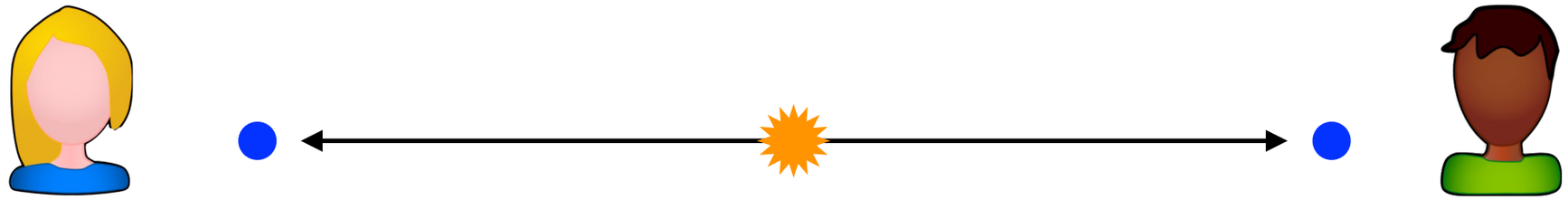


Alice	+	+	-	+	-	-	-	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+

?

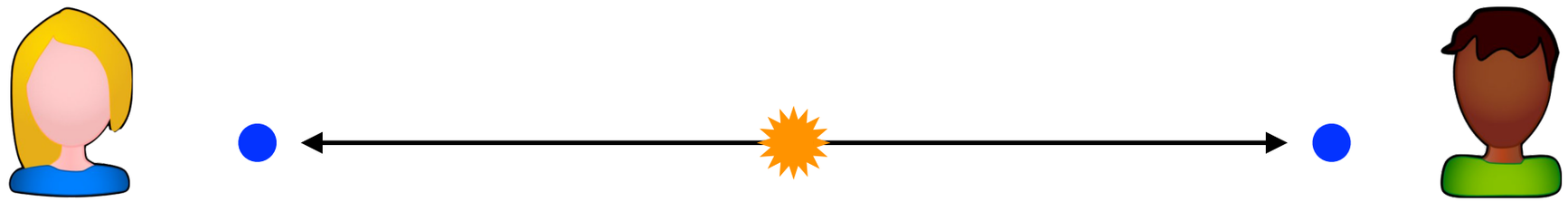


	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+



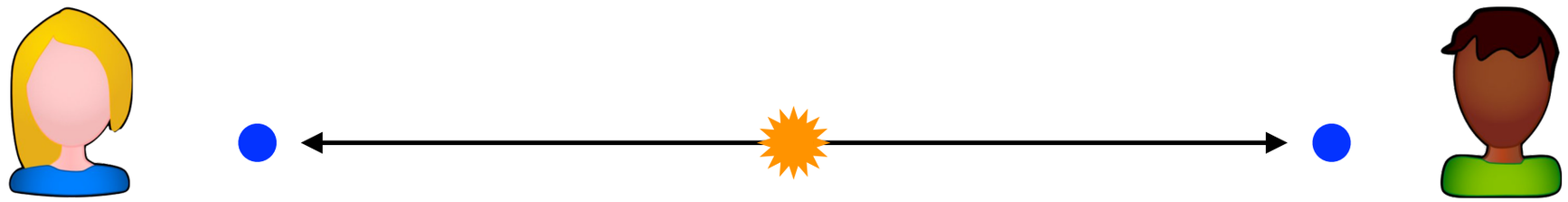
	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) = ?$



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

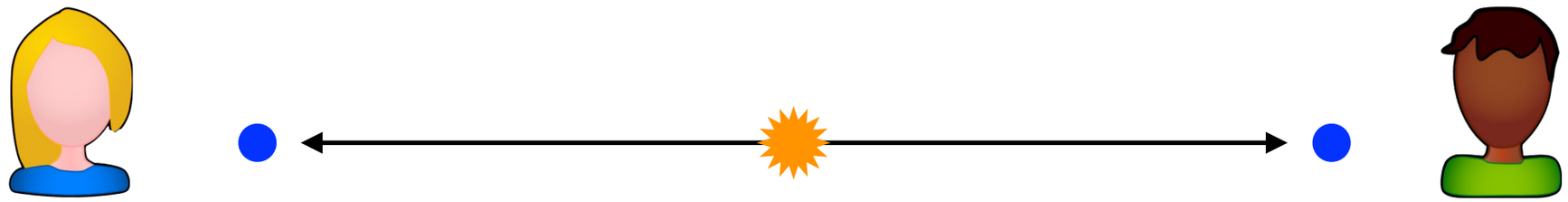
$$P(+z, +x) = P_3 + P_5$$



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) = P_3 + P_5$

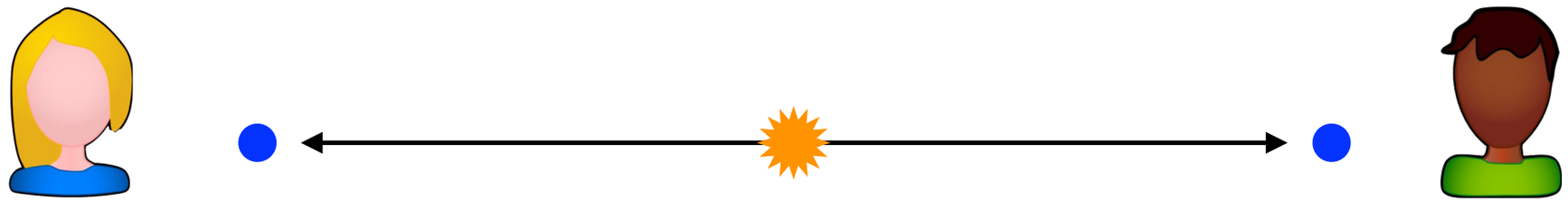
$P(+z, +\theta) = ?$



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) = P_3 + P_5$

$P(+z, +\theta) = P_4 + P_5$

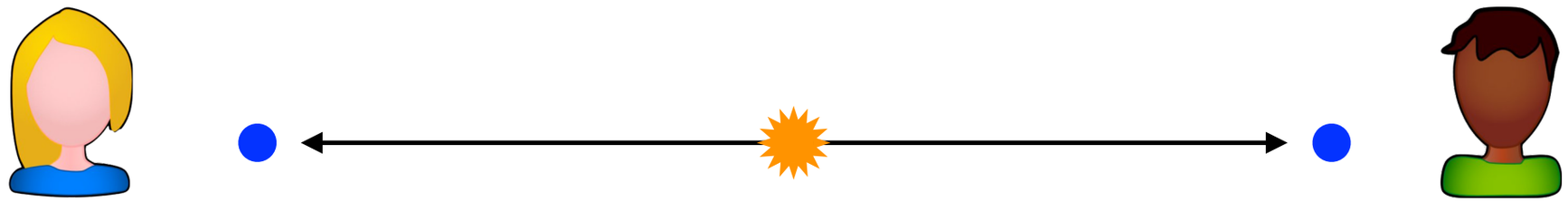


	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) = P_3 + P_5$

$P(+z, +\theta) = P_4 + P_5$

$P(+\theta, +x) = ?$

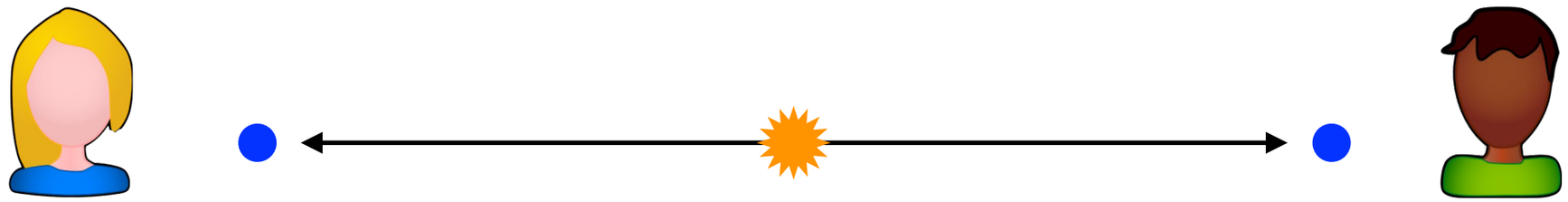


	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) = P_3 + P_5$

$P(+z, +\theta) = P_4 + P_5$

$P(+\theta, +x) = P_3 + P_7$

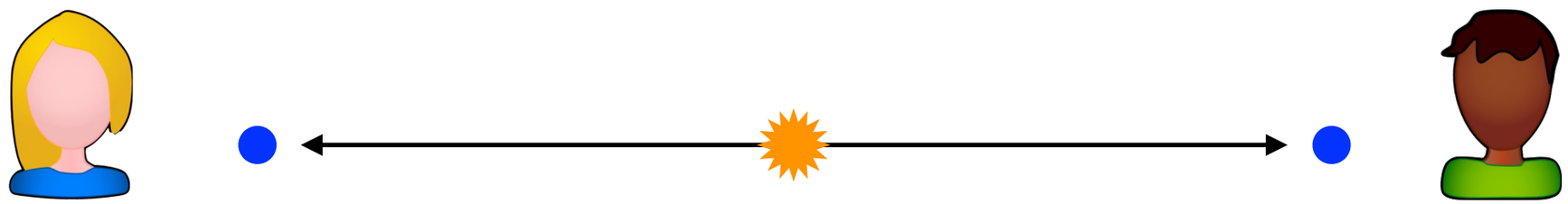


	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) = P_3 + P_5$

$P(+z, +\theta) = P_4 + P_5$

$P(+\theta, +x) = P_3 + P_7$



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$$P(+z, +x) = P_3 + P_5$$

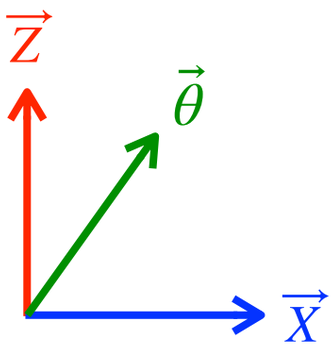
$$P(+z, +\theta) = P_4 + P_5$$

$$P(+\theta, +x) = P_3 + P_7$$

Bell inequality

$$P(+z, +\theta) + P(+\theta, +x) \geq P(+z, +x)$$

$$\mathbf{B}(\theta) = 4 \times [\, P(z+, \theta+) + P(\theta+, x+) - P(z+, x+) \,]$$

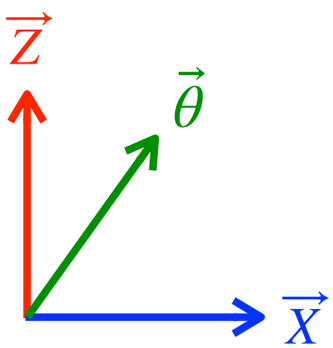


Bell inequality

$$\boxed{\mathbf{B}(\theta) \geq 0}$$

← for any θ

$$\mathbf{B}(\theta) = 4 \times [\, P(z+, \theta+) + P(\theta+, x+) - P(z+, x+) \,]$$



Bell inequality

$$\boxed{\mathbf{B}(\theta) \geq 0}$$

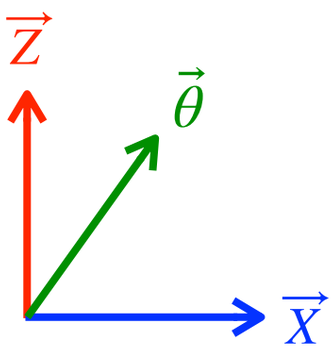
← for any θ

$$|\Psi^{(0,0)}\rangle = \frac{|+,-\rangle_z - |-,+\rangle_z}{\sqrt{2}}$$

QM →

$$\mathbf{B}(\theta) = 1 - \sqrt{2}\cos(\theta - \pi/4)$$

$$\mathbf{B}(\theta) = 4 \times [\, P(z+, \theta+) + P(\theta+, x+) - P(z+, x+) \,]$$



Bell inequality

$$\boxed{\mathbf{B}(\theta) \geq 0}$$

← for any θ

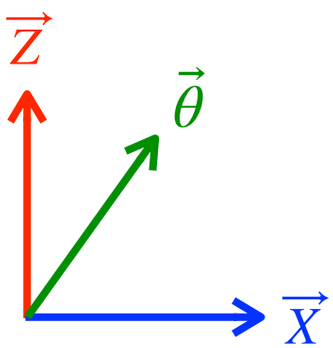
$$|\Psi^{(0,0)}\rangle = \frac{|+,-\rangle_z - |-,+\rangle_z}{\sqrt{2}}$$

QM →

$$\mathbf{B}(\theta) = 1 - \sqrt{2}\cos(\theta - \pi/4)$$

$$\mathbf{B}(\theta = \pi/4) \simeq -0.414 < 0 \;!!$$

$$\mathbf{B}(\theta) = 4 \times [\, P(z+, \theta+) + P(\theta+, x+) - P(z+, x+) \,]$$



Bell inequality

$$\boxed{\mathbf{B}(\theta) \geq 0}$$

← for any θ

$$|\Psi^{(0,0)}\rangle = \frac{|+,-\rangle_z - |-,+\rangle_z}{\sqrt{2}}$$

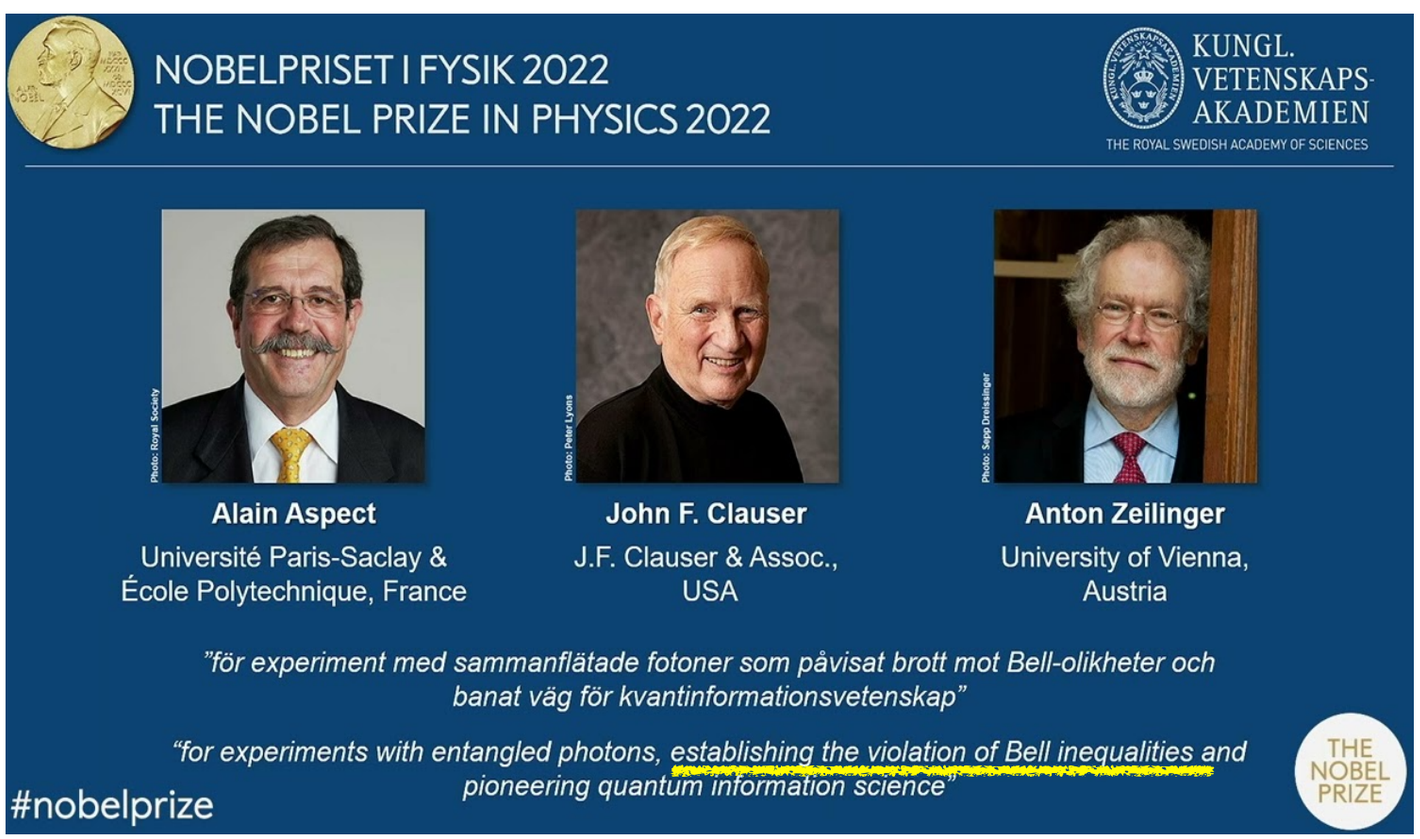
QM →

$$\mathbf{B}(\theta) = 1 - \sqrt{2}\cos(\theta - \pi/4)$$

$$\mathbf{B}(\theta = \pi/4) \simeq -0.414 < 0 \;!!$$

↑

Violation of Bell inequality has been experimentally confirmed!



Qubit

- The qubit system has two basis kets: $|0\rangle$ and $|1\rangle$. The general state is

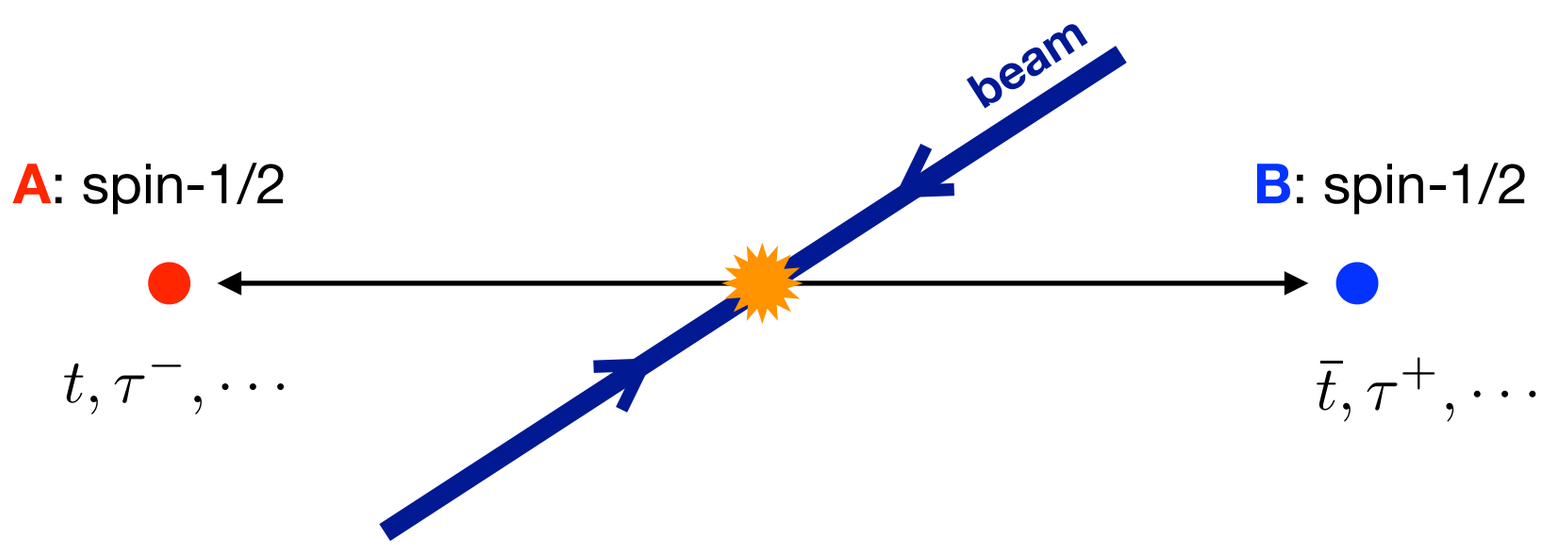
$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle \qquad (c_i \in \mathbb{C},\ |c_1|^2 + |c_2|^2 = 1)$$

- The n-qubit system has 2^n basis kets:

$$|\psi_1\psi_2\cdots\psi_n\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \qquad |\psi_i\rangle \in \{|0\rangle, |1\rangle\}$$

Ex.) The general state in the 2-qubit system:

$$|\Psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \qquad \left(c_{ij} \in \mathbb{C},\ \sum_{i,j} |c_{ij}|^2 = 1\right)$$



Separable or Entangled

Q.) Can we write

$$
\begin{aligned}
|\Psi\rangle_{AB} &= c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \\
&\stackrel{?}{=} (\underbrace{\alpha_0|0\rangle + \alpha_1|1\rangle}_{|\Psi\rangle_A}) \otimes (\underbrace{\beta_0|0\rangle + \beta_1|1\rangle}_{|\Psi\rangle_B})
\end{aligned}
$$

Separable or Entangled

Q.) Can we write

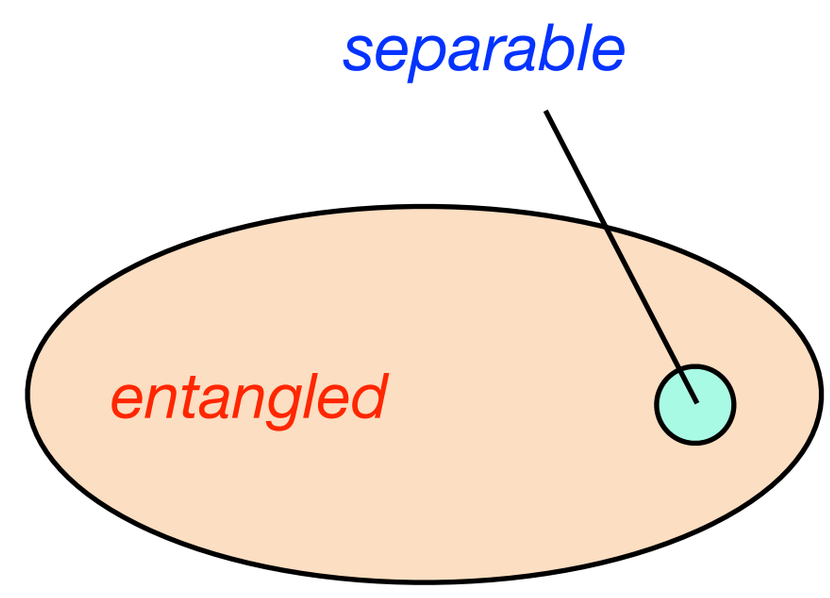
$$|\Psi\rangle_{AB} = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$
$$\stackrel{?}{=} \underbrace{(\alpha_0|0\rangle + \alpha_1|1\rangle)}_{|\Psi\rangle_A} \otimes \underbrace{(\beta_0|0\rangle + \beta_1|1\rangle)}_{|\Psi\rangle_B}$$

If the answer is
- **Yes** → $|\Psi\rangle_{AB}$ is *separable*
- **No** → $|\Psi\rangle_{AB}$ is *entangled*

$$|\Psi\rangle_{AB} = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

is separable iff: $c_{00}c_{11} - c_{01}c_{10} = 0$

→ Separable states are **rare!**



Mixed States

- We are uncertain about the quantum state and only know it is $|\Psi_i\rangle$ with probability p_i . $(i = 1, \cdots, N)$

  → We say the state is ***mixed*** and describe it by the ***density operator*** :

$$\rho = \sum_{i=1}^{N} p_i |\Psi_i\rangle\langle\Psi_i| \qquad \left(p_i \geq 0,\ \sum_i p_i = 1\right)$$

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- The density operators are ***Hermitian***, ***Tr = 1***, ***positive definite*** :

$$\rho = \rho^\dagger,\ \mathrm{Tr}\rho = 1,\ \rho \geq 0$$

- The expectation value of an observable $\hat{\mathcal{O}}$ is can be computed as:

$$\langle\hat{\mathcal{O}}\rangle = \mathrm{Tr}\left[\hat{\mathcal{O}}\rho\right]$$

- The mixed state ρ_{AB} of the A-B system is saied to be ***separable***, if it's possible to write

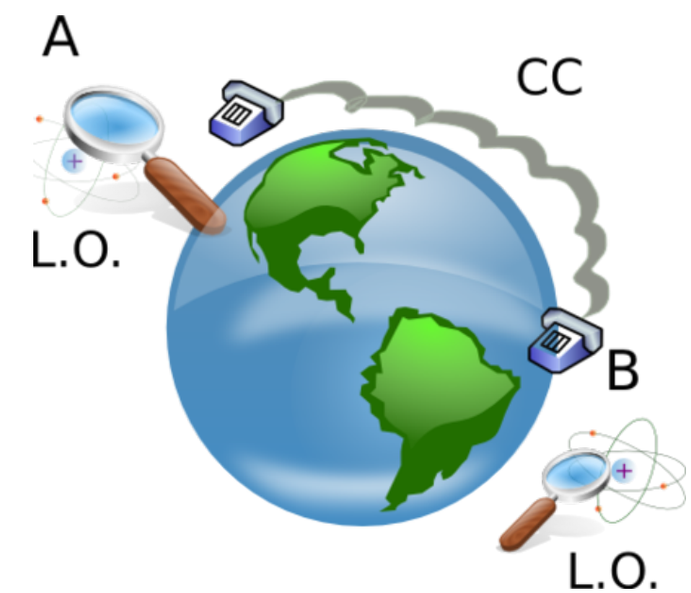
$$\rho_{AB} = \sum_i q_i \cdot \rho_A^i \otimes \rho_B^i \qquad \left(q_i \geq 0,\ \sum_i q_i = 1\right)$$

  Otherwise, the state is said to be ***entangled***.

Entanglement as a Resource

- Separable states will never be entangled by Local Operation and Classical Communication (LOCC)
- LOCC introduces an *order* among states

$\Rightarrow$ Entanglement is *defined* s.t. it decreases under non-invertible LOCC:



$$\underset{\text{maximally entangled}}{\rho_{\max}} \xrightarrow{\text{LOCC}} \rho \xrightarrow{\text{LOCC}} \underset{\text{separable state}}{\rho_{\text{sep}}}$$

$$\frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

ex) $|00\rangle + |11\rangle \xrightarrow[\text{local measurement}]{} |00\rangle$

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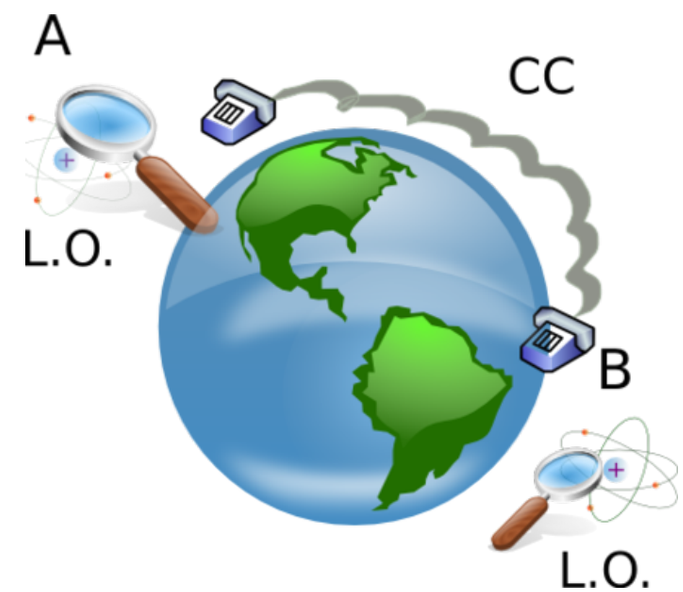

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- Entanglement measures (monotones), $E(\rho)$
  - monotonically decreases under LOCC
  - $E(\rho) = 0$ if ρ is separable and $E(\rho) > 0$ otherwise
  - $E(\rho) = 1$ if ρ is maximally entangled states

Entanglement Measures

- Entanglement measures are often defined nicely for pure states $|\Psi\rangle_{AB}$

Von Neumann Entropy: $S_V(|\Psi\rangle_{\mathrm{AB}}) = -\mathrm{Tr}[\rho_A \log_2 \rho_A]$

$$\rho_A = \mathrm{Tr}_B\left(|\Psi\rangle\langle\Psi|_{AB}\right)$$

Linear Entropy: $S_L(|\Psi\rangle_{\mathrm{AB}}) = 2[1 - \mathrm{Tr}(\rho_A^2)]$

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minimising over all possible decompositions

- For 2-qubit systems, the concurrence admits the analytical expression: [Wootters '98]

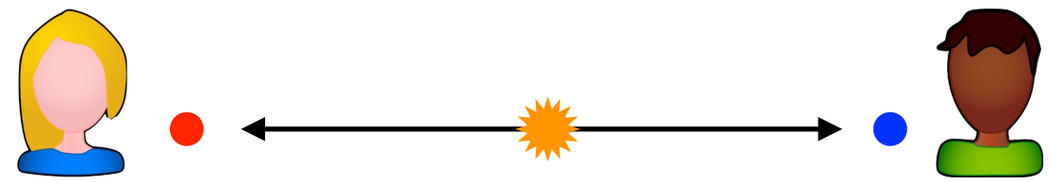
$$\mathcal{C}[\rho] = \max(0, \eta_1 - \eta_2 - \eta_3 - \eta_4) \in [0, 1]$$

$\eta_1 \geq \eta_2 \geq \eta_3 \geq \eta_4$ are eigenvalues of $\sqrt{\rho\tilde{\rho}}$ with $\tilde{\rho} \equiv (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

measures axis: $x \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$

probability for a: $p_A(a\,|\,x)$



$y \in \{\vec{n}\}$

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Experimental data is described by the *joint distribution* $p(a, b\,|\,x, y)$

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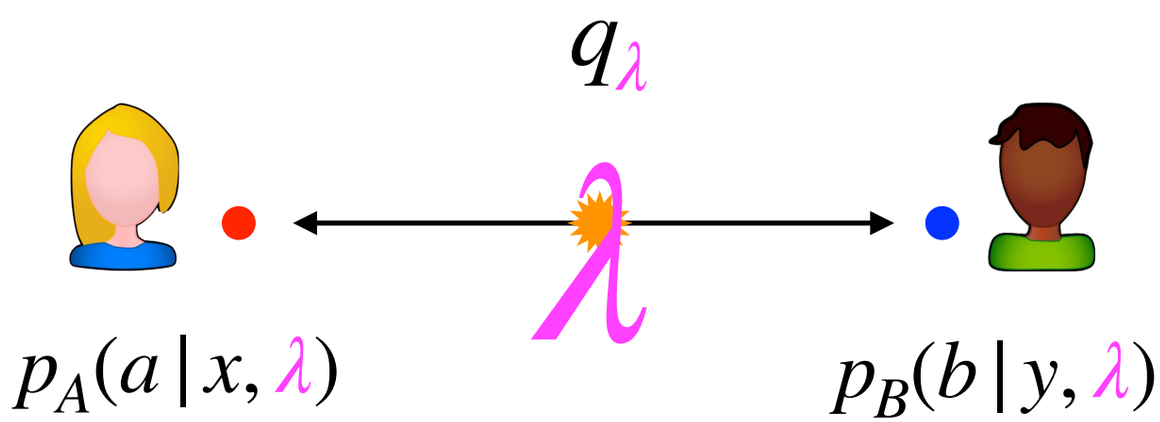

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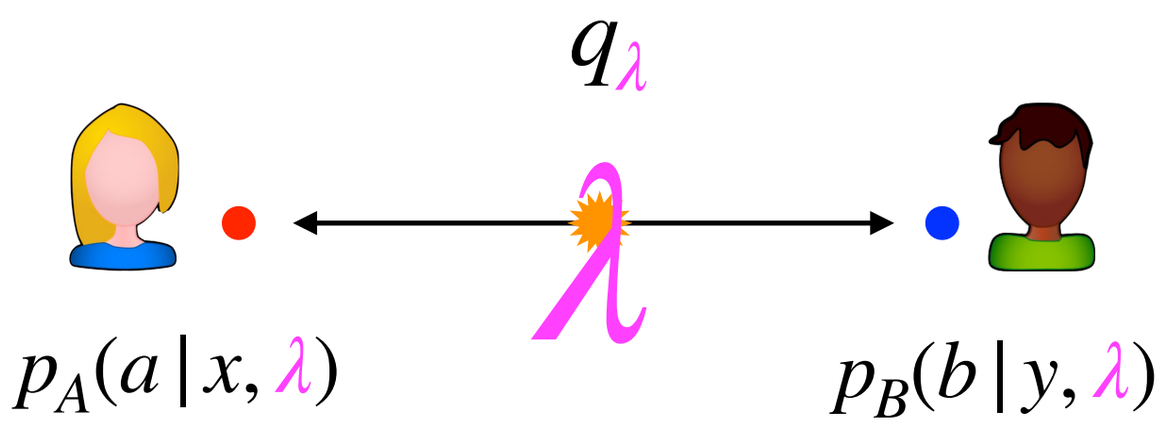

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Quantum Mechanics

density operator

$$p_Q(a, b\,|\,x, y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$$

For projective measurement:

$$M_{a|x} = |\,a_x\rangle\langle a_x\,|$$

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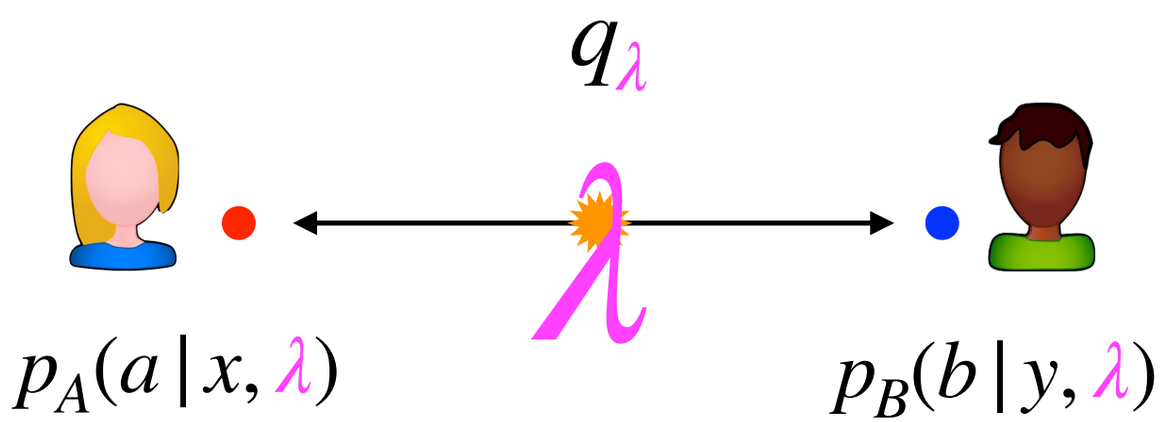

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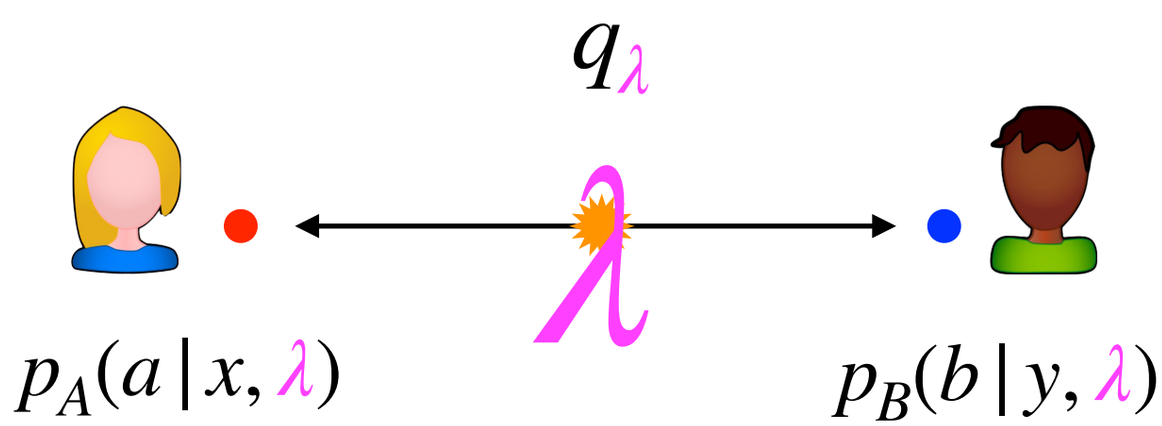

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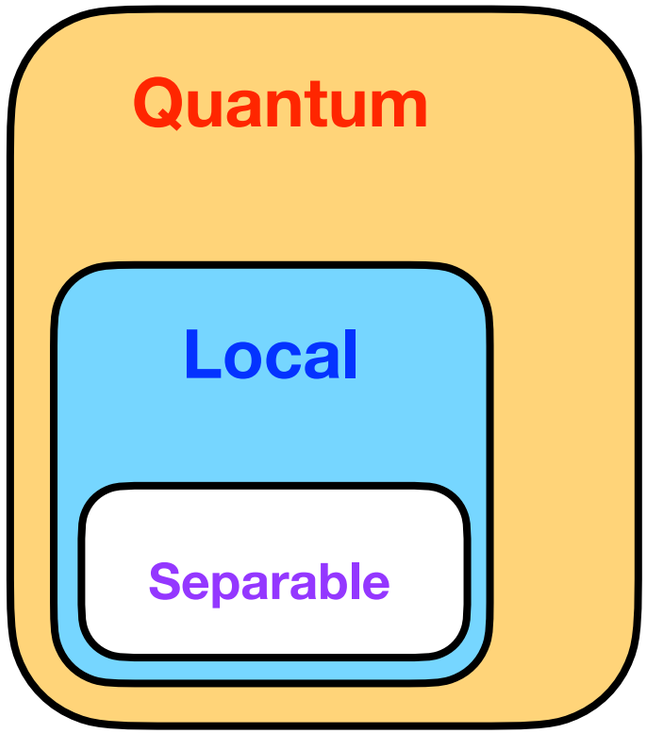

Quantum ⊃ Local ⊃ Separable

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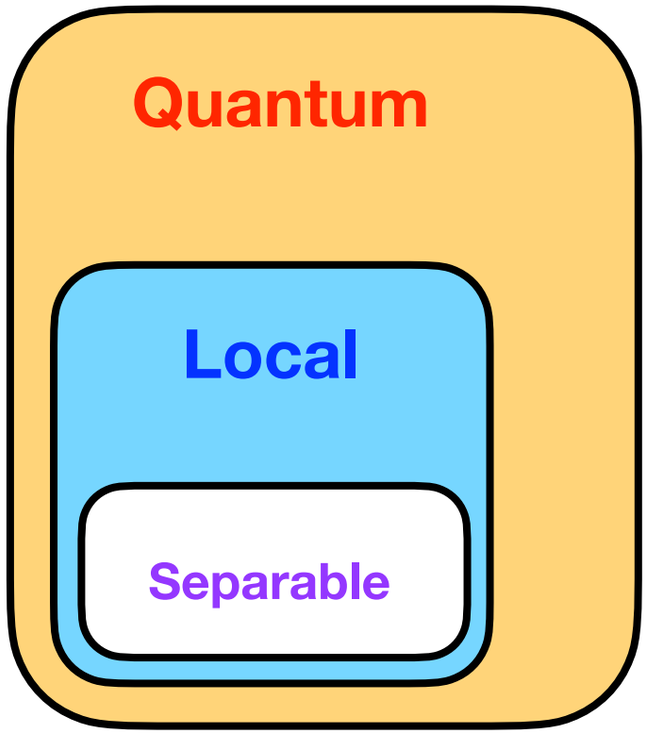

Quantum ⊃ **Local** ⊃ **Separable**

↕ ↕

Bell-nonlocal ⊂ **Entanglement**

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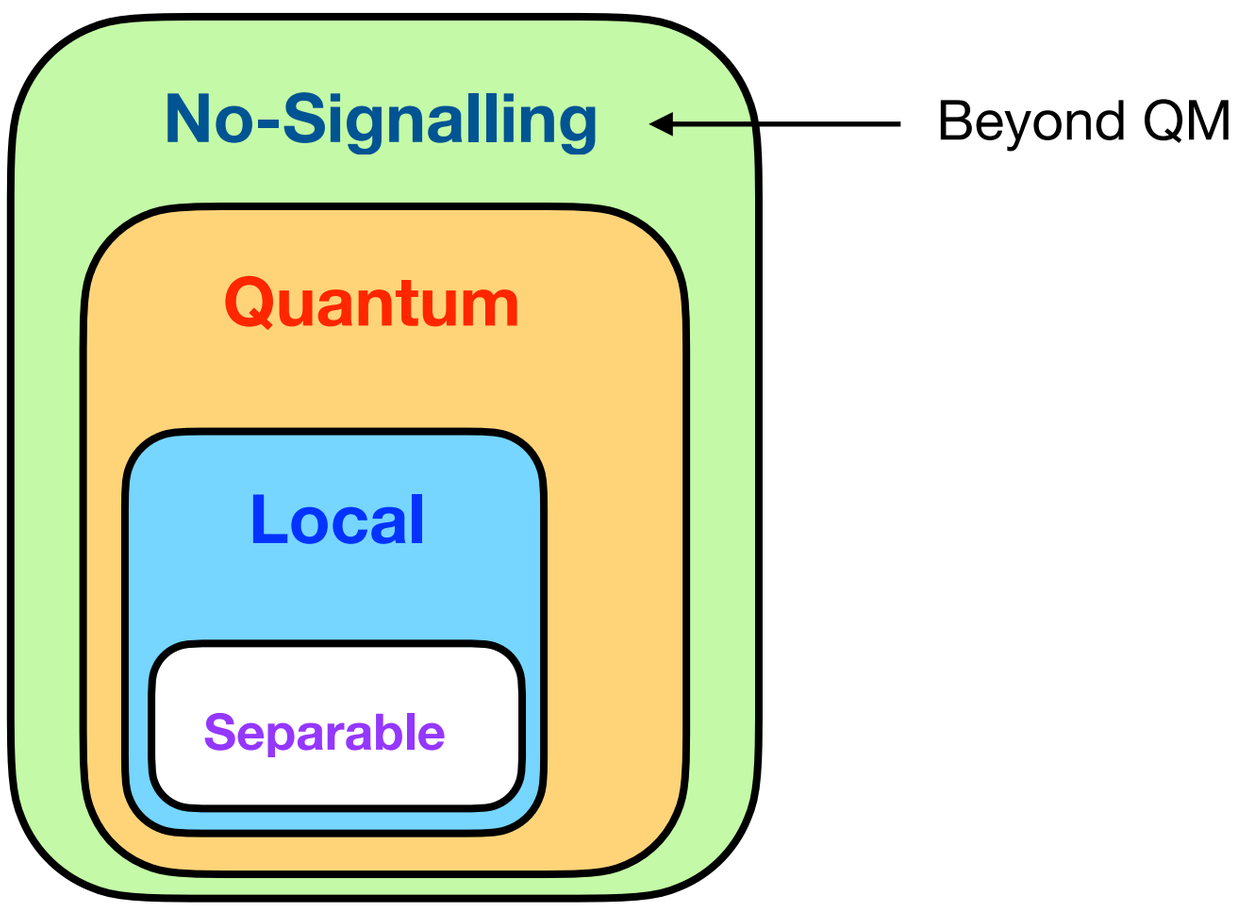
- Nonlocal states in QM does not violate causality

$$p(a\,|\,x,y) \equiv \sum_b p(a,b\,|\,x,y)$$

Condition for no causality violation: No-Signalling [Cirel'son(1980), Popescu, Rohrlich(1994)]

$$^{\forall}a,b,x,x',y,y' \quad \begin{cases} p(a\,|\,x,y) = p(a\,|\,x,y') & \text{Alice's dist. is indep. of Bob's choice for meas. axis} \\ p(b\,|\,x,y) = p(b\,|\,x',y) & \text{Bob's dist. is indep. of Alice's choice for meas. axis} \end{cases}$$

No-Signalling ⊃ Quantum ⊃ Local ⊃ Separable



CHSH Inequalities

- Bell-type inequalities (in general) are the inequalities that separate different types of distributions (No-signalling, Quantum, Local).

- Define the correlator $C_{xy} = \langle A_x B_y \rangle \equiv \sum_{a,b} abp(a,b\,|\,x,y)$

- CHSH quantity [Clauser-Horne-Shimony-Holt(1969)]

For $a, b \in \{\pm 1\}$, $x \in \{\mathbf{n}_1, \mathbf{n}_2\}$, $y \in \{\mathbf{e}_1, \mathbf{e}_2\}$

$$S_{\mathrm{CHSH}} \equiv C_{\mathbf{n}_1,\mathbf{e}_1} + C_{\mathbf{n}_1,\mathbf{e}_2} + C_{\mathbf{n}_2,\mathbf{e}_1} - C_{\mathbf{n}_2,\mathbf{e}_2}$$

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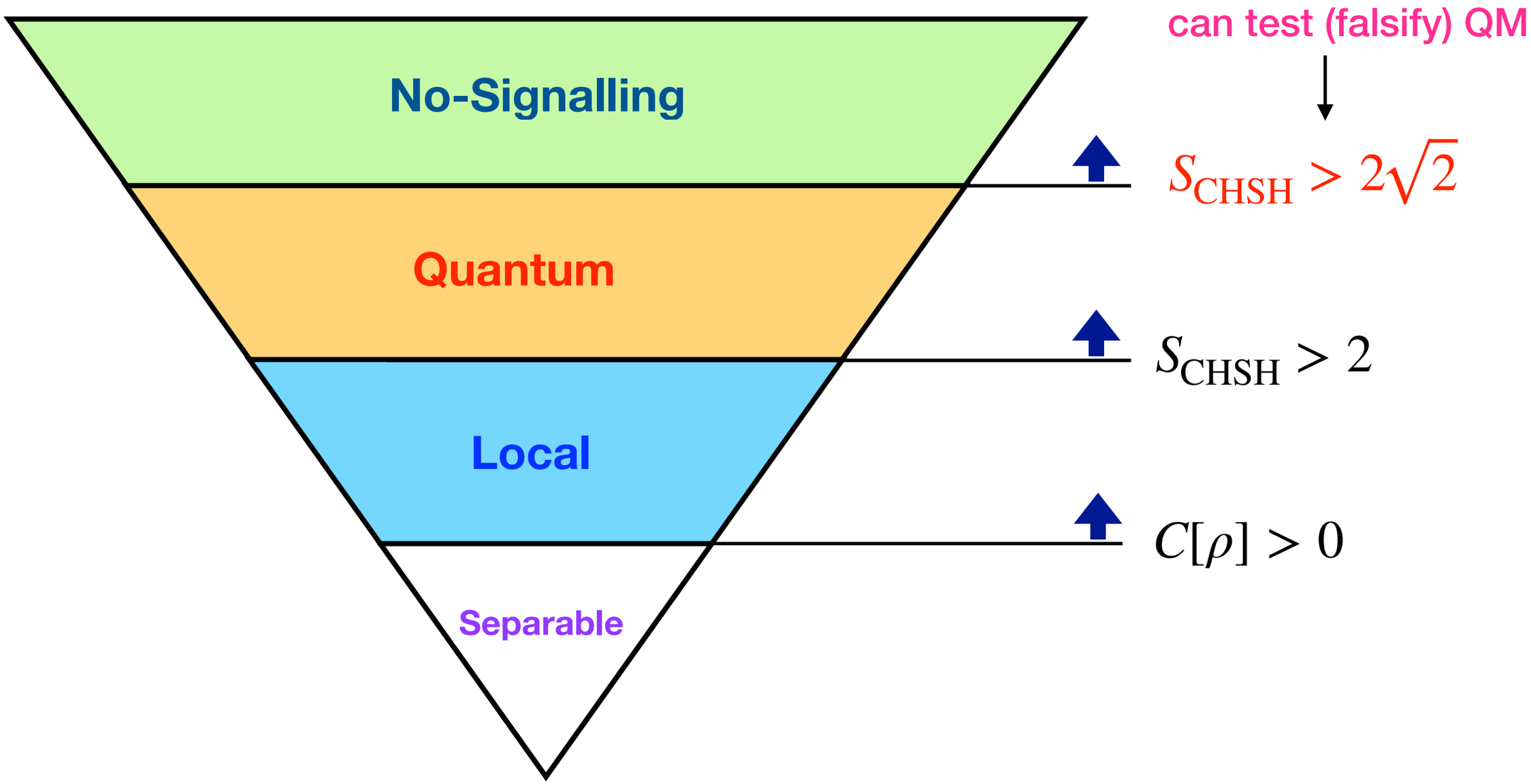
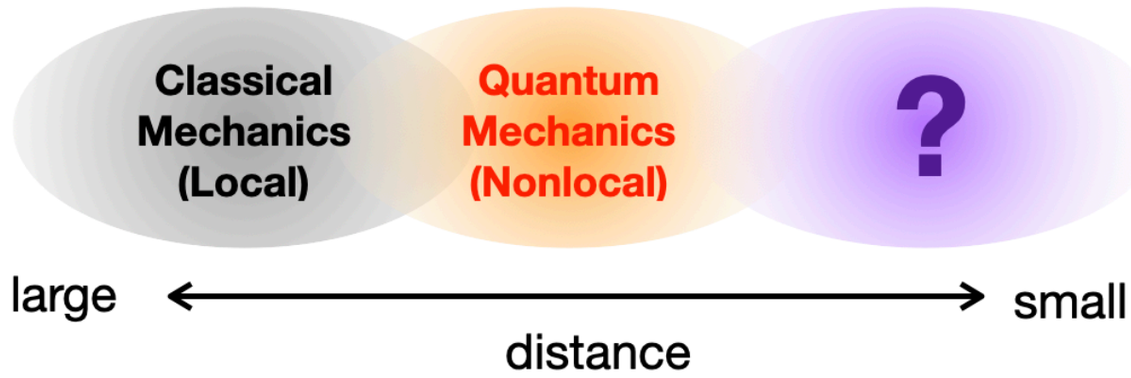
$$S_{\mathrm{CHSH}} \leq \begin{cases} 2 & \textbf{Local theories} \quad \text{[CHSH(1969)]} \\ 2\sqrt{2} & \textbf{Quantum Mechanics} \quad \text{[Tsirelson(1987)]} \\ 4 & \textbf{No-signalling} \quad \text{[Popescu, Rohrlich(1994)]} \end{cases}$$

No-signalling ⊃ Quantum ⊃ Local ⊃ Separable

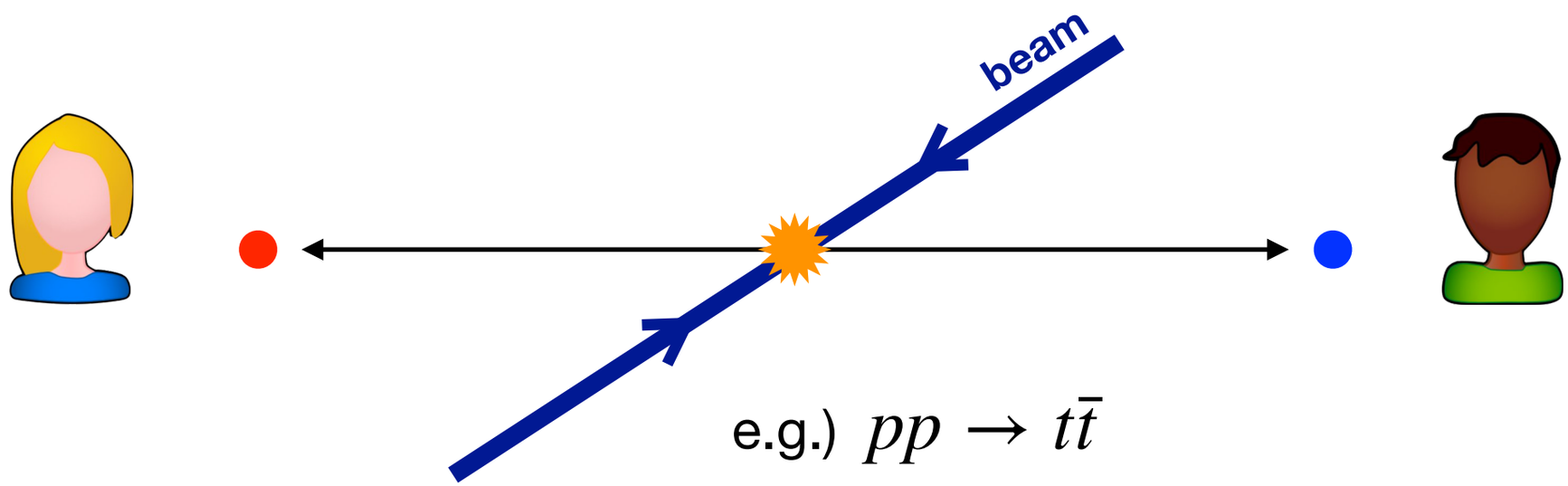


High Energy Test of QM

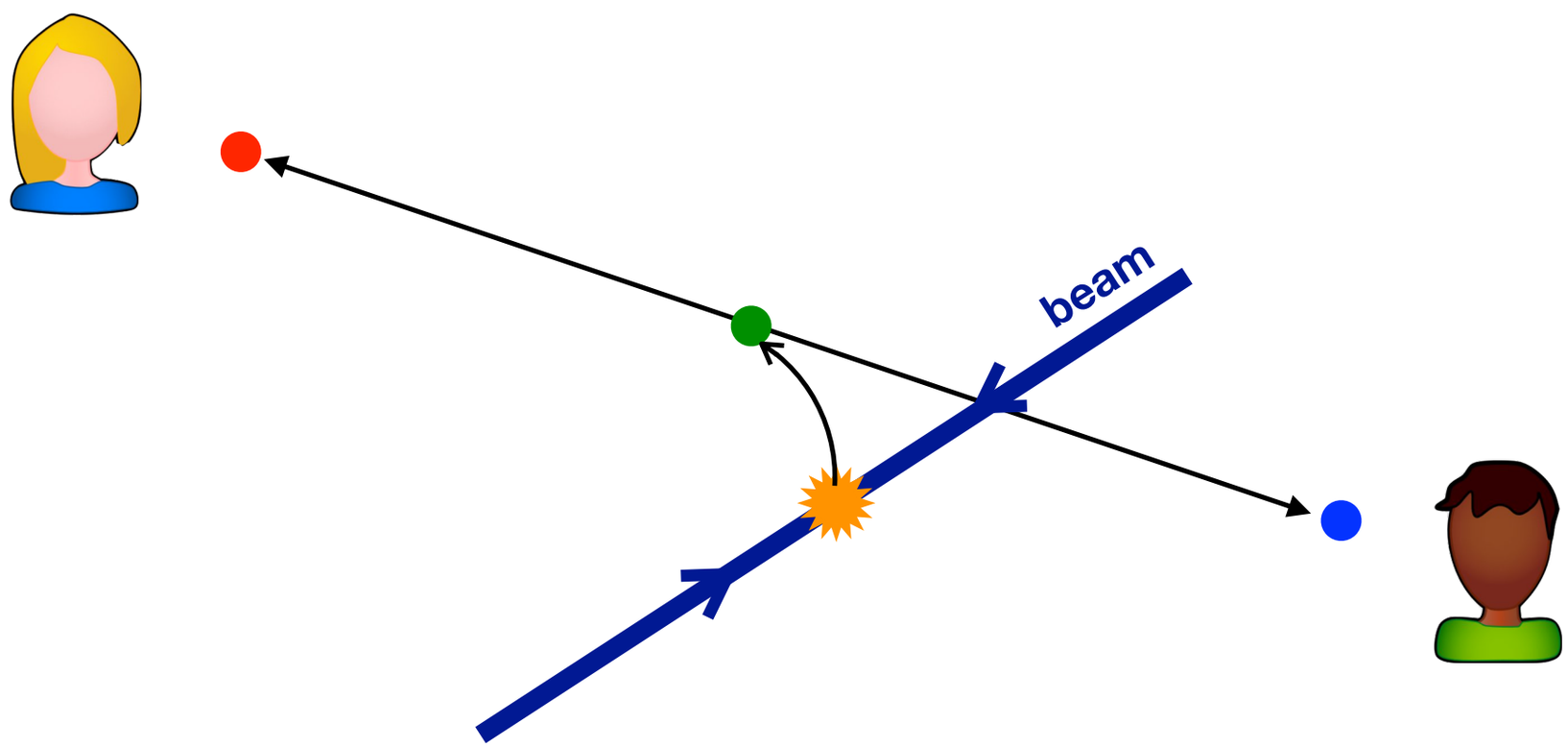


- At very short-distances (high-energies), QM might be modified.
  - sense of locality may change (again)
  - QM might be modified to be married with gravity
- Possible modification of QM?
  - No-signalling theories: $\langle \mathcal{B} \rangle_{\rm NS} \leq 4$ [Cirel'son (1980), Popescu, Rohrlich (1994)]
  - Non-linear extensions of QM: [Weinberg (1989), Polchinski (1991), D.E.Kaplan, S.Rajendran, (2021)]
- Quantum Information quantities (entanglement, CHSH quantity) are often sensitive to BSM and non-perturbative effects. They might open a door to "new" physics.

Entangled pairs at Colliders



e.g.) $pp \to t\bar{t}$



e.g.) $H \to \tau^+\tau^-, WW^*, ZZ^*$

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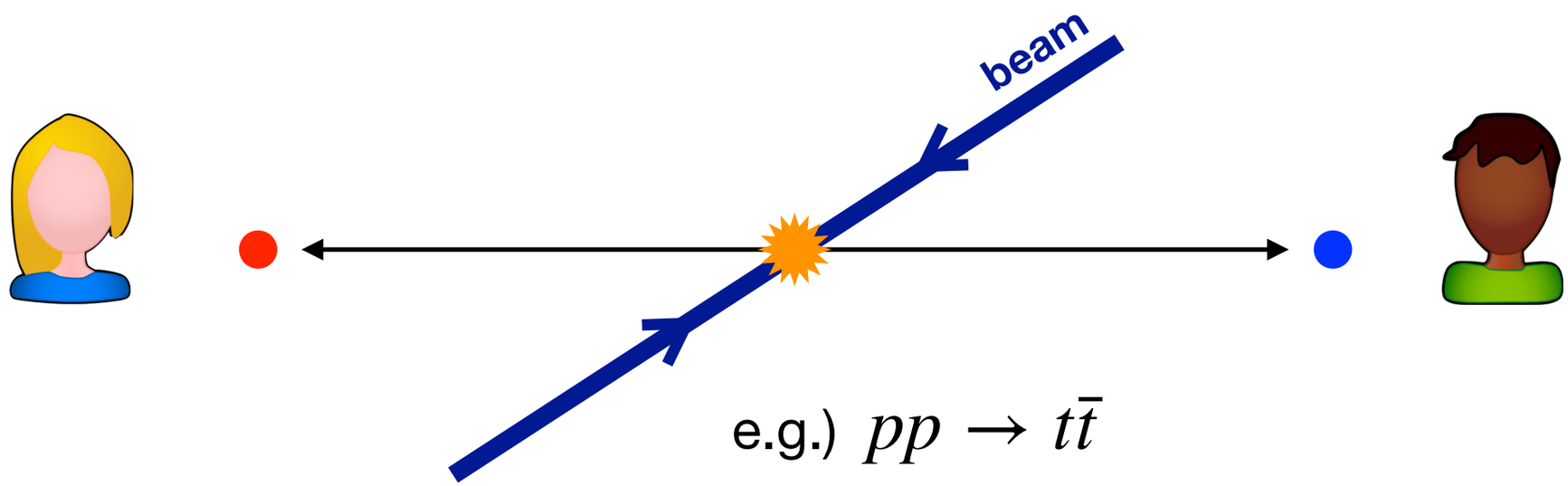
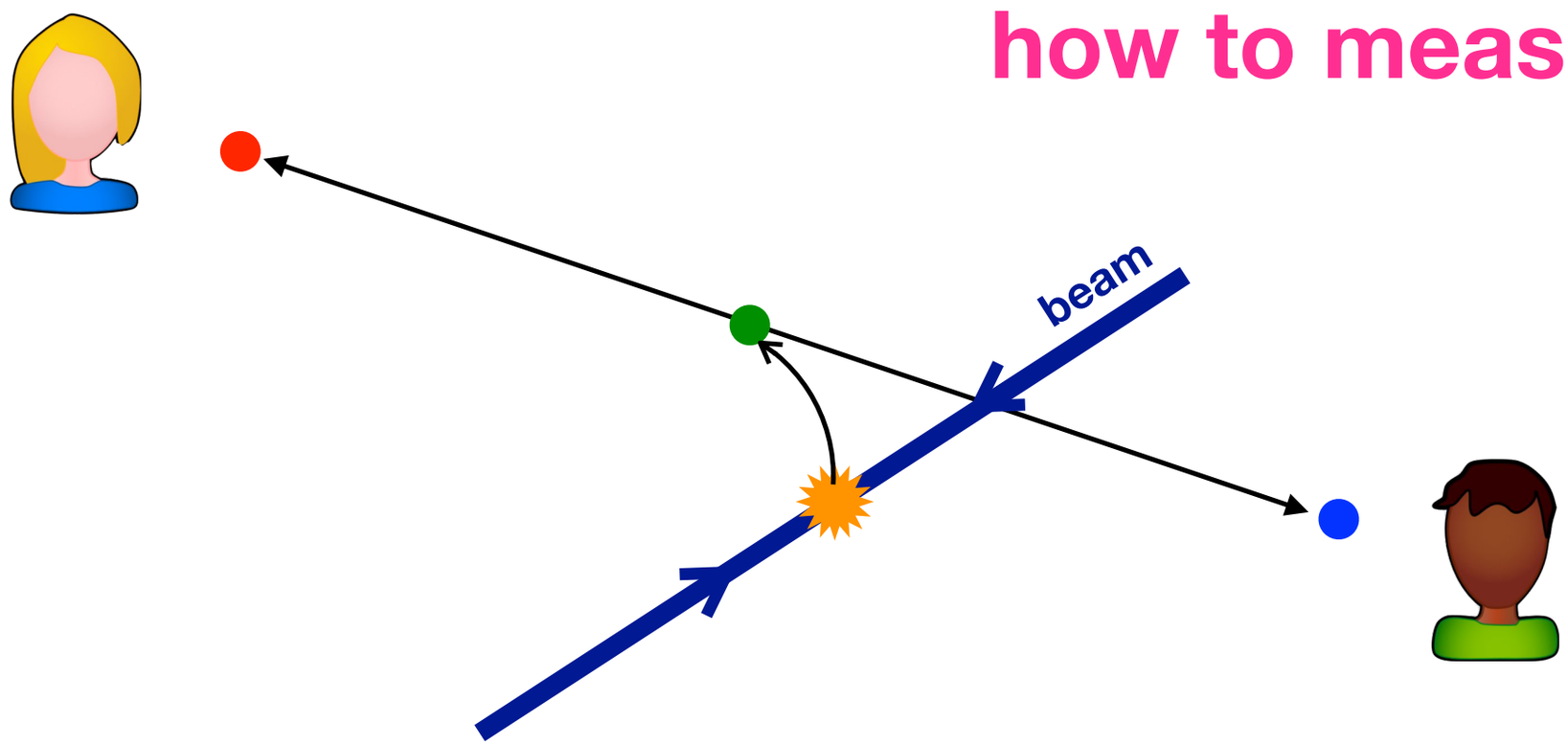


e.g.) $pp \to t\bar{t}$

how to measure spin?



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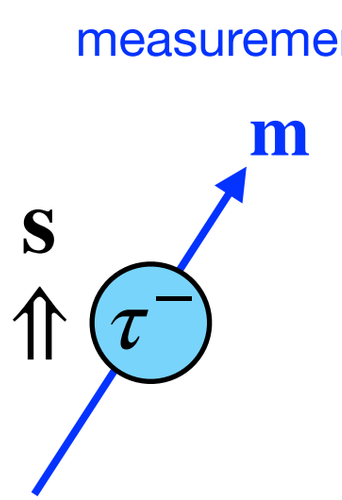
Particles with weak decays are their own polarimeters

e.g.) For $\tau^- \to \pi^- + \nu_\tau$ (τ^- rest frame), the spin of τ^- is measured in the direction of π^- ($\vec{\pi}$) and the outcome is +1.



$$p(+\,|\,\mathbf{m}) = |\langle +_{\mathbf{m}} \,|\, +_{\mathbf{s}} \rangle|^2$$

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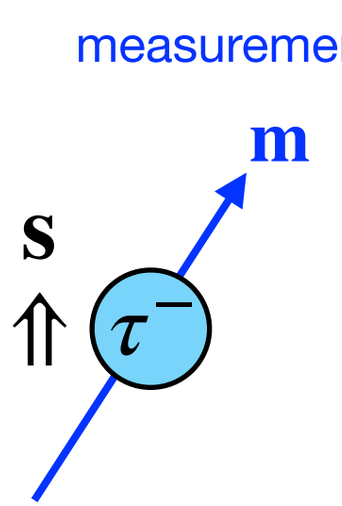

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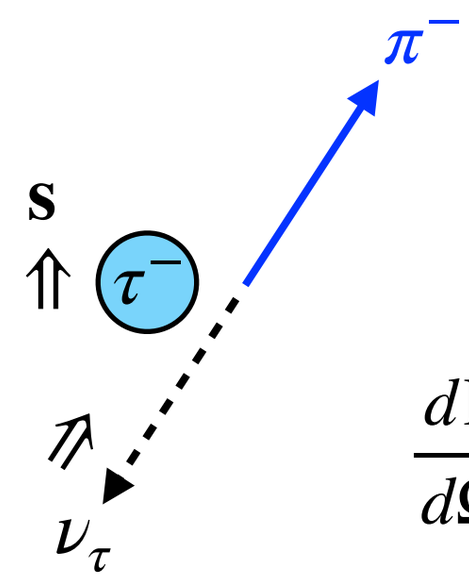

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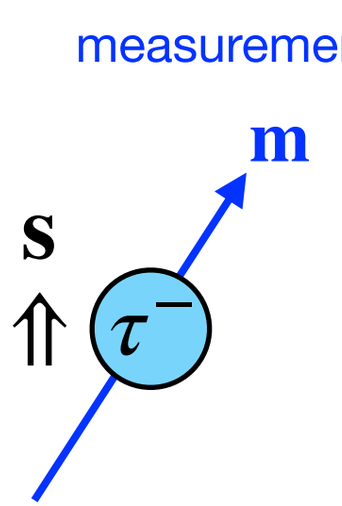

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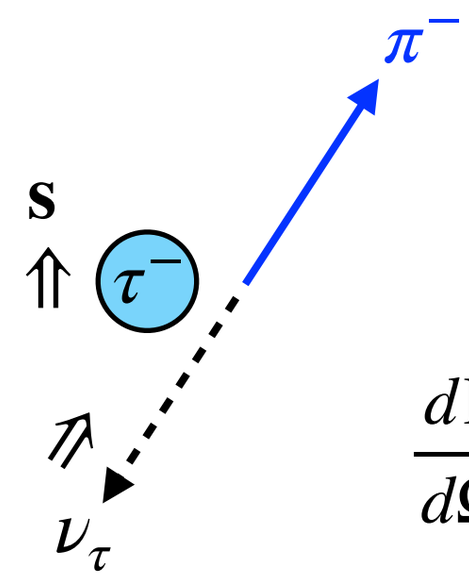

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Weakly decaying particles in the SM:

$$\tau,\ t,\ W^{\pm},\ Z^0$$

Particles with weak decays are their own polarimeters

More generally,

$$\frac{d\Gamma}{d\Omega} = \frac{1 + \alpha_x \cdot (\vec{x} \cdot \mathbf{s})}{2}$$

$\alpha_x \in [-1, +1]$: spin analyzing power

- tau decay

$\alpha_x = 1$ for ($x = \pi^-$ in $\tau^- \to \pi^- \nu$)

- top decay

decay product x	α_x
b	$-0.3925(6)$
W^+	$0.3925(6)$
ℓ^+ (from a W^+)	$0.999(1)$
$\bar{d}$, $\bar{s}$ (from a W^+)	$0.9664(7)$
u, c (from a W^+)	$-0.3167(6)$

Spin correlation:

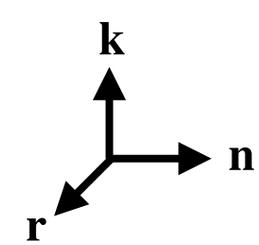
$\mathbf{n}, \mathbf{n}'$: spin measurement axes

$\vec{x}, \vec{y}$: direction of decay products

$$C_{\mathbf{n},\mathbf{n}'} \equiv \langle (\mathbf{s}_A \cdot \mathbf{n})(\mathbf{s}_B \cdot \mathbf{n}') \rangle = \frac{9}{\alpha_x \alpha_y} \langle (\vec{x} \cdot \mathbf{n})(\vec{y} \cdot \mathbf{n}') \rangle$$

$$S_{\mathrm{CHSH}} \equiv C_{\mathbf{n}_1,\mathbf{n}_1'} + C_{\mathbf{n}_1,\mathbf{n}_2'} + C_{\mathbf{n}_2,\mathbf{n}_1'} - C_{\mathbf{n}_2,\mathbf{n}_2'} \begin{cases} > 2 & \to \text{Bell-nonlocal} \\ > 2\sqrt{2} & \to \text{QM violation} \end{cases}$$

$$D \equiv \mathrm{Tr}[C]/3 < -\frac{1}{3}$$ → sufficient cond. for entanglement



Recent activities to look into entanglements, etc. in HEP

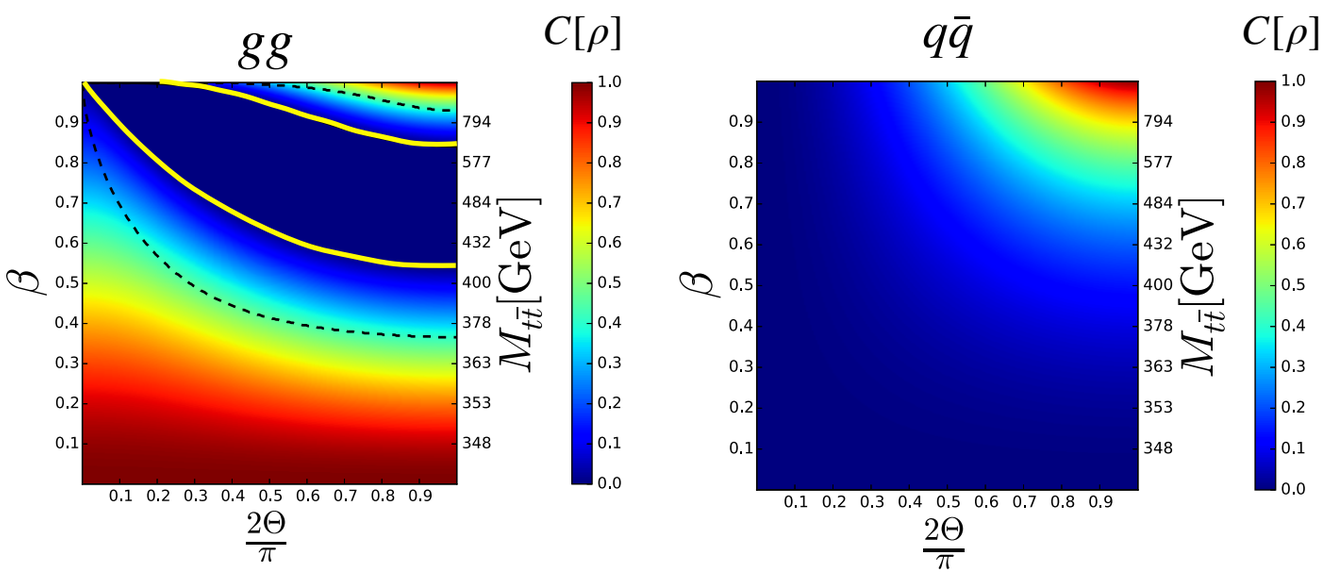
❖ Prospects to measure entanglement and Bell-nonlocality @ LHC

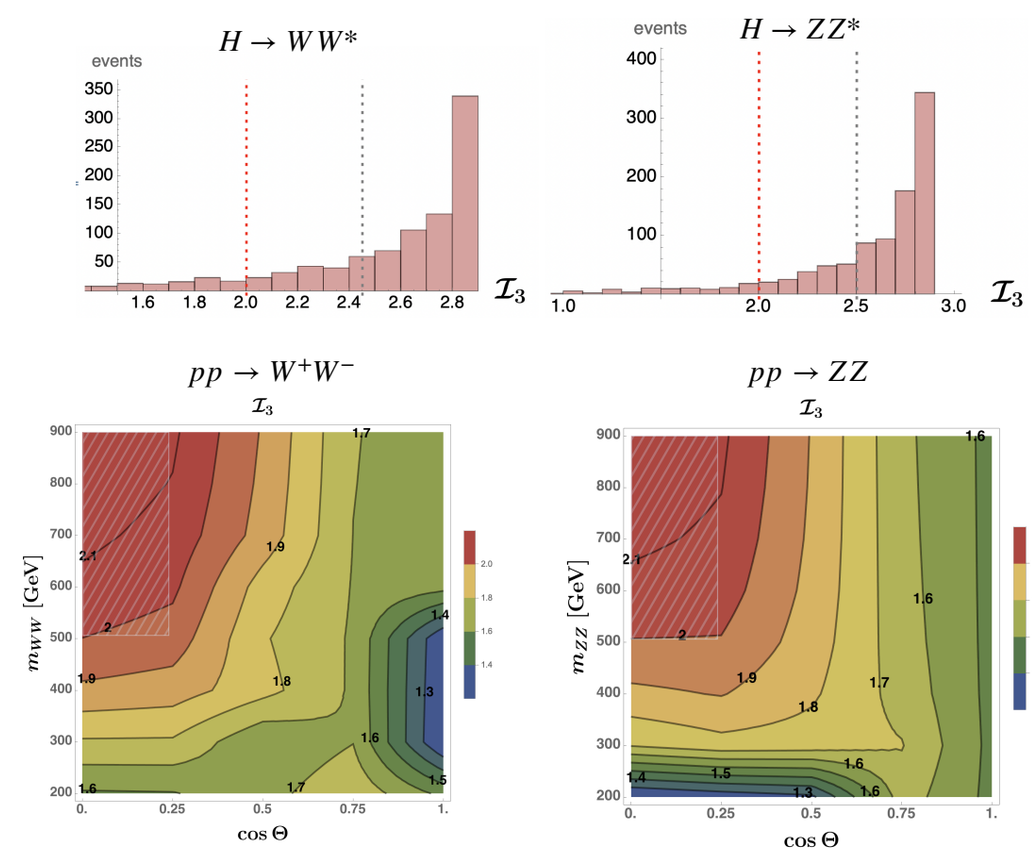
- $pp \to t\bar{t}$ — Y. Afik and J. R. M. de Nova ’21, ’22, M. Fabbrichesi, R. Floreanini, G. Panizzo ‘21 Z. Dong, D. Gonçalves, K. Kong, A. Navarro ‘23

- $H \to WW, ZZ$ — A. J. Barr ’21, J.A. Aguilar-Saavedra, A. Bernal, J.A. Casas, J.M. Moreno ’22, A. Bernal, P. Caban, J. Rembieliński ’23, M. Fabbrichesi, R. Floreanini, E. Gabrielli, Luca Marzola ‘23

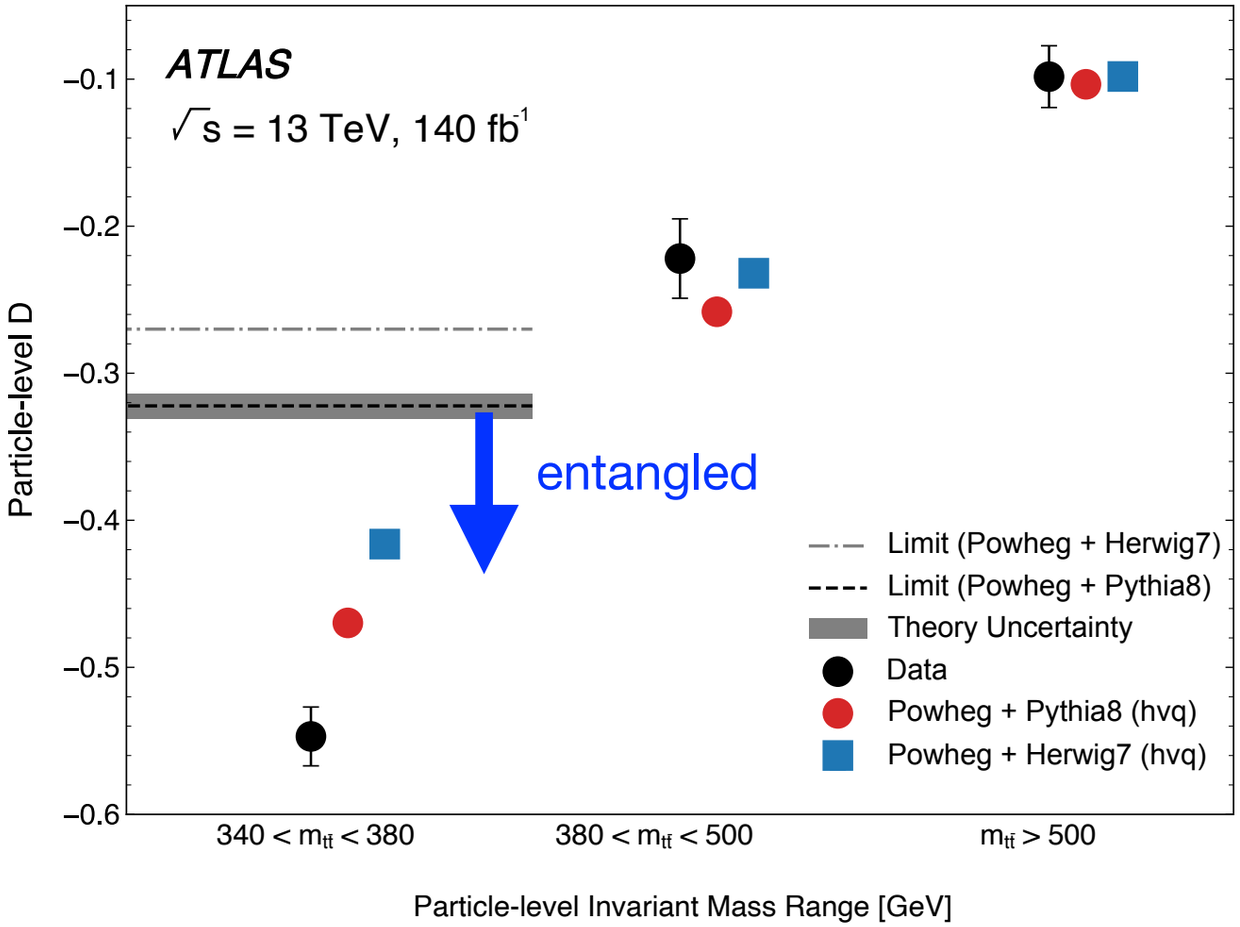
  $pp \to WW, ZZ$

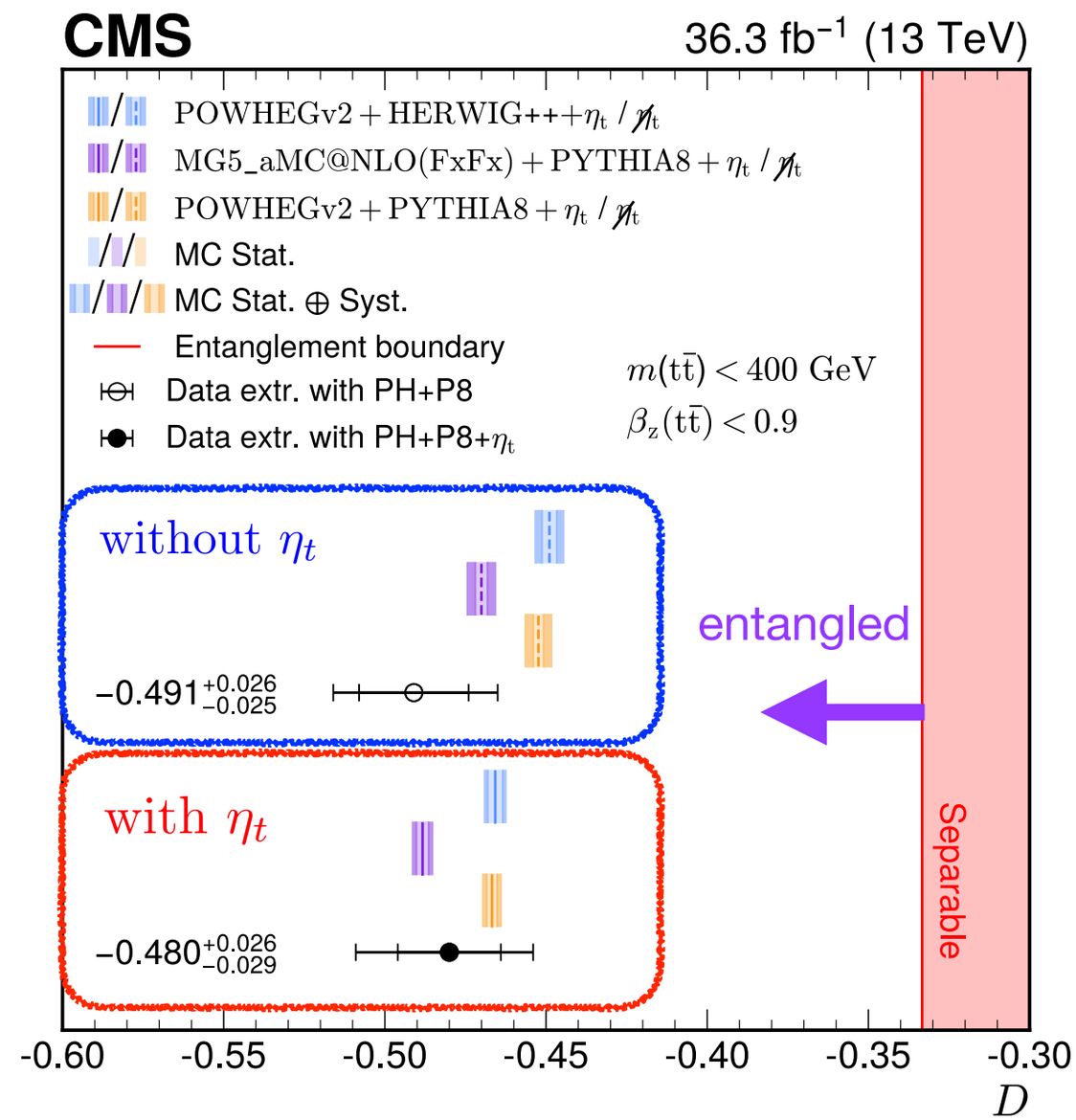
- $H \to \tau^+\tau^-$ (@ e^+e^- colliders) — M. Fabbrichesi, R. Floreanini, E. Gabrielli 22, M. Altakach, P. Lamba, F. Maltoni, K. Mawatari, KS ’22, K. Ma, T. Li ‘23





Entanglement Observation at LHC $pp \to t\bar{t}$

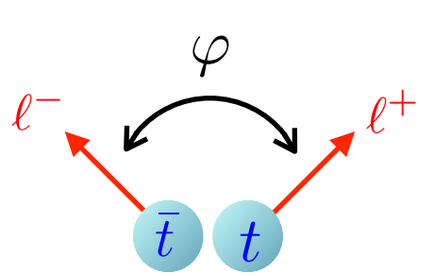




$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\varphi} = \frac{1}{2}(1 - D\cos\varphi)$$

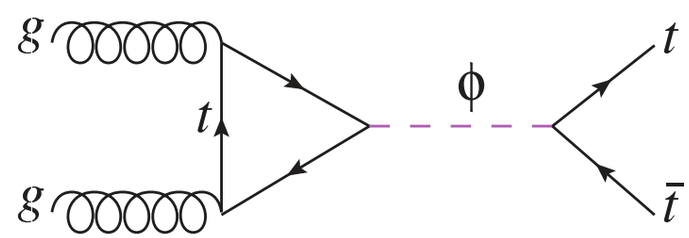
$$\rightarrow \quad D = -3 \cdot \langle \cos\varphi \rangle = \frac{\mathrm{tr}[\mathbf{C}]}{3} < -\frac{1}{3}$$

(sufficient condition for entanglement)



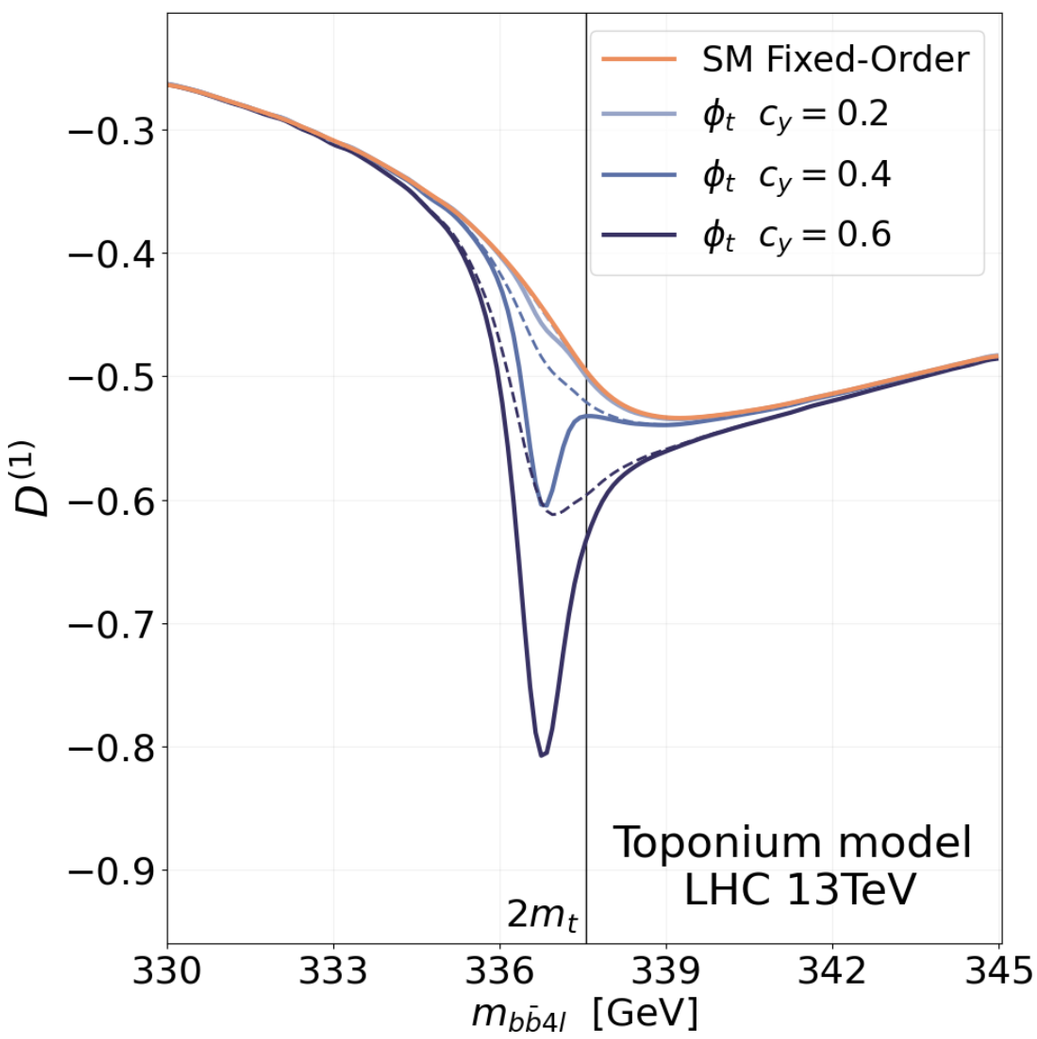
Maltoni, Severi, Tentori, Vryonidou [2401.08751]

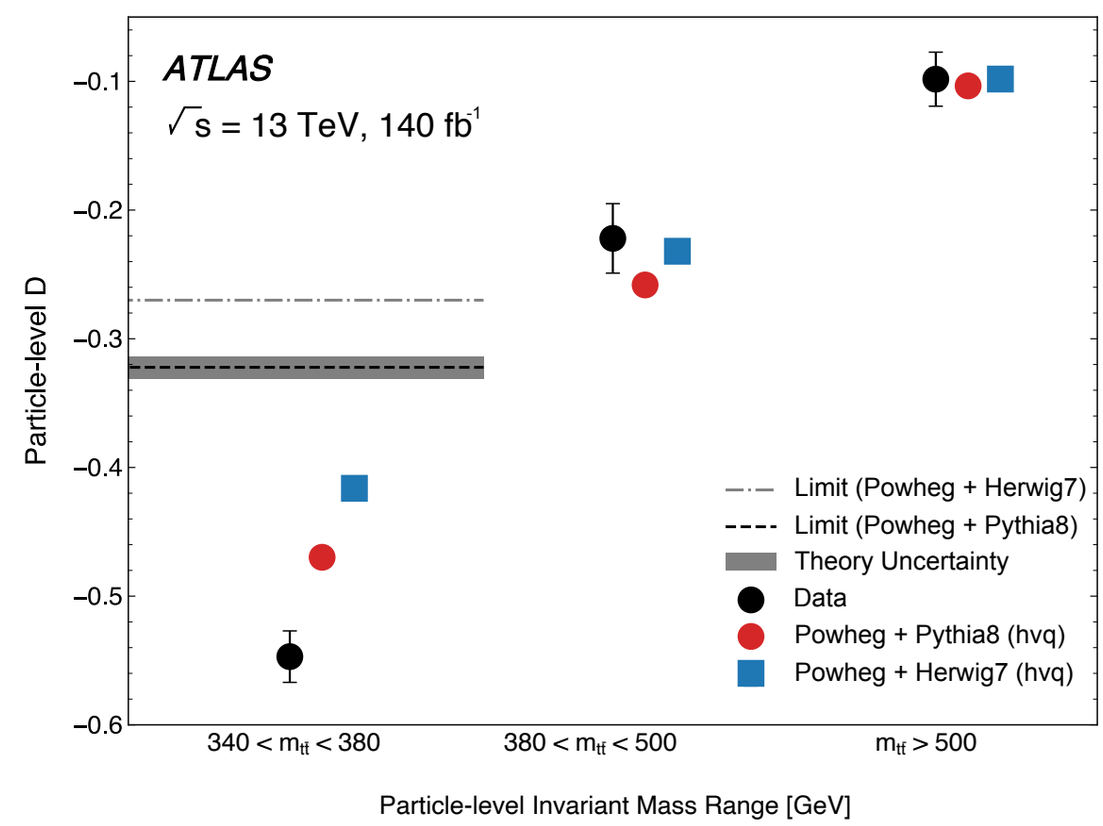
$$\mathcal{L} = \mathcal{L}_{\rm SM} - \frac{1}{2}\phi(\partial^2 + M_\phi^2)\phi + c_y \frac{y_t}{\sqrt{2}}\,\phi\,\bar{t}\,(\cos\alpha + i\gamma^5 \sin\alpha)\,t$$



$$|\phi\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

$M_\phi = 343.5\,\text{GeV}, \quad \alpha = \pi/2$

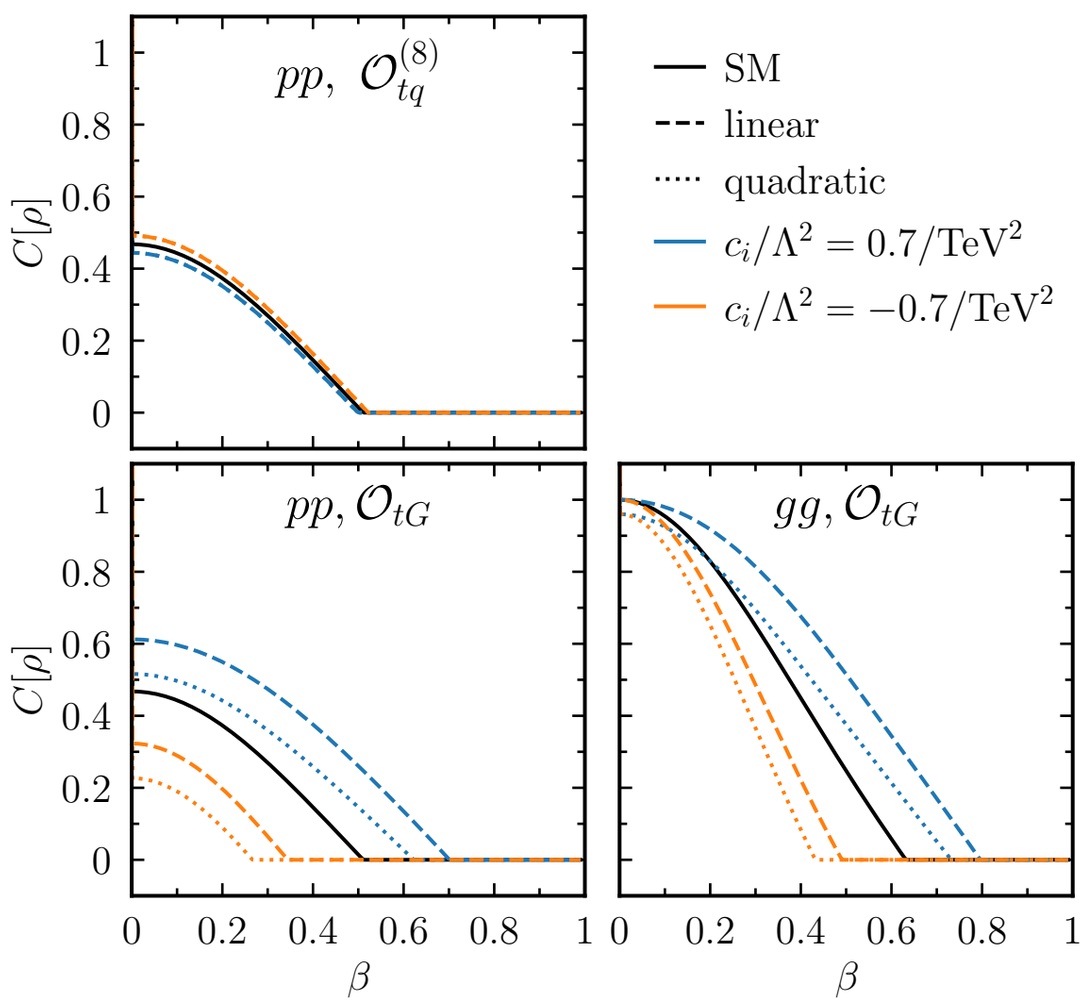




Effect of BSM $pp \to t\bar{t}$

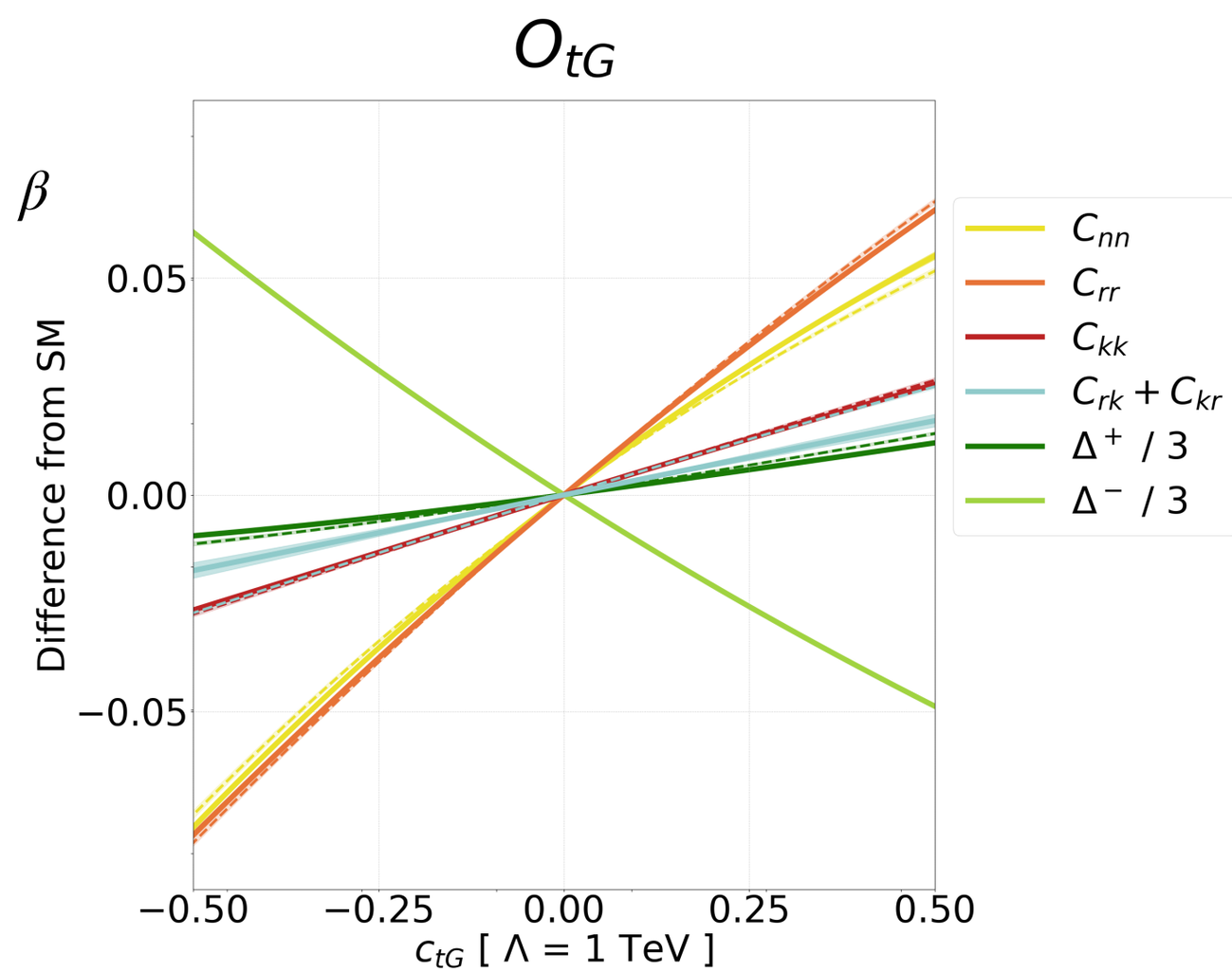
$$\mathcal{O}_{tG} = g_S\, \overline{Q} T_A \tilde{\varphi} \sigma^{\mu\nu} t\, G^A_{\mu\nu} \qquad \mathcal{O}^8_{tq} = \sum_{\mathrm{f}=1}^{2} (\bar{q}_{\mathrm{f}} \gamma_\mu T_A q_{\mathrm{f}})(\bar{t} \gamma^\mu T^A t)$$

[Aoude Madge Maltoni Mantani (2022)]



β: top velocity in the $t\bar{t}$ rest frame

[Severi Vryonidou (2023)]



$$\Delta^{\pm} = \pm(C_{kk} + C_{rr}) - C_{nn} - 1$$

$H \to \tau^+\tau^-$ @ e^+e^- colliders

- M. Altakach, F. Maltoni, K. Mawatari, P. Lamba, KS, [2211.10513] (PRD)

3-particle entanglement

- KS, M. Spannowsky [2310.01477] (PRL)

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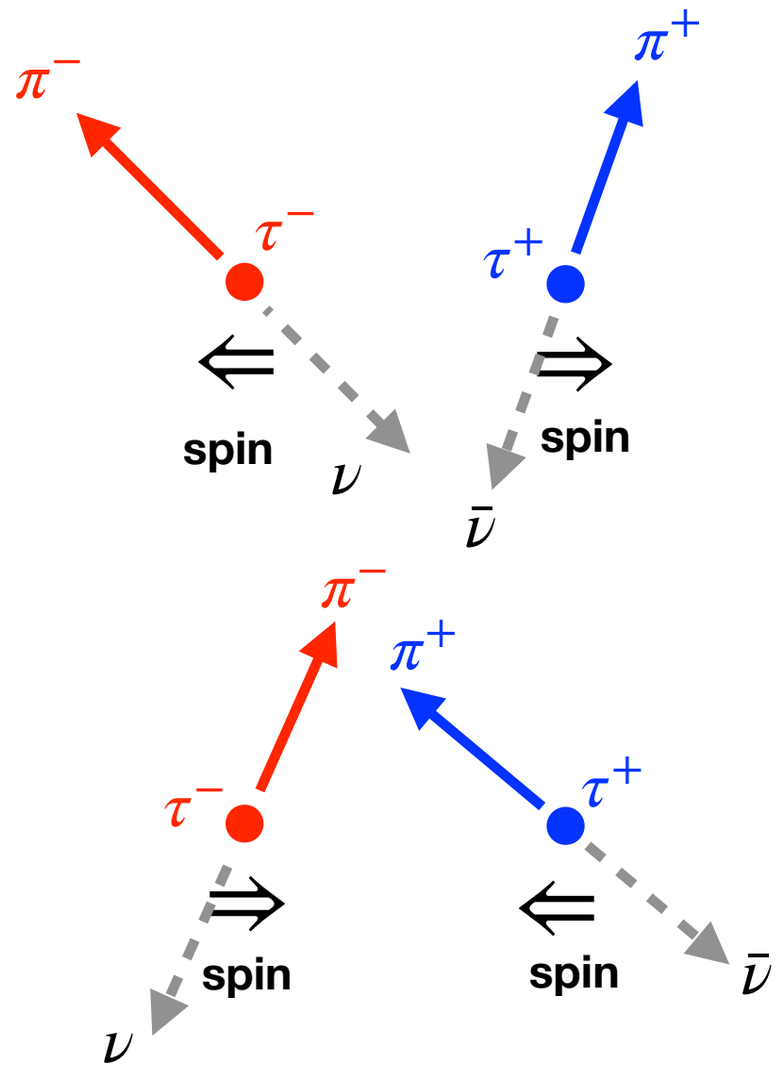
Experimental Challenge in $H \to \tau^+\tau^-$

Among weakly decaying particles, τ, t, $W^\pm$, Z^0, the tau-lepton is special because $m_\tau \ll m_H$

One has to measure the direction of pions at the rest frame of each tau.

→ Reconstruction of the tau rest frames (i.e. neutrino reconstruction) is necessary



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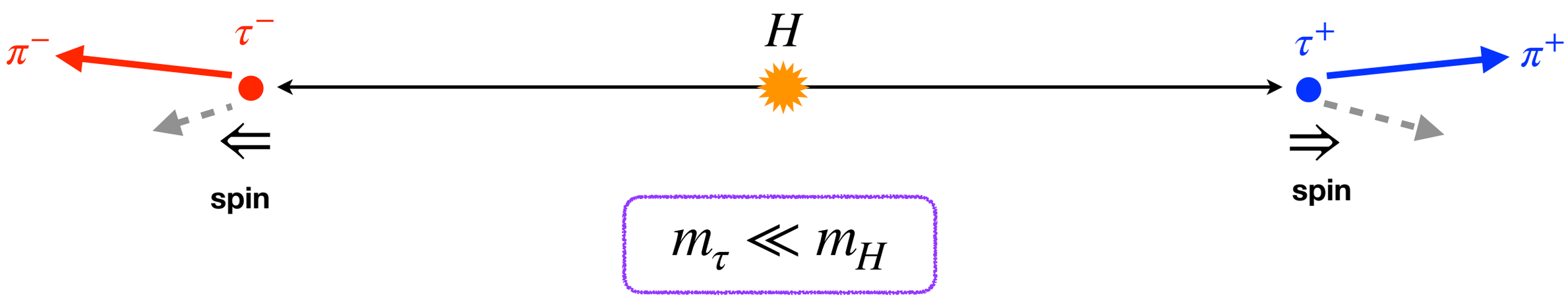
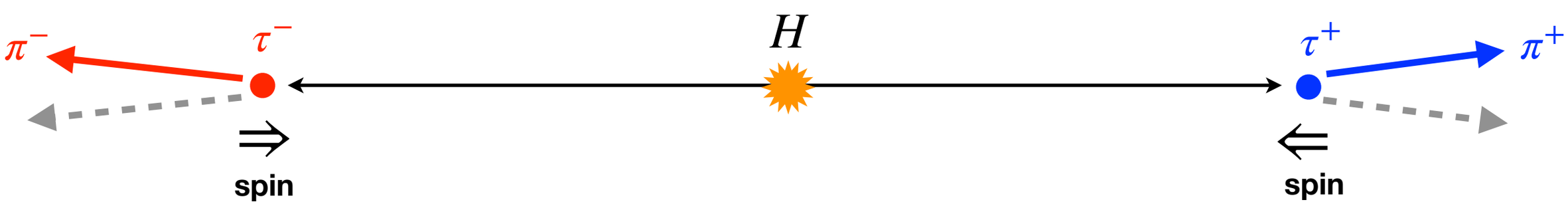
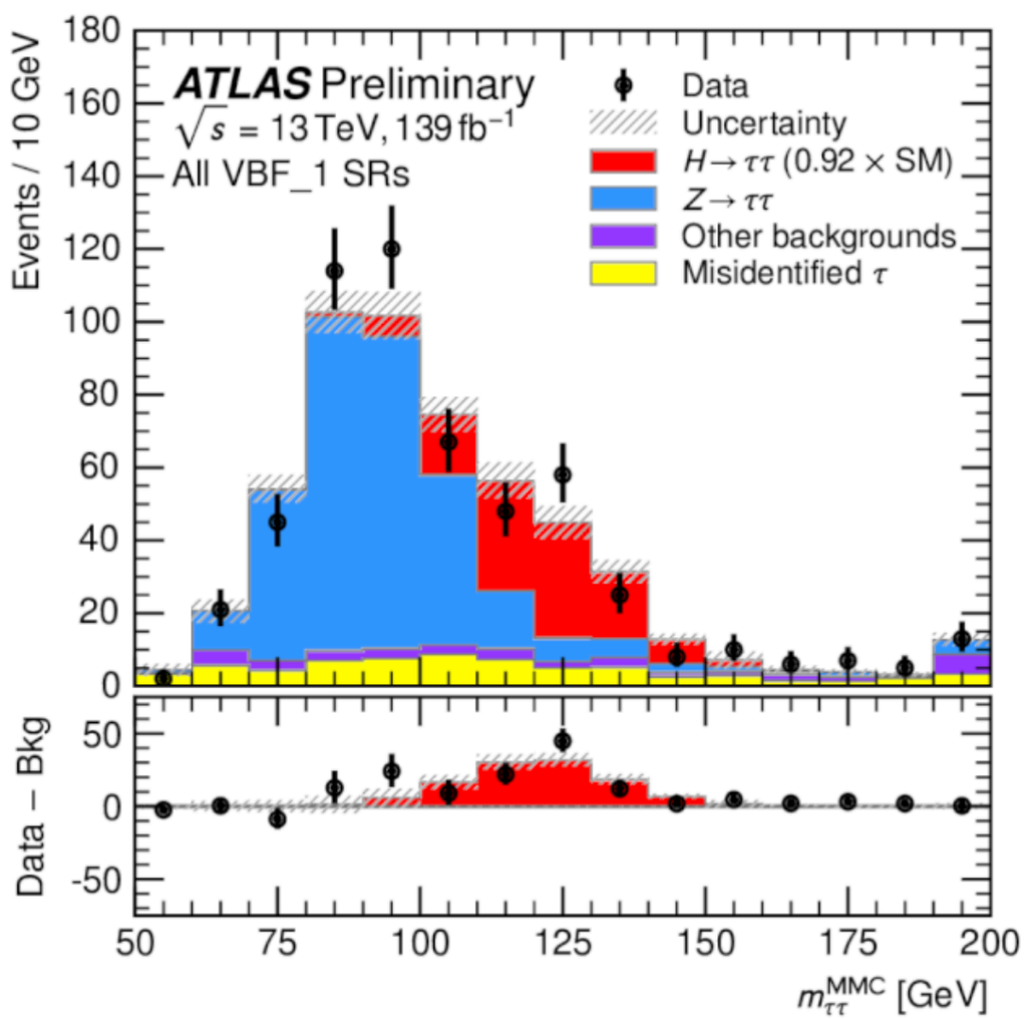




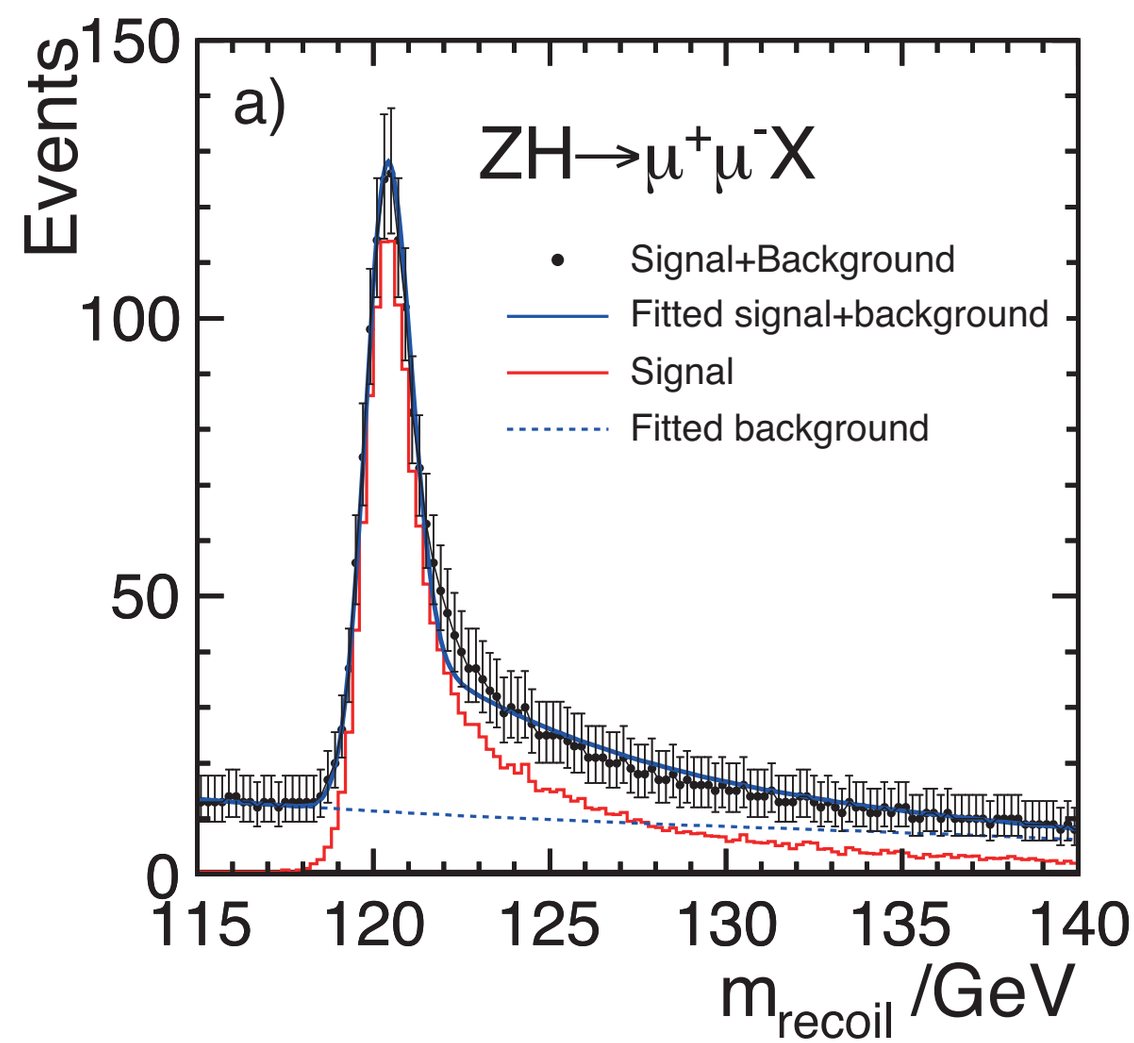
$H \to \tau^+\tau^-$ @ lepton colliders

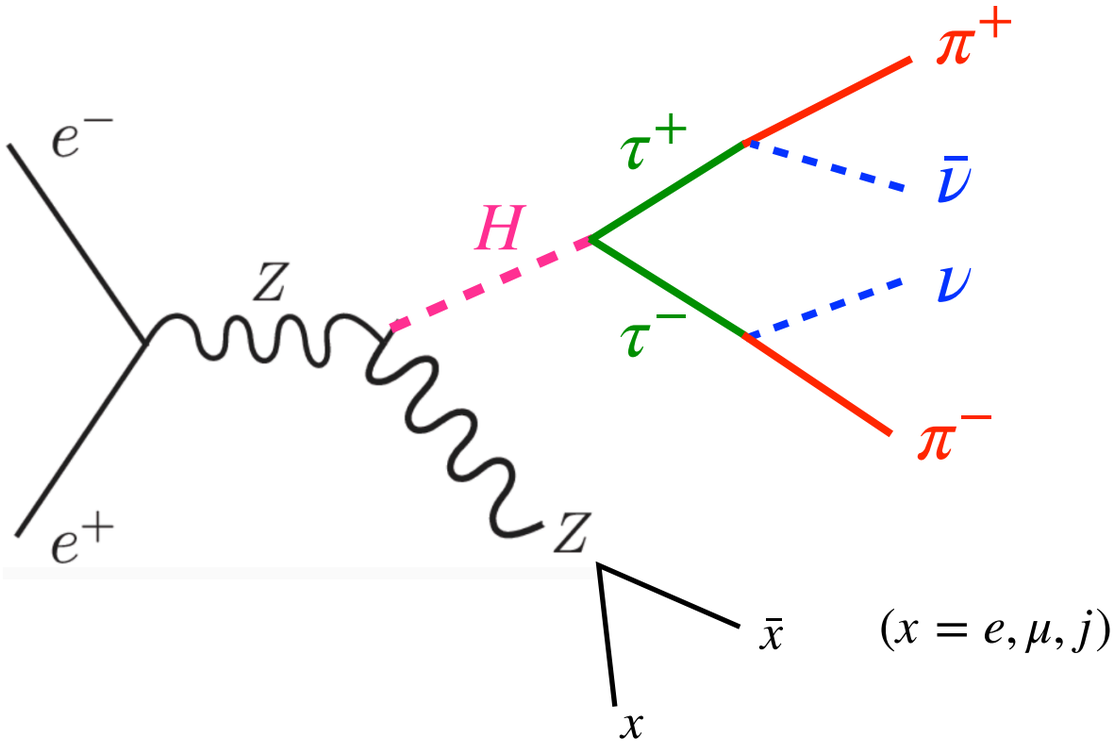
- For precise event reconstruction and for much smaller background, we consider lepton colliders.

LHC



ILC





$$(P_H^{\rm reco})^\mu \equiv P^\mu_{e^+e^-} - P^\mu_{Z\to x\bar{x}} \qquad M^2_{\rm recoil} \equiv (P_H^{\rm reco})^2$$

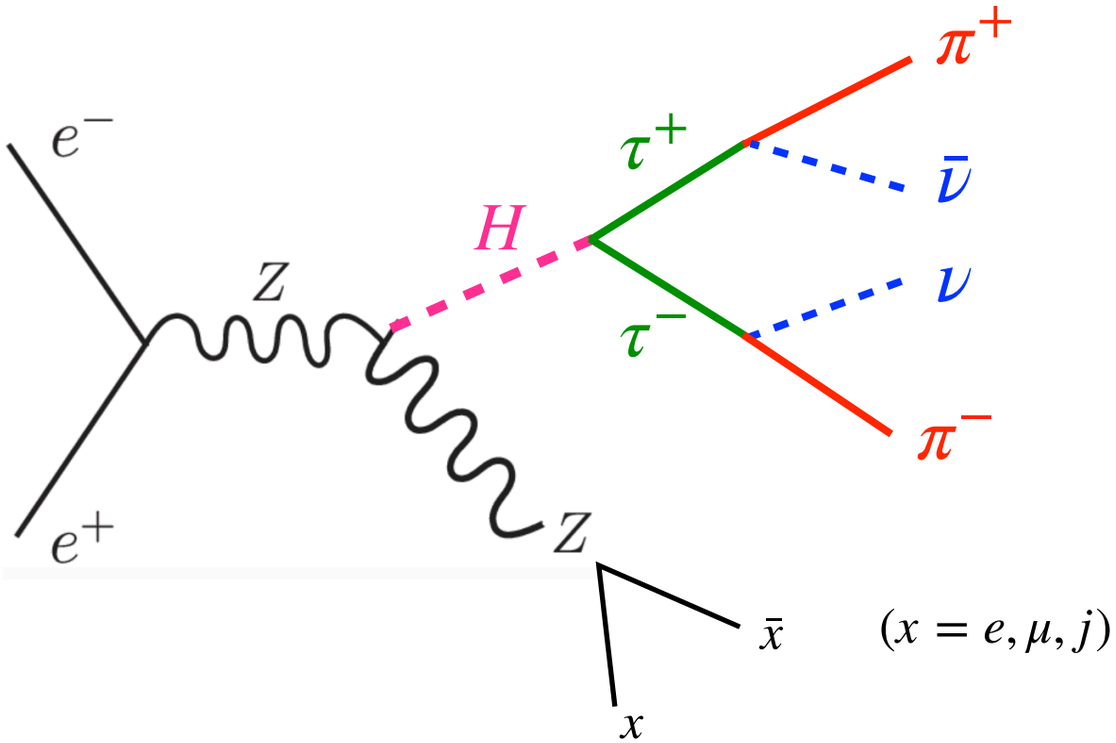
Event selection: $|M_{\rm recoil} - 125\,{\rm GeV}| < 5\,{\rm GeV}$

$$e^+e^- \to Z + (Z^*/\gamma^*) \to f\bar{f} + \tau^+\tau^-$$

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	0.83×10^{-4}
beam resolution e^- (%)	0.27	0.83×10^{-4}
$\sigma(e^+e^- \to HZ)$ (fb)	240.1	240.3
# of signal ($\sigma \cdot \mathrm{BR} \cdot L \cdot \epsilon$)	385	663
# of background ($\sigma \cdot \mathrm{BR} \cdot L \cdot \epsilon$)	20	36

- Generate the SM events $(\kappa, \delta) = (1,0)$ with **MadGraph5**.

- **100 pseudo-experiments** to estimate the statistical uncertainties



$$(P_H^{\text{reco}})^\mu \equiv P^\mu_{e^+e^-} - P^\mu_{Z\to x\bar{x}} \qquad M^2_{\text{recoil}} \equiv (P_H^{\text{reco}})^2$$

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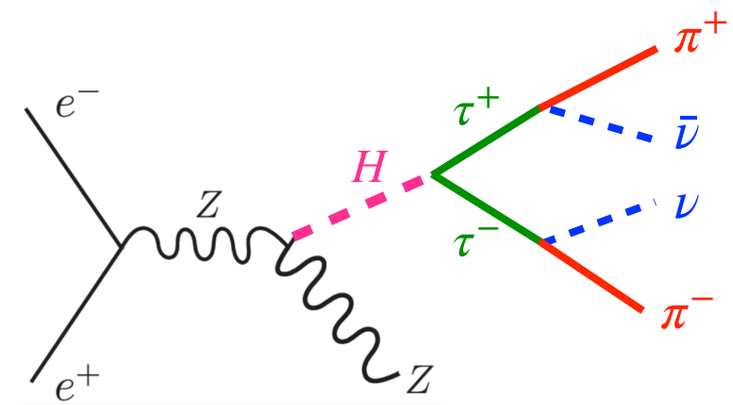
- Generate the SM events $(\kappa, \delta) = (1,0)$ with **MadGraph5**.
- **100 pseudo-experiments** to estimate the statistical uncertainties

- To determine the tau momenta, we have to reconstruct the unobserved neutrino momenta $(p_x^\nu, p_y^\nu, p_z^\nu)$, $(p_x^{\bar{\nu}}, p_y^{\bar{\nu}}, p_z^{\bar{\nu}})$.



- **6** unknowns can be constrained by **2** mass-shell conditions and **4** energy-momentum conservation.

$$m_\tau^2 = (p_{\tau^+})^2 = (p_{\pi^+} + p_{\bar{\nu}})^2$$

$$m_\tau^2 = (p_{\tau^-})^2 = (p_{\pi^-} + p_\nu)^2$$

$$(p_{ee} - p_Z)^\mu = p_H^\mu = \left[(p_{\pi^-} + p_\nu) + (p_{\pi^+} + p_{\bar{\nu}})\right]^\mu$$

=> 2-fold solutions.

$$|\Psi^{\rm SM}_{\tau^+\tau^-}\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$

$$C_{ij}^{\rm SM} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$C_{\rm SM}[\rho] = 1$$

$$S_{\rm CHSH} = 2\sqrt{2}$$

reproduced very accurately in the simulation

→ we found that false solutions also give the correct correlations! (?)

Effect of momentum mismeasurement

$$E_i^{\text{true}} \to E_i^{\text{obs}} = (1 + \sigma_E \cdot \omega) \cdot E_i^{\text{true}} \qquad \sigma_E = 0.03 \qquad (i = \pi^\pm, e^\pm, \mu^\pm, j)$$

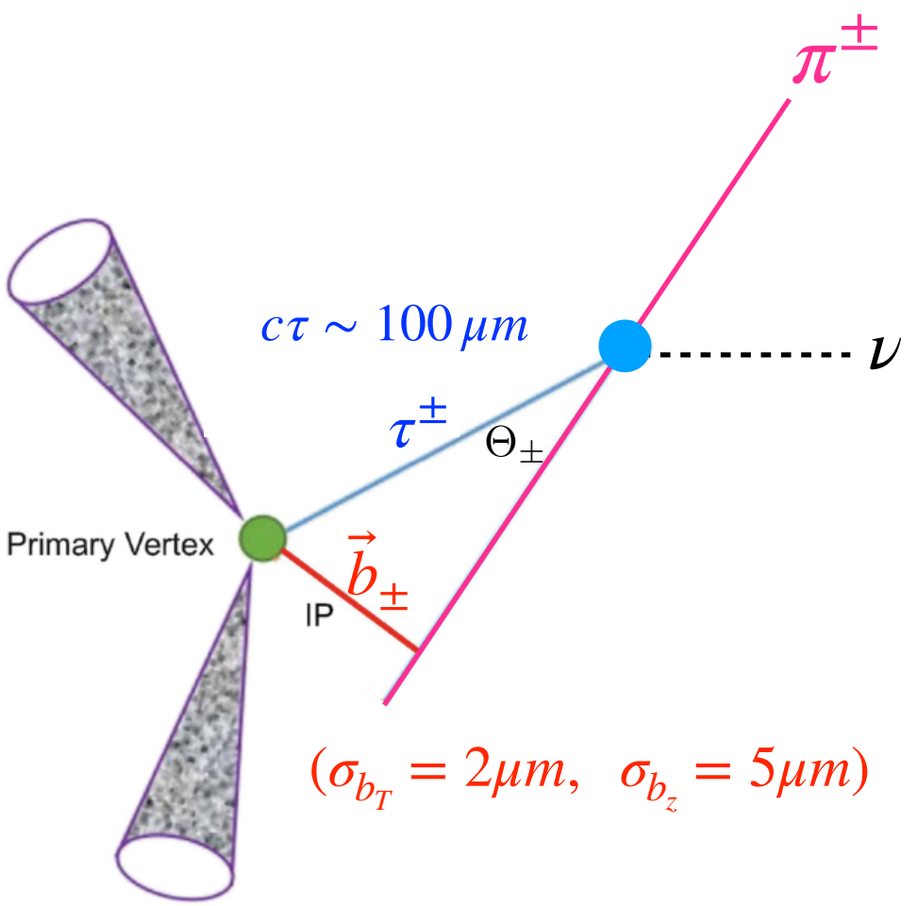
random number drawn from the normal distribution

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} -0.600 \pm 0.210 & 0.003 \pm 0.125 & 0.020 \pm 0.149 \\ 0.003 \pm 0.125 & -0.494 \pm 0.190 & 0.007 \pm 0.128 \\ 0.048 \pm 0.174 & 0.0007 \pm 0.156 & 0.487 \pm 0.193 \end{pmatrix}$	$\begin{pmatrix} -0.559 \pm 0.143 & -0.010 \pm 0.095 & -0.014 \pm 0.122 \\ -0.010 \pm 0.095 & -0.494 \pm 0.139 & -0.002 \pm 0.111 \\ 0.012 \pm 0.124 & 0.020 \pm 0.105 & 0.434 \pm 0.134 \end{pmatrix}$
$\mathcal{C}[\rho]$	0.030 ± 0.071	0.005 ± 0.023
$S_{\text{CHSH}}/2$	0.769 ± 0.189	0.703 ± 0.134

$$C_{ij}^{\text{SM}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\text{SM}}[\rho] = 1 \qquad S_{\text{CHSH}}^{\text{SM}}/2 = \sqrt{2}$$

Momentum smearing spoils the previous good result…

Use impact parameter information



$(\sigma_{b_T} = 2\mu m, \quad \sigma_{b_z} = 5\mu m)$

Goal:

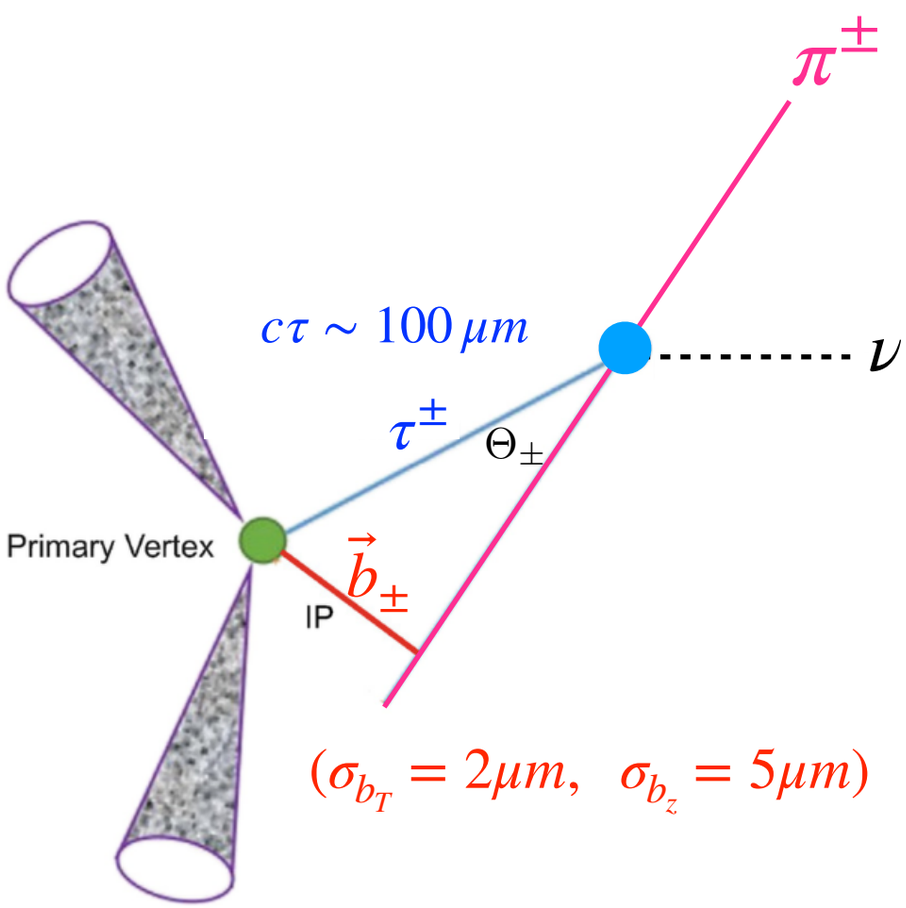
$$E_i^{\text{true}} \rightarrow E_i^{\text{obs}} \rightarrow E_i^{\text{true}} \quad (i = \pi^{\pm}, e^{\pm}, \mu^{\pm}, j)$$

What we do:

- modify E_i^{obs} for some amount by δ

$$E_i^{\text{obs}} \rightarrow E_i(\delta_i) = (1 + \delta_i\,\sigma_E) \cdot E_i^{\text{obs}}$$

Use impact parameter information



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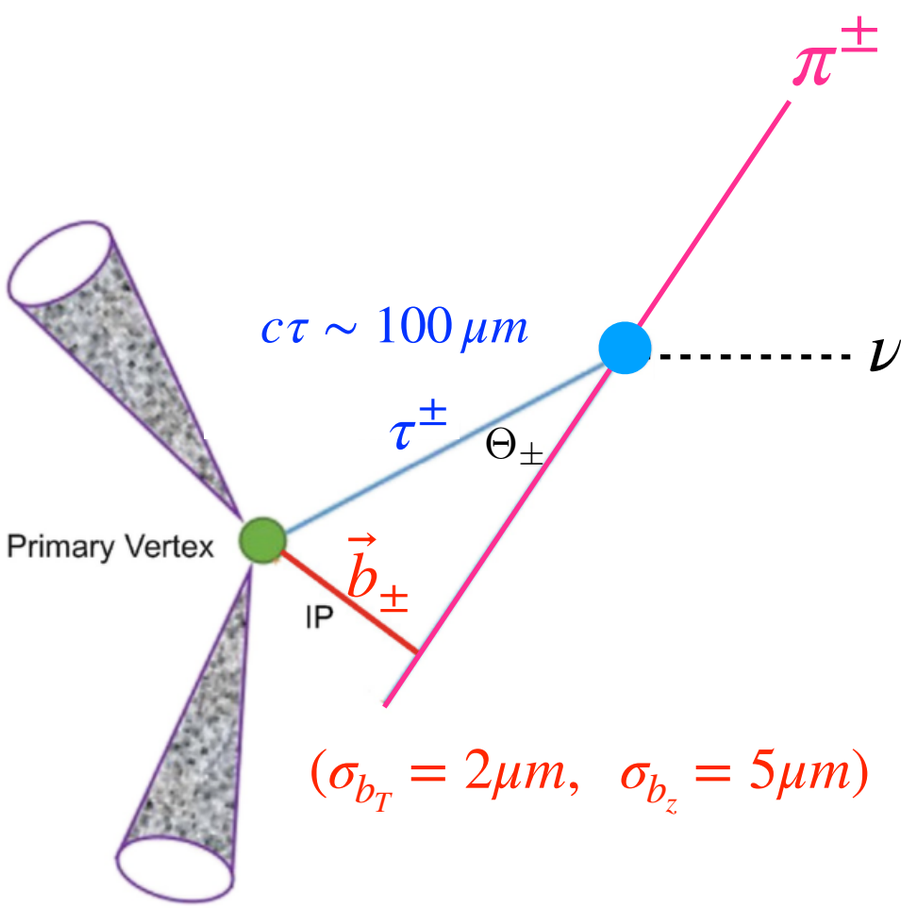
$$E_i^{\text{obs}} \rightarrow E_i(\delta_i) = (1 + \delta_i\,\sigma_E) \cdot E_i^{\text{obs}}$$

- solve tau direction $\mathbf{e}_{\tau^{\pm}}(\boldsymbol{\delta})$

→ lets us calculate $\vec{b}_{\pm}$ as functions of δ

$$\vec{b}_{\pm}^{\text{reco}}(\mathbf{e}_{\tau^{\pm}}) \;=\; |\vec{b}_{\pm}| \cdot \left[\mathbf{e}_{\tau^{\pm}} \cdot \sin^{-1}\Theta_{\pm} \;-\; \mathbf{e}_{\pi^{\pm}} \cdot \tan^{-1}\Theta_{\pm}\right]$$

Use impact parameter information



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$$\vec{b}_{\pm}^{\text{reco}}(\mathbf{e}_{\tau^{\pm}}) \;=\; |\vec{b}_{\pm}| \cdot \left[\mathbf{e}_{\tau^{\pm}} \cdot \sin^{-1}\Theta_{\pm} \;-\; \mathbf{e}_{\pi^{\pm}} \cdot \tan^{-1}\Theta_{\pm}\right]$$

- compare the calculated $\vec{b}_{\pm}^{\text{reco}}(\boldsymbol{\delta})$ and measured $\vec{b}_{\pm}^{\text{obs}}$ and construct the likelihood function

2 fold solutions: $i_s = 1,2$

$$\vec{\Delta}_{b_{\pm}}^{i_s}(\boldsymbol{\delta}) \;\equiv\; \vec{b}_{\pm} - \vec{b}_{\pm}^{\text{reco}}\left(\mathbf{e}_{\tau^+}^{i_s}(\boldsymbol{\delta})\right)$$

$$L_{\pm}^{i_s}(\boldsymbol{\delta}) \;=\; \frac{[\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_x^2 + [\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_y^2}{\sigma_{b_T}^2} + \frac{[\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_z^2}{\sigma_{b_z}^2} + \delta_{\pi^+}^2 + \delta_{\pi^-}^2 + \delta_x^2 + \delta_{\bar{x}}^2\,.$$

minimizing $L^{i_s}(\boldsymbol{\delta})$ would give us the correct set of $\boldsymbol{\delta}s$ and solution i_s

Result

2211.10513

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} 0.830 \pm 0.176 & 0.020 \pm 0.146 & -0.019 \pm 0.159 \\ -0.034 \pm 0.160 & 0.981 \pm 0.1527 & -0.029 \pm 0.156 \\ -0.001 \pm 0.158 & -0.021 \pm 0.155 & -0.729 \pm 0.140 \end{pmatrix}$	$\begin{pmatrix} 0.925 \pm 0.109 & -0.011 \pm 0.110 & 0.038 \pm 0.095 \\ -0.009 \pm 0.110 & 0.929 \pm 0.113 & 0.001 \pm 0.115 \\ -0.026 \pm 0.122 & -0.019 \pm 0.110 & -0.879 \pm 0.098 \end{pmatrix}$
$\mathcal{C}[\rho]$	0.778 ± 0.126	0.871 ± 0.084
$S_{\mathrm{CHSH}}/2$	1.103 ± 0.163	1.276 ± 0.094

$$C_{ij}^{\mathrm{SM}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\mathrm{SM}}[\rho] = 1 \qquad S_{\mathrm{CHSH}}^{\mathrm{SM}}/2 = \sqrt{2}$$

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$\mathcal{C}[\rho]$	0.778 ± 0.126 $\sim 5\sigma$	0.871 ± 0.084 $\gg 5\sigma$
$S_{\mathrm{CHSH}}/2$	1.103 ± 0.163	1.276 ± 0.094 $\sim 3\sigma$

$$C_{ij}^{\mathrm{SM}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\mathrm{SM}}[\rho] = 1 \qquad S_{\mathrm{CHSH}}^{\mathrm{SM}}/2 = \sqrt{2}$$

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$$C_{ij}^{\rm SM} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\rm SM}[\rho] = 1 \qquad S_{\rm CHSH}^{\rm SM}/2 = \sqrt{2}$$

Superiority of FCC-ee over ILC is due to a better beam resolution

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	$0.83 \cdot 10^{-4}$
beam resolution e^- (%)	0.27	$0.83 \cdot 10^{-4}$

CP measurement

- Under CP, the spin correlation matrix transforms: $C \xrightarrow{CP} C^T$
- This can be used for a *model-independent* test of CP violation. We define:

$$A \equiv (C_{rn} - C_{nr})^2 + (C_{nk} - C_{kn})^2 + (C_{kr} - C_{rk})^2 \ \geq 0$$

- Observation of $A \neq 0$ immediately confirms CP violation.

- From our simulation, we observe

$$A = \begin{cases} 0.204 \pm 0.173 & \text{(ILC)} \\ 0.112 \pm 0.085 & \text{(FCC-ee)} \end{cases}$$

← consistent with absence of CPV

- This model independent bounds can be translated to the constraint on the CP-phase δ

$$\mathscr{L}_{\text{int}} \propto H\bar{\psi}_\tau(\cos\delta + i\gamma_5 \sin\delta)\psi_\tau \Rightarrow C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow A(\delta) = 4\sin^2 2\delta$$

CP measurement

- Focusing on the region near $|\delta| = 0$, we find the 1-σ bounds:

$$|\delta| < \begin{cases} 8.9^o & \text{(ILC)} \\ 6.4^o & \text{(FCC-ee)} \end{cases}$$

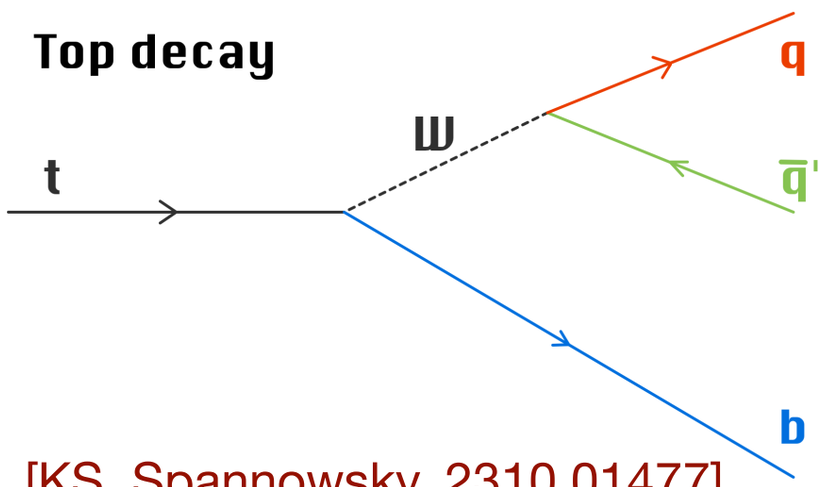
- Other studies:

$$\Delta\delta \sim 11.5^o \quad \text{(HL-LHC)}$$ [Hagiwara, Ma, Mori 2016]

$$\Delta\delta \sim 4.3^o \quad \text{(ILC)}$$ [Jeans and G. W. Wilson 2018]

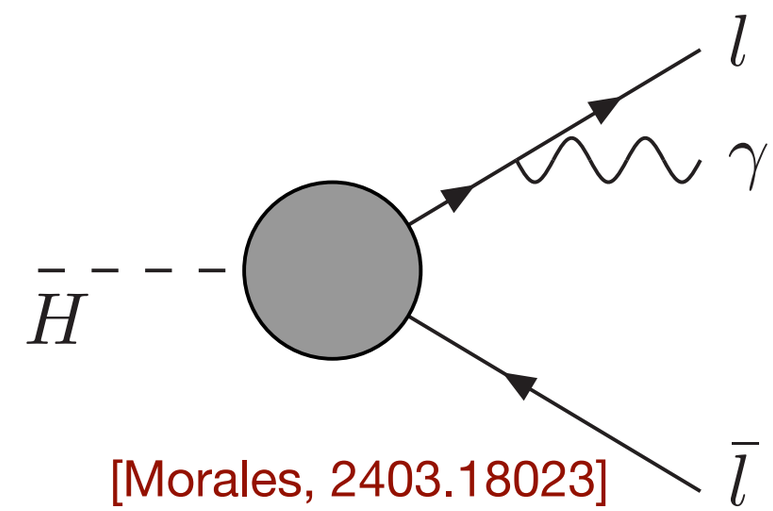
3-particle entanglement

$$2 \to 222$$



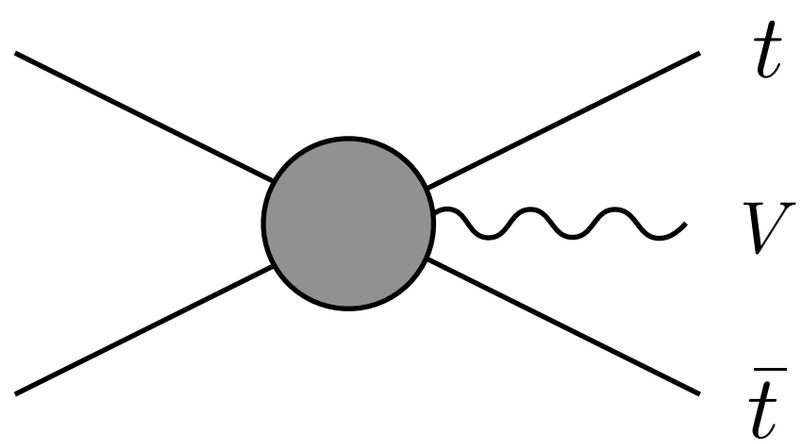
[KS, Spannowsky, 2310.01477]

$$0 \to 222$$



[Morales, 2403.18023]

$$22 \to 223$$



[Subba, Rahaman, 2404.03292]

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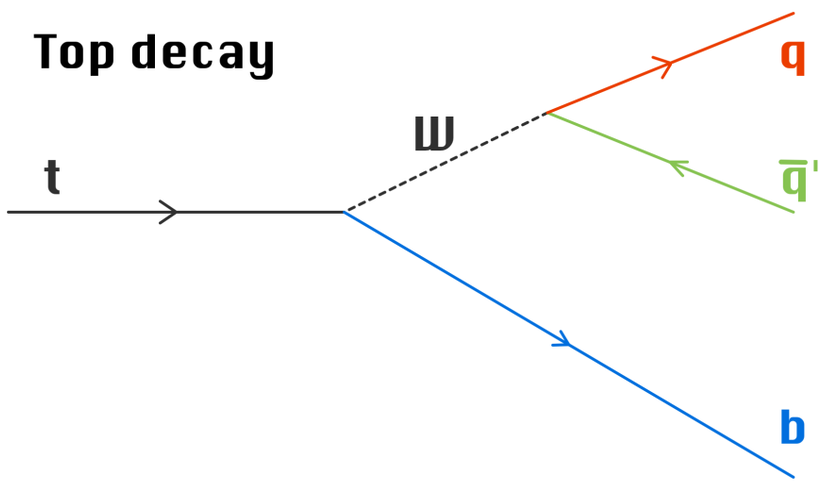

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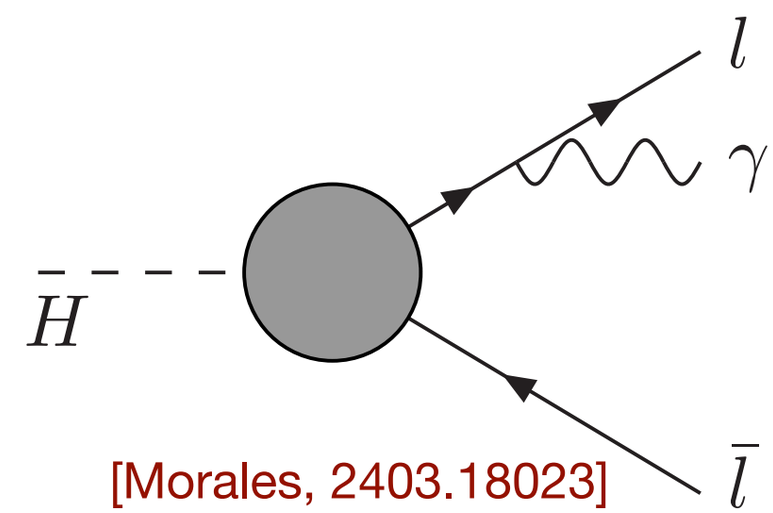

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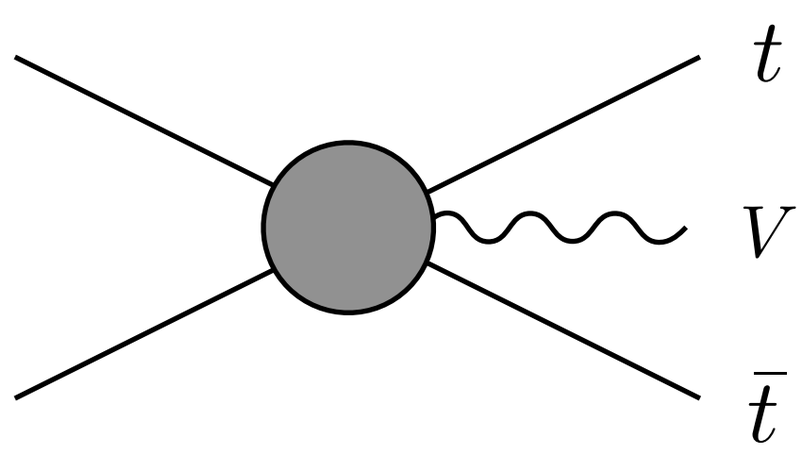


[Subba, Rahaman, 2404.03292]

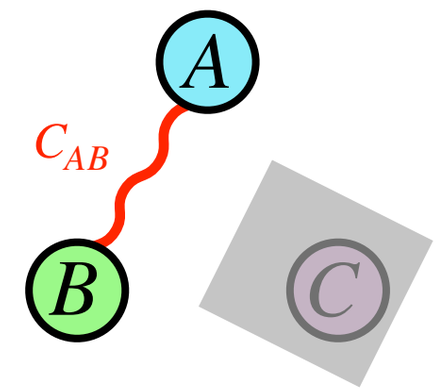
Three qubit system: 222

- $2^3 = 8$ basis kets:

$$|\Psi_{ABC}\rangle = c_{000}|000\rangle + c_{001}|001\rangle + c_{010}|010\rangle + \cdots$$

- Entanglement among 2-individual particles:

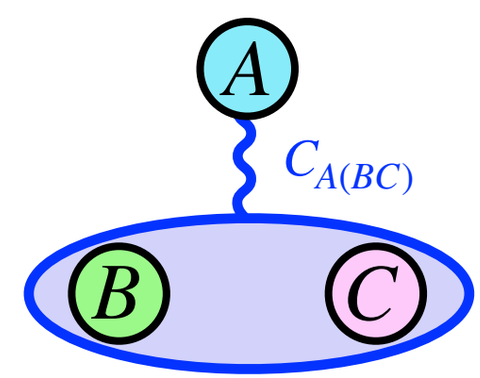
$$\mathcal{C}_{AB}[|\Psi_{ABC}\rangle] = \mathcal{C}[\rho_{AB}] \qquad \rho_{AB} = \mathrm{Tr}_C|\Psi\rangle\langle\Psi|_{ABC}$$



- Entanglement among one-to-other:

$$|\Psi_{ABC}\rangle = |0\rangle_A \otimes (c_{000}|00\rangle_{BC} + c_{001}|01\rangle_{BC} + \cdots)$$
$$+|1\rangle_A \otimes (c_{100}|00\rangle_{BC} + c_{101}|01\rangle_{BC} + \cdots)$$

$$\mathcal{C}_{A(BC)}[|\Psi_{ABC}\rangle] = \sqrt{2[1 - \mathrm{Tr}\rho_{BC}^2]} \qquad \rho_{BC} = \mathrm{Tr}_A|\Psi\rangle\langle\Psi|_{ABC}$$



Classification

Fully-separable: $|\phi^{\mathrm{fs}}\rangle_{A|B|C} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C,$

$\mathcal{C}_{ij} = \mathcal{C}_{i(jk)} = 0$

Bi-separable: $|\phi^{\mathrm{bs}}\rangle_{A|BC} = |\alpha\rangle_A \otimes |\delta\rangle_{BC}$

$|\phi^{\mathrm{bs}}\rangle_{B|AC} = |\beta\rangle_B \otimes |\delta\rangle_{AC}$

$|\phi^{\mathrm{bs}}\rangle_{C|AB} = |\gamma\rangle_C \otimes |\delta\rangle_{AB}$

$\mathcal{C}_{A(BC)} = 0$

$\mathcal{C}_{BC}, \mathcal{C}_{B(AC)}, \mathcal{C}_{C(AB)} \neq 0$

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complement

Genuinely **M**ultipartite **E**ntangled (**GME**):

$$|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle$$

maximally GME

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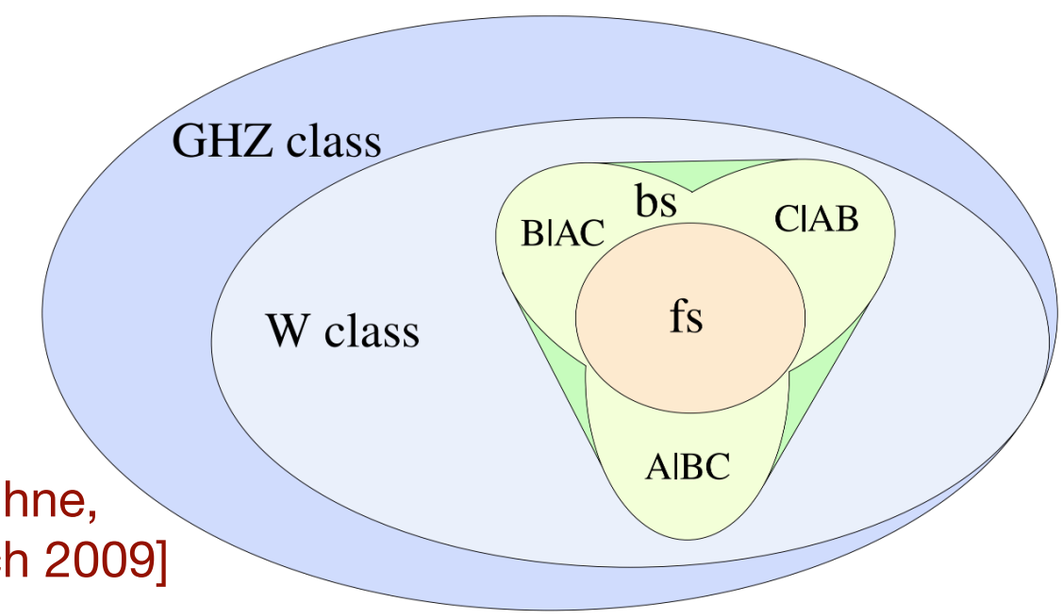
maximally GME

- All GME states can be classified either ***GHZ*** or ***W*** classes! [Dur, Vidal, Cirac 2000]

$$|\phi^{\rm GME}\rangle \longrightarrow \hat{A} \otimes \hat{B} \otimes \hat{C}\,|\phi^{\rm GME}\rangle = \begin{cases} |GHZ_3\rangle \\ |W_3\rangle \end{cases}$$

$\hat{A} \in I(\mathcal{H}_A), \hat{B} \in I(\mathcal{H}_B), \hat{C} \in I(\mathcal{H}_C)$

$I(\mathcal{H})$: set of intertible operators in $\mathcal{H}$



[Guhne, Toch 2009]

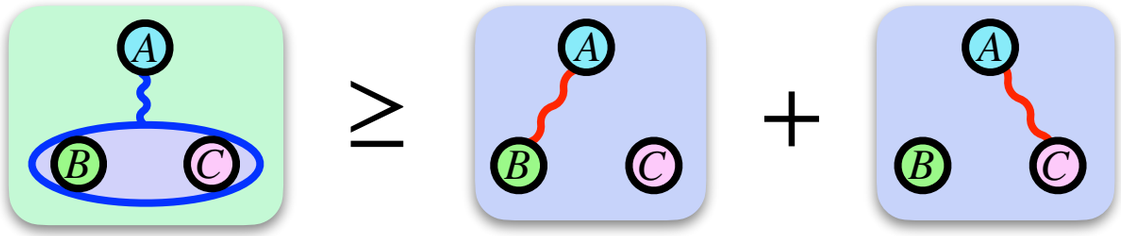
Monogamy

- All 3-qubit pure states can be transformed by a local unitary to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \qquad \lambda_i \geq 0, \sum_i \lambda_i^2 = 1 \text{ and } \theta \in [0;\pi]$$

- **1-2** and **1-1** entanglements are related to by the **monogamy** relations: [Coffman, Kundu, Wootters '99]

$$\mathcal{C}^2_{A(BC)} = \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} + \tau \qquad \tau = 4\lambda_0^2\lambda_4^2 \geq 0 \longleftarrow$$ **3-tangle**



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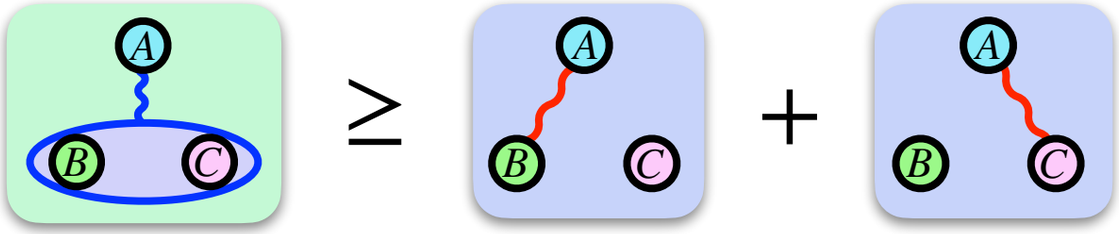

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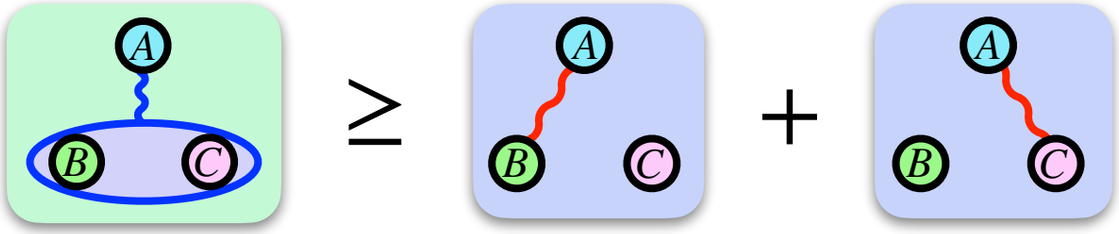

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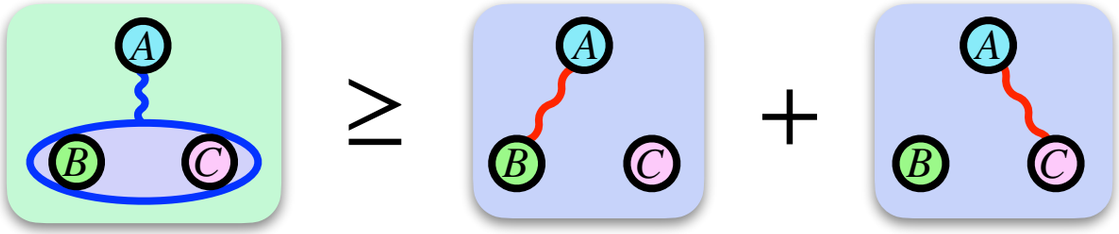
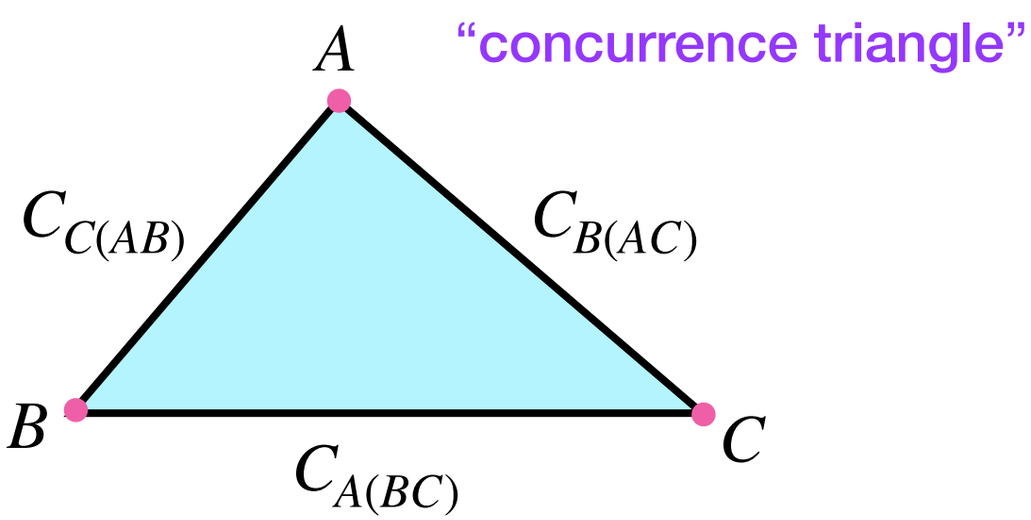


- 1-2 concurrence inequalities

$$\mathcal{C}^2_{A(BC)} + \mathcal{C}^2_{B(AC)} \geq \mathcal{C}^2_{C(AB)}$$

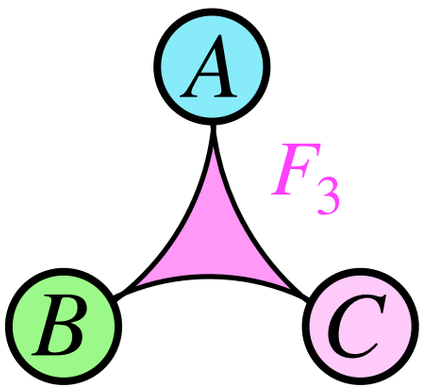
$$\downarrow$$

$$\mathcal{C}_{A(BC)} + \mathcal{C}_{B(AC)} \geq \mathcal{C}_{C(AB)} \quad \Rightarrow$$



GME measure

Genuine Multi-particle Entanglement (**GME**) measure: [Dur, Vidal, Cirac '00, Ma, Chen, Chen, Spengler, Gabriel, Huber '11, Xie, Eberly '21]



The measure should satisfy:

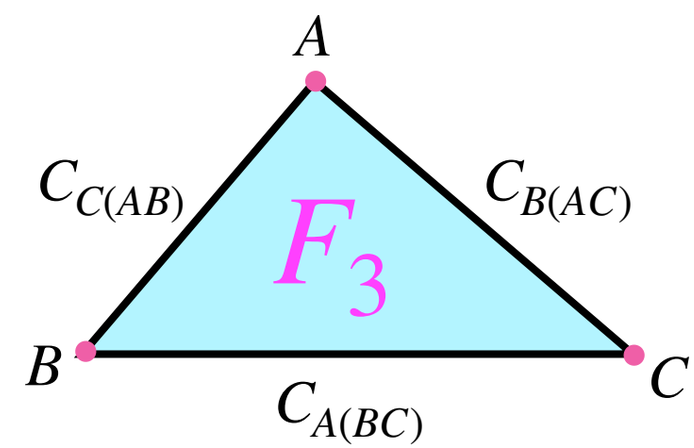
(1) vanishes for all fully- and bi-separable states

(2) positive for all GME states

(3) non-increasing under LOCC

- The **area** of the "**concurrence triangle**" satisfies (1), (2), (3) **!**

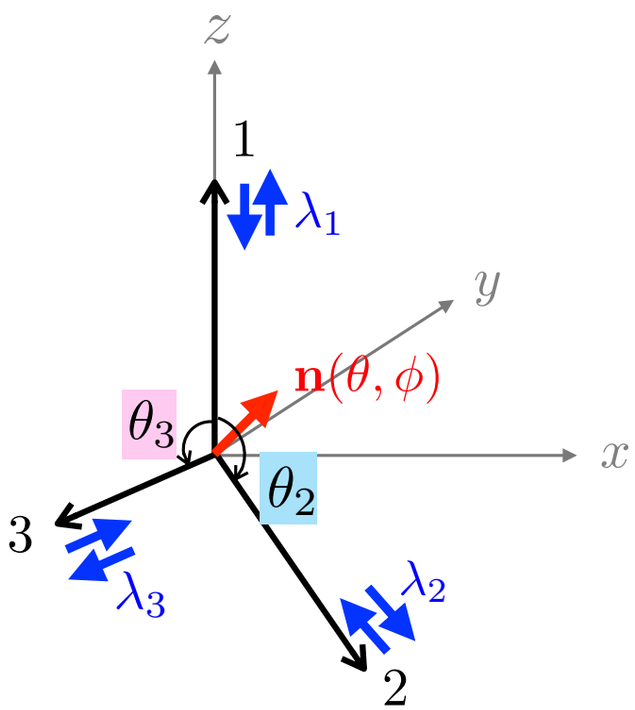
[Jin, Tao, Gui, Fei, Li-Jost, Qiao (2023)]



$$F_3 \equiv \left[\tfrac{16}{3} Q\big(Q - \mathcal{C}_{A(BC)}\big)\big(Q - \mathcal{C}_{B(AC)}\big)\big(Q - \mathcal{C}_{C(AB)}\big)\right]^{\frac{1}{2}} \in [0,1]$$

$$Q \equiv \frac{1}{2}\left[\mathcal{C}_{A(BC)} + \mathcal{C}_{B(AC)} + \mathcal{C}_{C(AB)}\right]$$

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$



[KS, M.Spannowsky 2310.01477]

Assumptions:

- all particles have spin 1/2
- all final particles 1,2,3 are massless

Kinematics:

- rest frame of the initial particle 0
- p_1 is in the z-axis
- decay is in the x-z plane

$$p_1^\mu = p_1(1, 0, 0, 1)$$
$$p_2^\mu = p_2(1, \sin\theta_2, 0, \cos\theta_2)$$
$$p_3^\mu = p_3(1, -\sin\theta_3, 0, \cos\theta_3)$$

$\mathbf{n}(\theta, \phi)$: polarisation of initial spin

$\lambda_1, \lambda_2, \lambda_3 \in (+, -)$: helicities of 1,2,3

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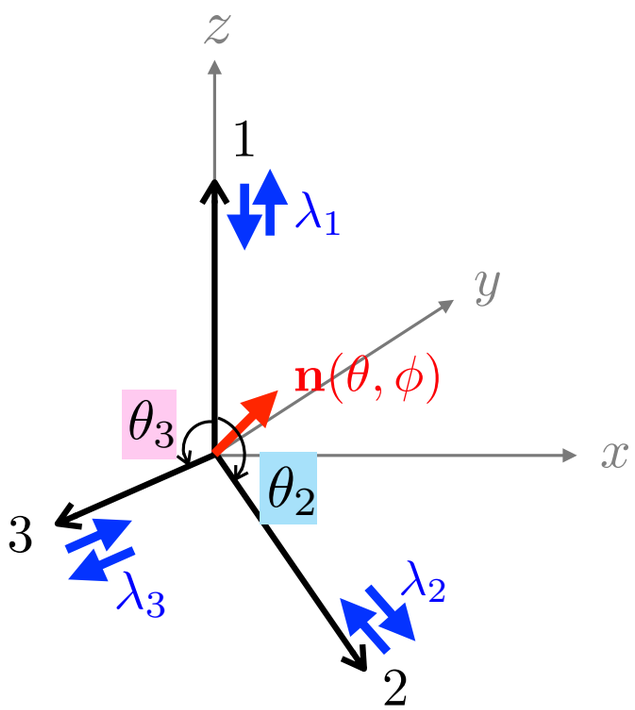

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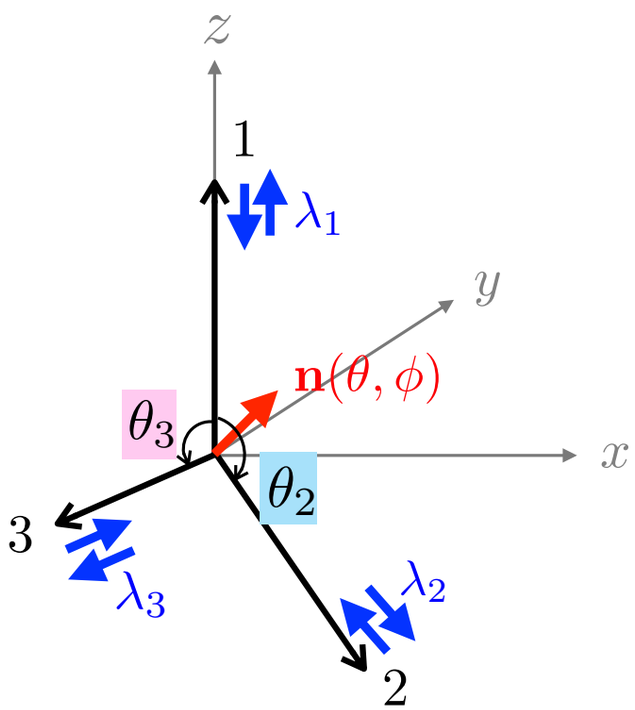
initial state

$$|\mathbf{n}(\theta,\phi)\rangle$$

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$



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amplitude

$$\hat{\mathbf{1}} = \sum_{\lambda_1,\lambda_2,\lambda_3} |\lambda_1,\lambda_2,\lambda_3\rangle\langle\lambda_1,\lambda_2,\lambda_3| \qquad \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3} = \langle\lambda_1,\lambda_2,\lambda_3|\mathbf{n}(\theta,\phi)\rangle$$

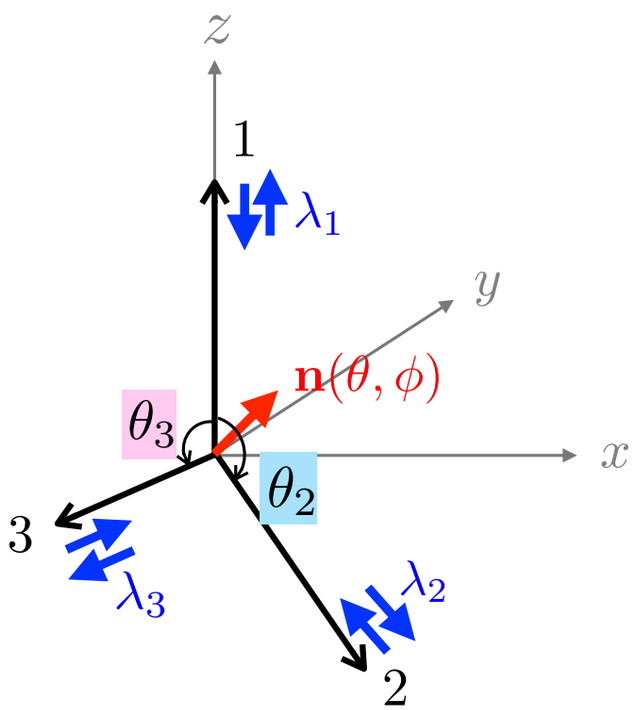
initial state

$$|\mathbf{n}(\theta,\phi)\rangle = \sum_{\lambda_1,\lambda_2,\lambda_3} \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3}|\lambda_1,\lambda_2,\lambda_3\rangle + \cdots$$

final state

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$

[KS, M.Spannowsky 2310.01477]



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pure (entangled) **3-spin state**

Interaction

- Consider **most general** Lorentz invariant 4-fermion interactions

$$\mathcal{L}_{\rm int} = (\bar{\psi}_1 \Gamma_A \psi_0)(\bar{\psi}_3 \Gamma_B \psi_2)$$

$$\Gamma_{A/B} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$$

$$\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$$

❖ **Scalar-type**

$$[\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

$$c \equiv c_S + ic_A = e^{i\delta_1}$$
$$d \equiv d_S + id_A = e^{i\delta_2}$$

❖ **Vector-type**

$$[\bar{\psi}_1\gamma_\mu(c_L P_L + c_R P_R)\psi_0][\bar{\psi}_3\gamma^\mu(d_L P_L + d_R P_R)\psi_2]$$

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$
$$c_L, c_R, d_L, d_R \in \mathbb{R}$$

❖ **Tensor-type**

$$[\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

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[KS, M.Spannowsky 2310.01477]

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$$|\Psi\rangle = \frac{cd}{\sqrt{2}} \cdot e^{i\phi} s\tfrac{\theta}{2}|---\rangle - \frac{cd^*}{\sqrt{2}} \cdot e^{i\phi} s\tfrac{\theta}{2}|-++\rangle + \frac{c^*d}{\sqrt{2}} \cdot c\tfrac{\theta}{2}|+--\rangle - \frac{c^*d^*}{\sqrt{2}} \cdot c\tfrac{\theta}{2}|+++\rangle$$

[KS, M.Spannowsky 2310.01477]

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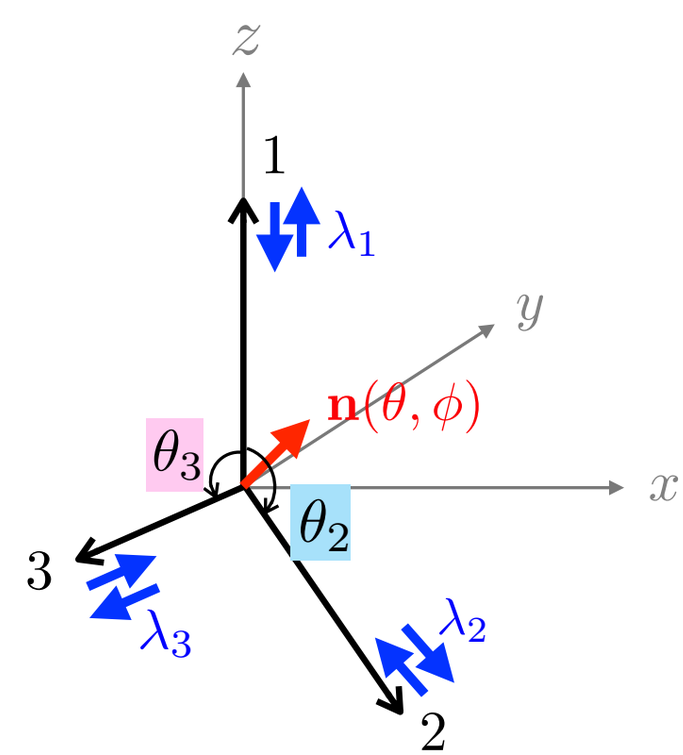
independent of final state momenta θ_2, θ_3



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independent of final state momenta θ_2, θ_3

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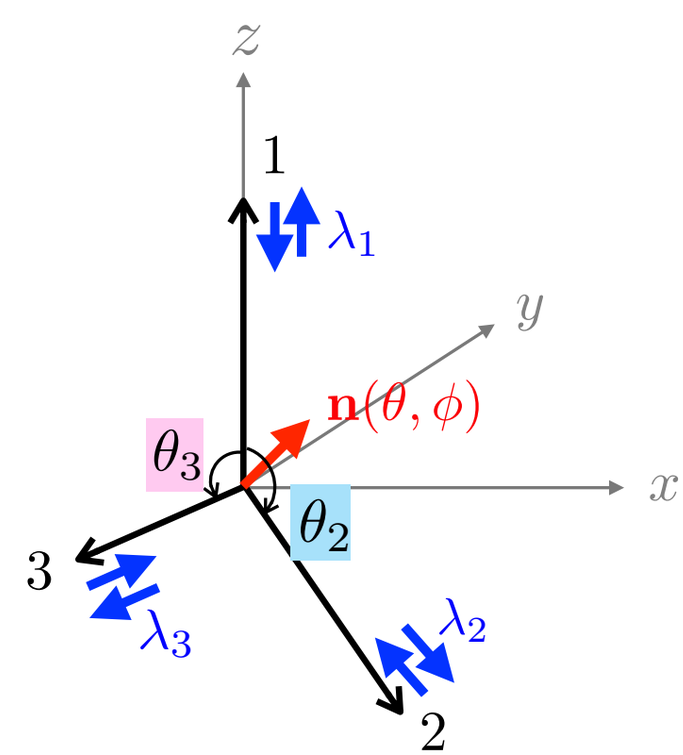
bi-separable



[KS, M.Spannowsky 2310.01477]

Scalar

$$\mathcal{L}_{\rm int} = [\bar\psi_1(c_S + ic_A\gamma_5)\psi_0][\bar\psi_3(d_S + id_A\gamma_5)\psi_2]$$

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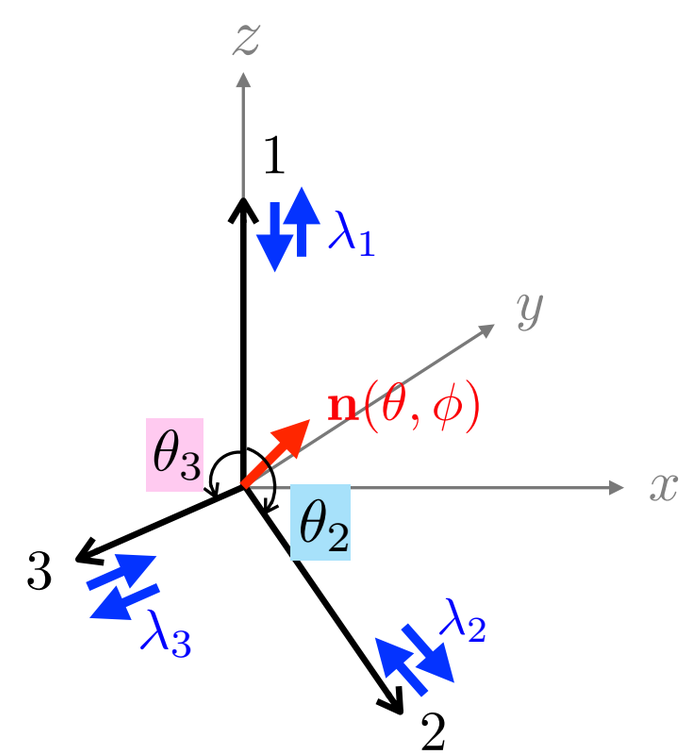
$$\Rightarrow F_3 = 0$$



[KS, M.Spannowsky 2310.01477]

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✤ **1** is **not entangled** with **2** and **3** in any way: $\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$

[KS, M.Spannowsky 2310.01477]

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[KS, M.Spannowsky 2310.01477]

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$$\mathcal{C}^2_{2(13)} = \underset{\substack{\| \\ 0}}{\mathcal{C}^2_{12}} + \underset{\substack{\| \\ 1}}{\mathcal{C}^2_{23}} + \tau \qquad \mathcal{C}^2_{3(12)} = \underset{\substack{\| \\ 0}}{\mathcal{C}^2_{13}} + \underset{\substack{\| \\ 1}}{\mathcal{C}^2_{23}} + \tau$$

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[KS, M.Spannowsky 2310.01477]

Vector

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$

$$\mathcal{L}_{\text{int}} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R) \psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R) \psi_2]$$

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[KS, M.Spannowsky 2310.01477]

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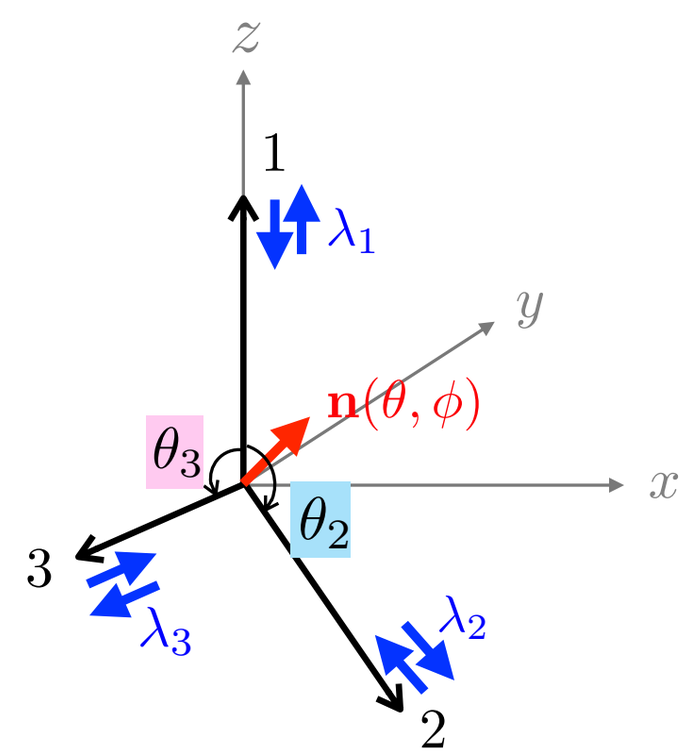
$$+ \quad c_R d_L s\tfrac{\theta_2}{2}\left[c\tfrac{\theta}{2}s\tfrac{\theta_3}{2} - e^{i\phi}s\tfrac{\theta}{2}c\tfrac{\theta_3}{2}\right]|++-\rangle \;+\; c_R d_R s\tfrac{\theta_3}{2}\left[c\tfrac{\theta}{2}s\tfrac{\theta_2}{2} - e^{i\phi}s\tfrac{\theta}{2}c\tfrac{\theta_2}{2}\right]|+-+\rangle$$



[KS, M.Spannowsky 2310.01477]

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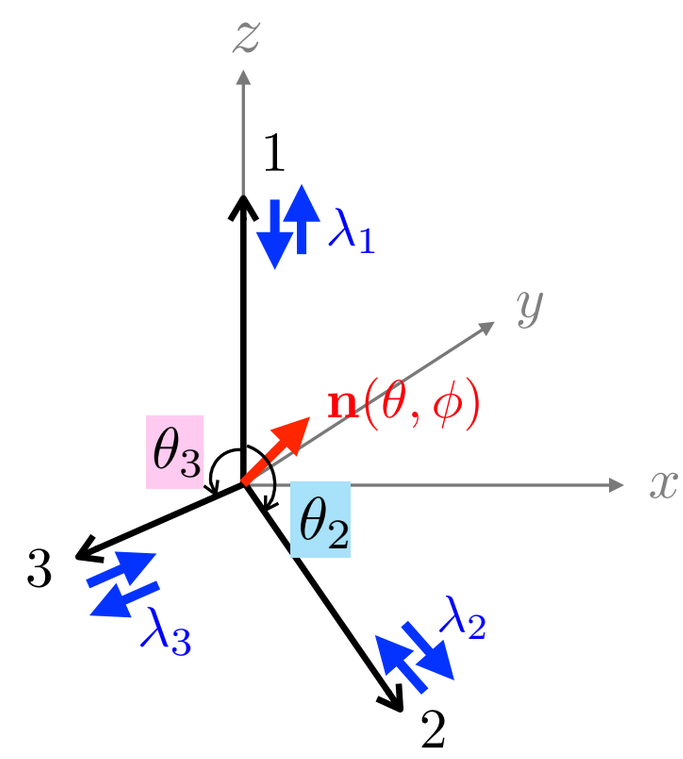
$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M_{LR}^* + M_{RL}M_{RR}^*|$$



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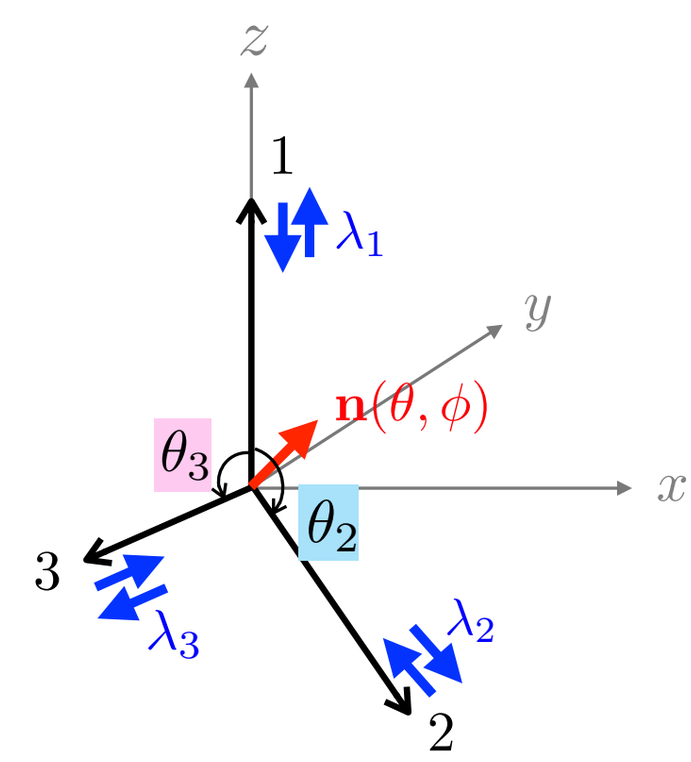

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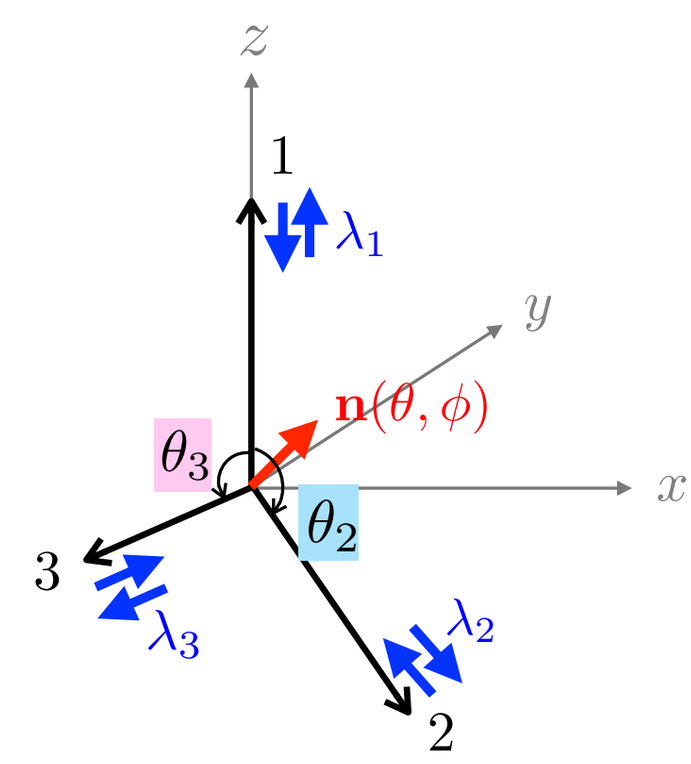
✤ 3-tangle

$$\tau = \mathcal{C}_{1(23)}^2 - [\underbrace{\mathcal{C}_{12}^2}_{=0} + \underbrace{\mathcal{C}_{13}^2}_{=0}] = \mathcal{C}_{1(23)}^2$$



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$$\mathcal{C}_{1(23)} = 2\big|M_{RR}M_{LL} - M_{LR}M_{RL}\big| \quad \longleftarrow \text{vanish if } c_L c_R d_L d_R = 0$$

✤ 3-tangle

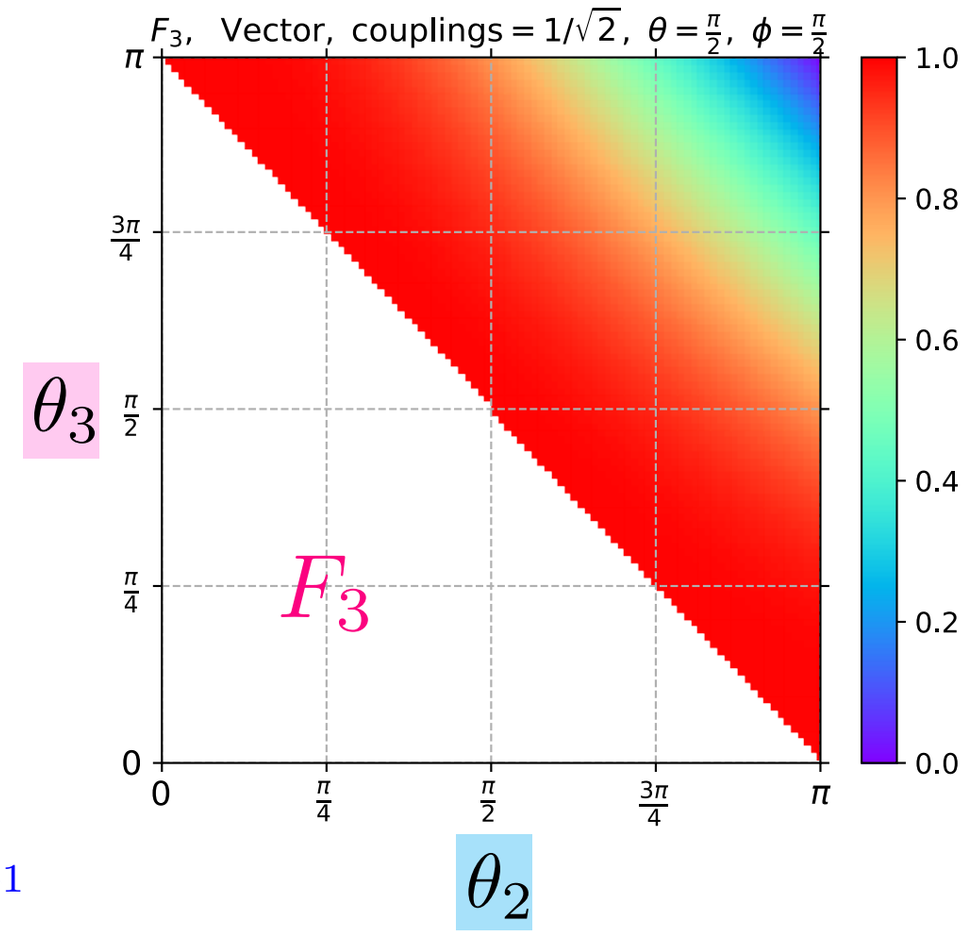
$$\tau = \mathcal{C}_{1(23)}^2 - [\underbrace{\mathcal{C}_{12}^2}_{=\,0} + \underbrace{\mathcal{C}_{13}^2}_{=\,0}] = \mathcal{C}_{1(23)}^2$$

➡ **All** entanglements **vanish** for **weak decays**

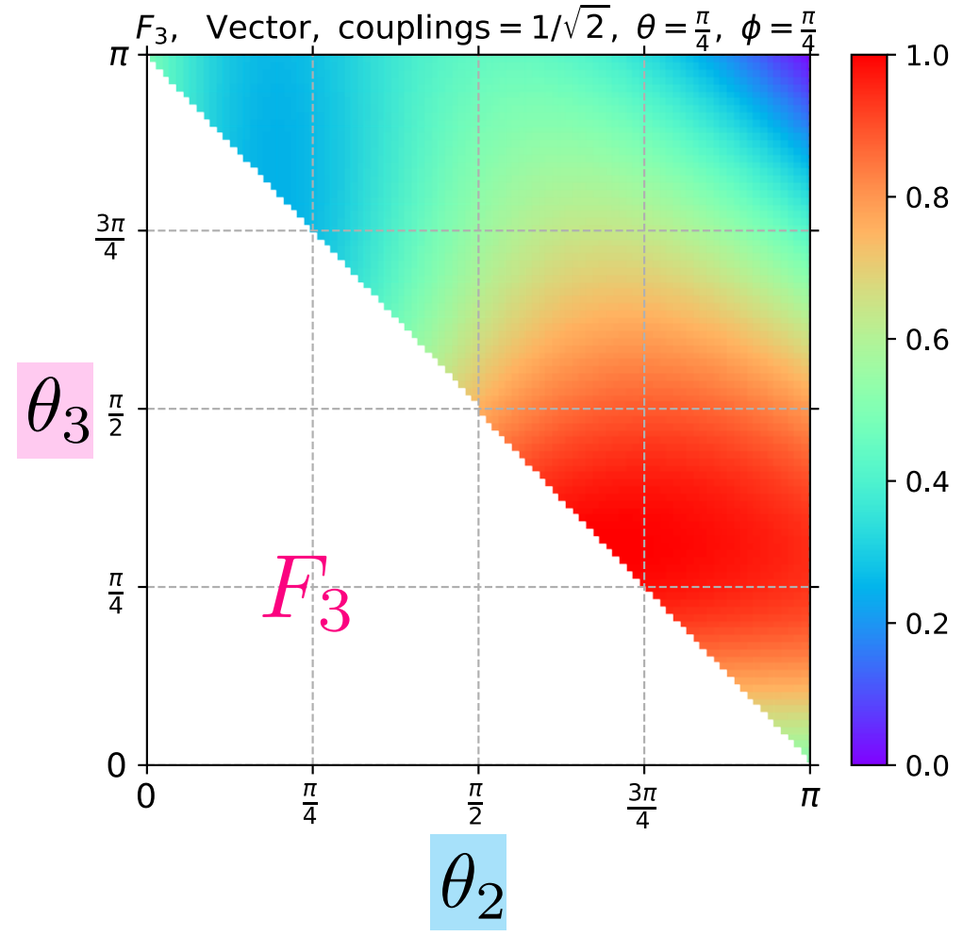
$$c_R = d_R = 0$$

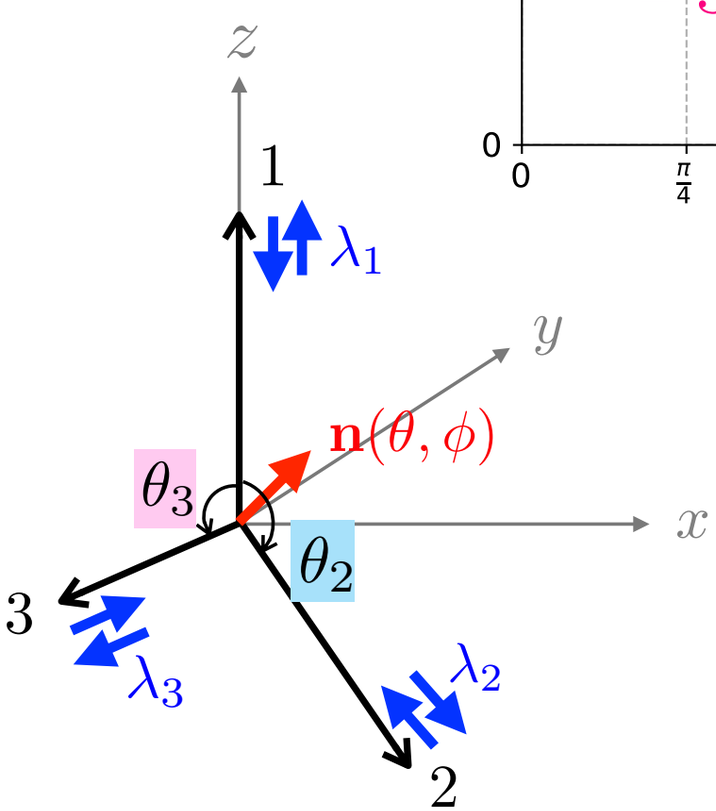
F_3 for Vector

$$\mathbf{n} = \mathbf{e}_y$$

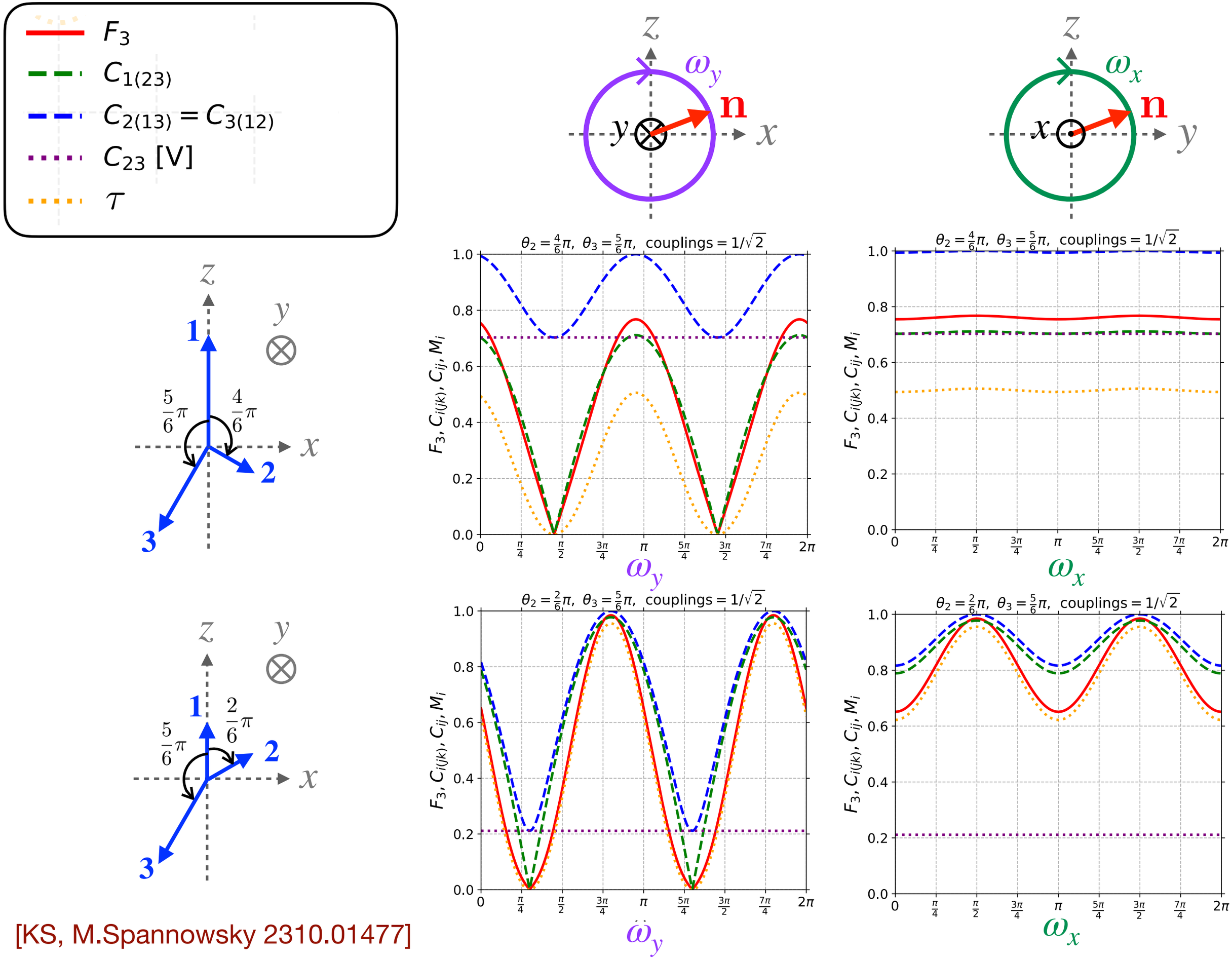


$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$$





[KS, M.Spannowsky 2310.01477]



[KS, M.Spannowsky 2310.01477]

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
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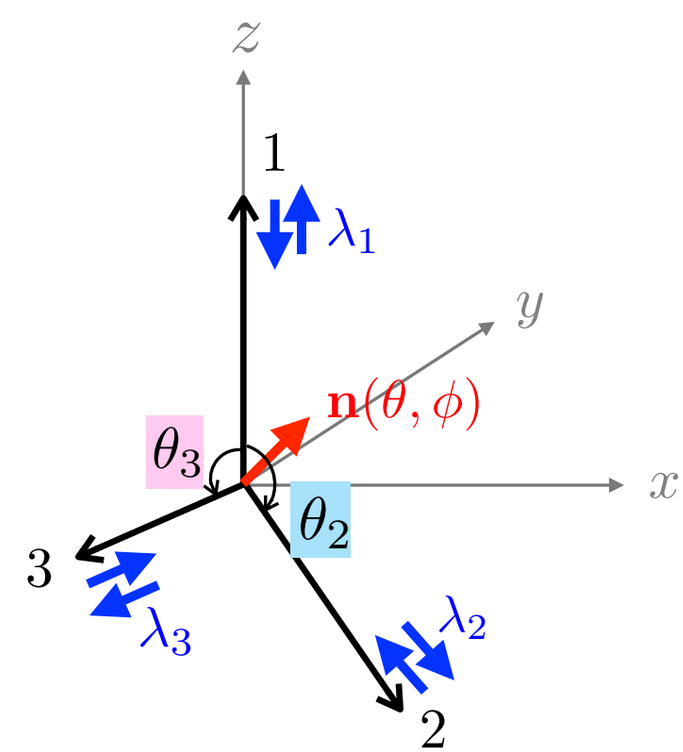
$$|\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto c^* d^* [2e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2} + c\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2}]\,|+++\rangle + cd\,[-e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2} + 2c\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2}]\,|---\rangle$$



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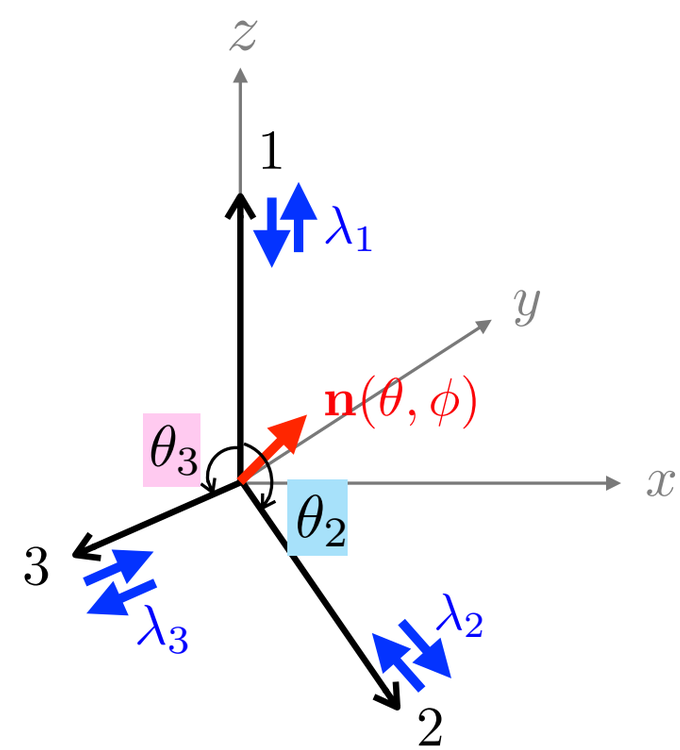
$$d \equiv d_M + id_E = e^{i\omega_2}$$

$$|\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto c^* d^* [2e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2} + c\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2}]\,|+++\rangle + cd\,[-e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2} + 2c\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2}]\,|---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$$(M_R M_L = 0) \qquad |\pm\pm\pm\rangle \longleftrightarrow |\mathrm{GHZ}\rangle = \tfrac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \qquad (M_R = M_L)$$



[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
$$d \equiv d_M + id_E = e^{i\omega_2}$$

$$\Rightarrow \quad |\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto \; c^* d^* \left[2e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2} + c\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2}\right]|+++\rangle + cd\left[-e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2} + 2c\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2}\right]|---\rangle$$

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- **No** individual 2-party entanglements:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
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$$|\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto\ c^* d^* \left[2e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2} + c\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2}\right]|+++\rangle + cd\left[-e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2} + 2c\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2}\right]|---\rangle$$

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✤ **No** individual 2-party entanglements:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0 \longrightarrow$$

3-tangle

$$\tau = \mathcal{C}^2_{i(jk)} - [\underbrace{\mathcal{C}^2_{ij}}_{=0} + \underbrace{\mathcal{C}^2_{ik}}_{=0}] = \mathcal{C}^2_{i(jk)}$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
$$d \equiv d_M + id_E = e^{i\omega_2}$$

$$|\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto\ c^* d^* \left[2e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2} + c\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2}\right]|+++\rangle + cd\left[-e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2} + 2c\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2}\right]|---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$$(M_R M_L = 0) \qquad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \tfrac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \qquad (M_R = M_L)$$

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3-tangle

$$\tau = \mathcal{C}^2_{i(jk)} - [\underbrace{\mathcal{C}^2_{ij}}_{=0} + \underbrace{\mathcal{C}^2_{ik}}_{=0}] = \mathcal{C}^2_{i(jk)}$$

✤ one-to-other entanglements are **universal**:

$$\mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2|M_R M_L|$$

$$F_3 = 4|M_R M_L|^2$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar\psi_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar\psi_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

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$$\propto\ c^* d^* \left[2e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2} + c\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2}\right]|+++\rangle + cd\left[-e^{i\phi} s\tfrac{\theta}{2} s\tfrac{\theta_3-\theta_2}{2} + 2c\tfrac{\theta}{2} s\tfrac{\theta_2}{2} s\tfrac{\theta_3}{2}\right]|---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$$(M_R M_L = 0) \qquad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \tfrac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \qquad (M_R = M_L)$$

✤ **No** individual 2-party entanglements:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0 \longrightarrow$$

3-tangle

$$\tau = \mathcal{C}^2_{i(jk)} - [\underbrace{\mathcal{C}^2_{ij}}_{=0} + \underbrace{\mathcal{C}^2_{ik}}_{=0}] = \mathcal{C}^2_{i(jk)}$$

✤ one-to-other entanglements are **universal**:

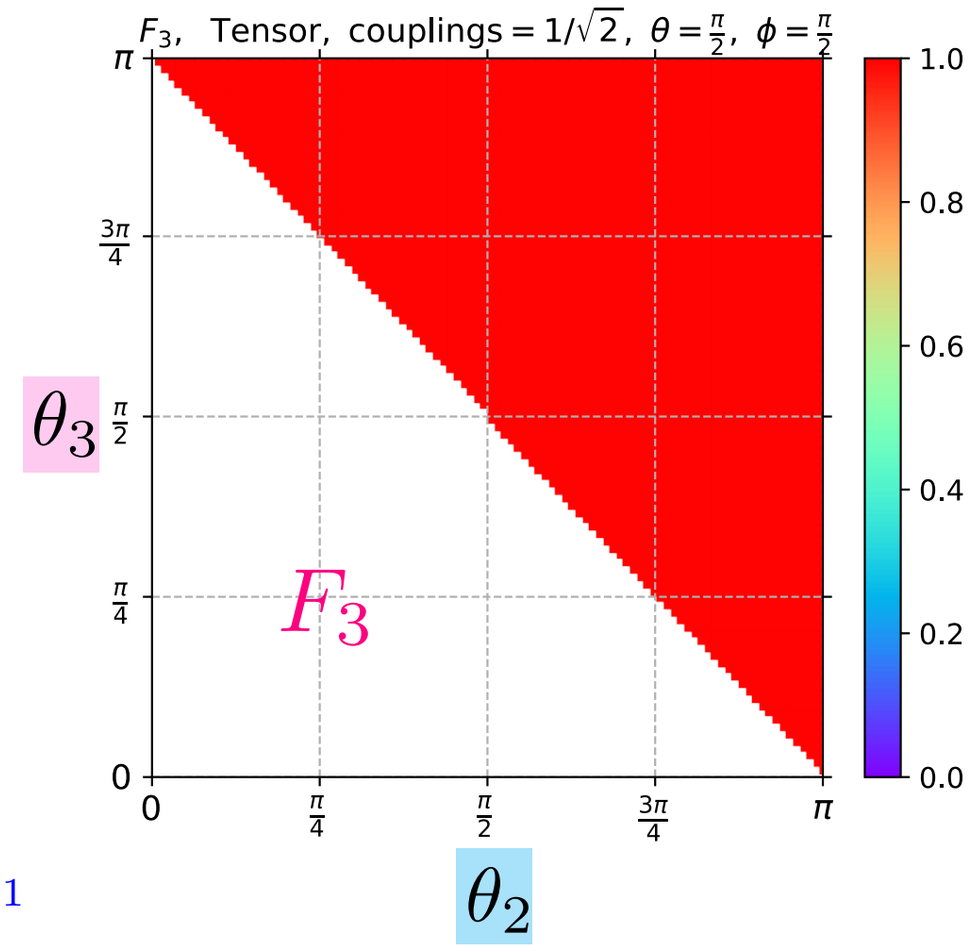
$$\left.\begin{aligned} \mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} &= 2|M_R M_L| \\ F_3 &= 4|M_R M_L|^2 \end{aligned}\right\}$$

independent of the coupling structure (CP phases)

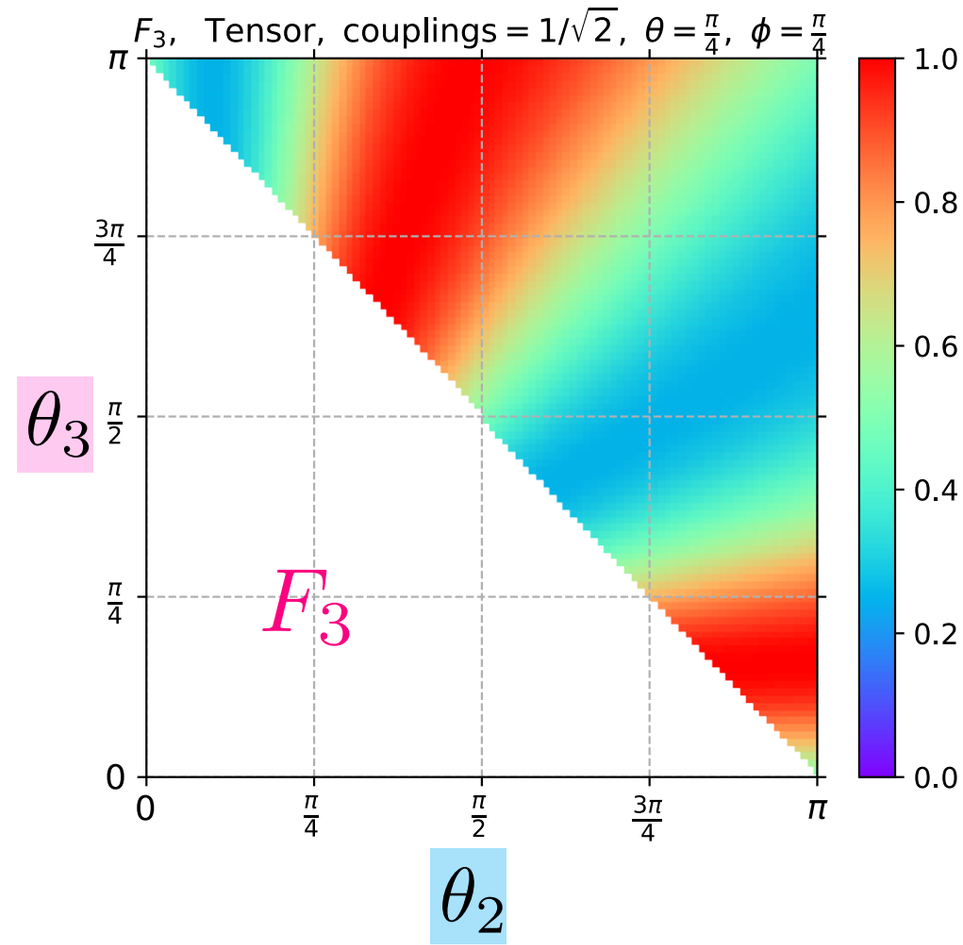
ω_1, ω_2

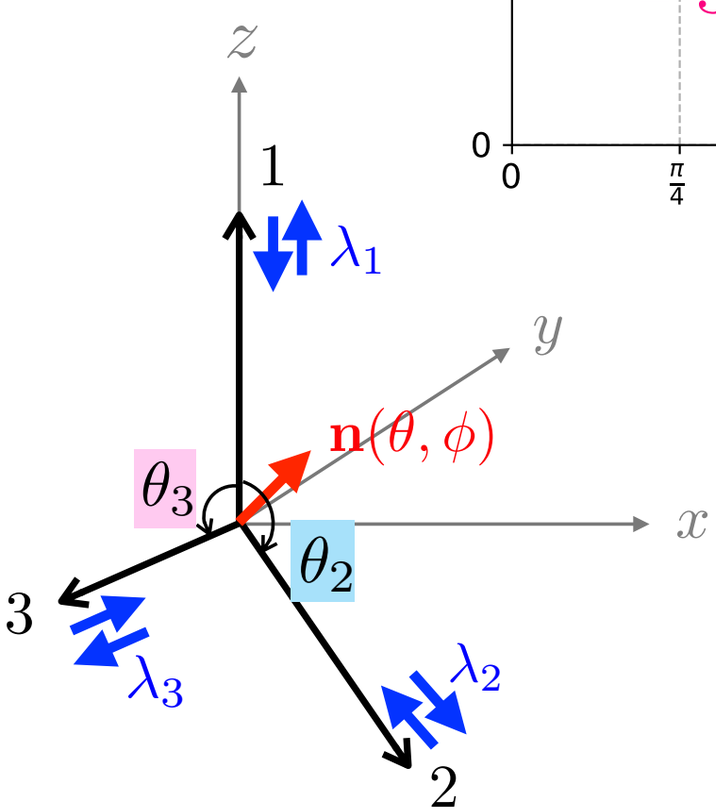
F_3 for Tensor

$$\mathbf{n} = \mathbf{e}_y$$



$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$$





[KS, M.Spannowsky 2310.01477]

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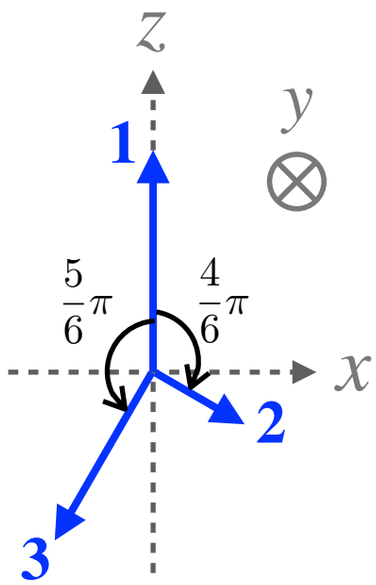
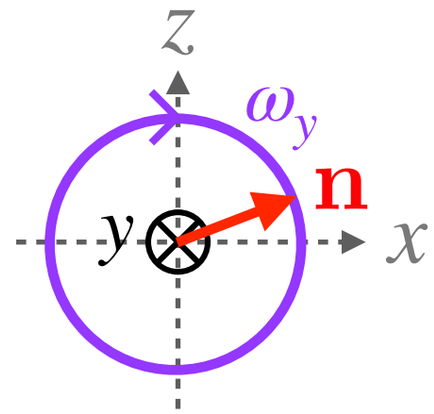
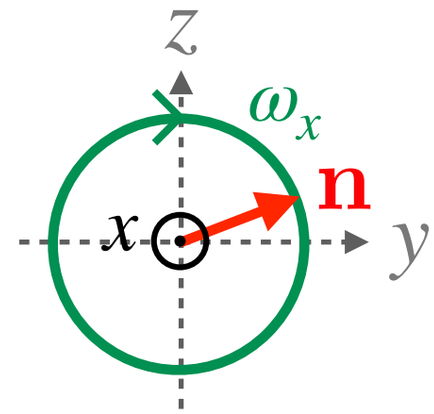
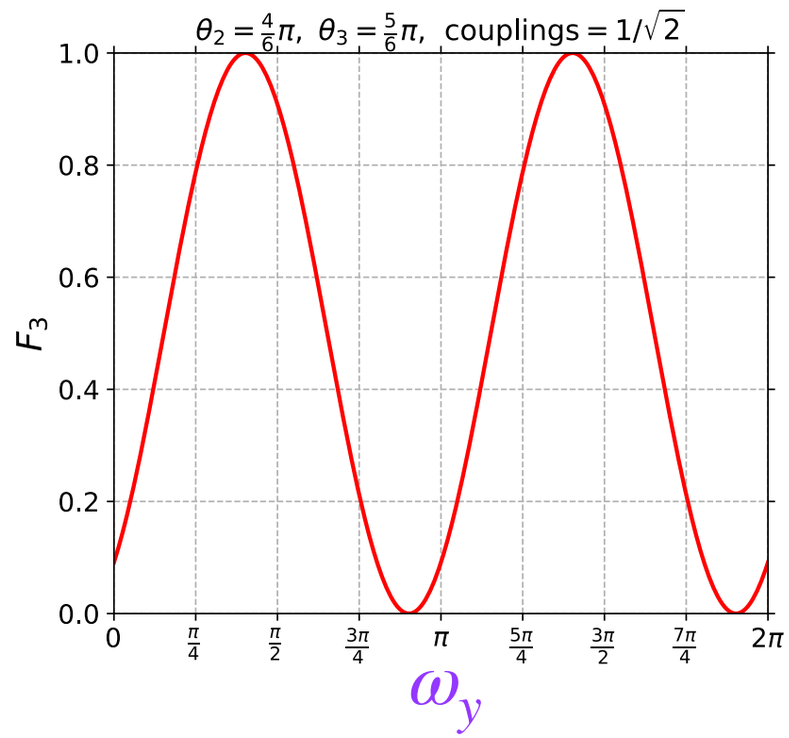
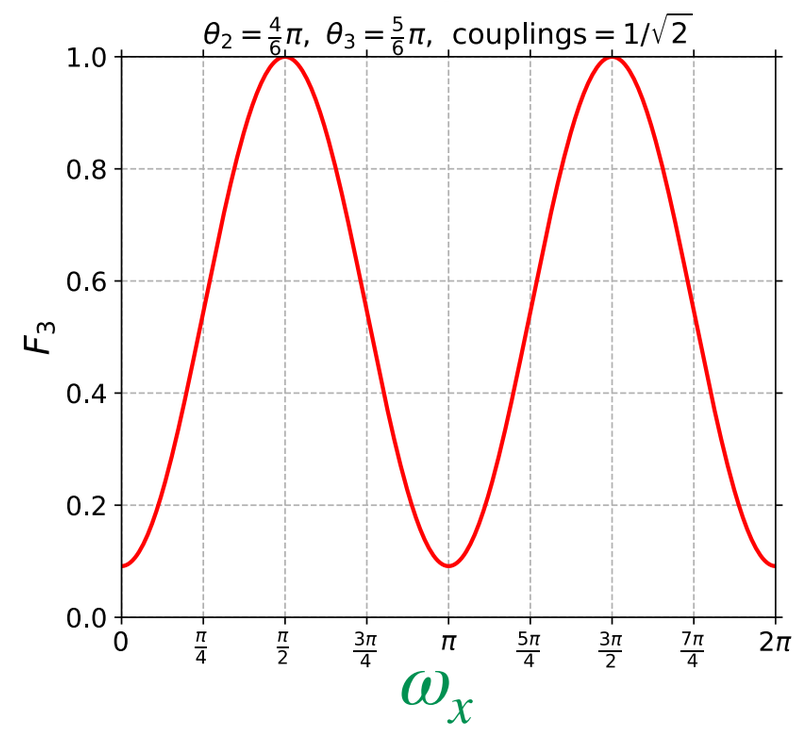
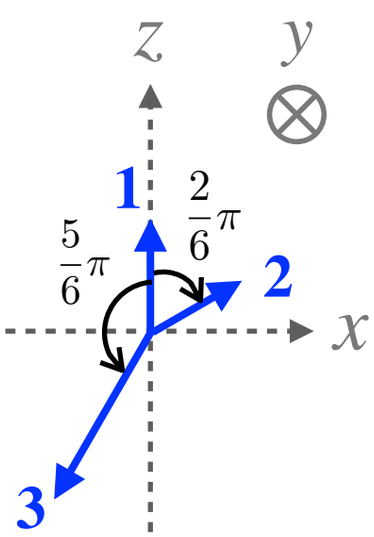


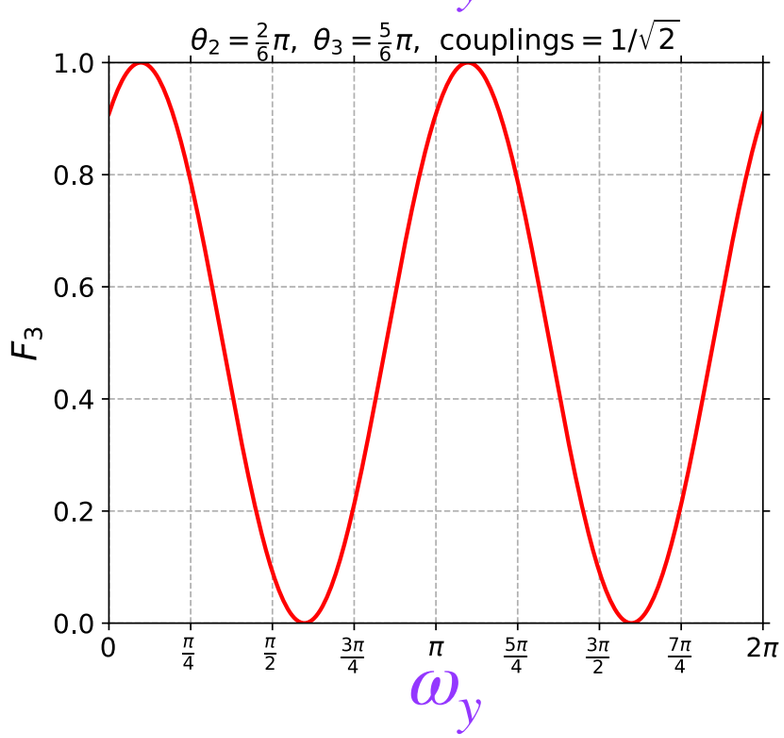


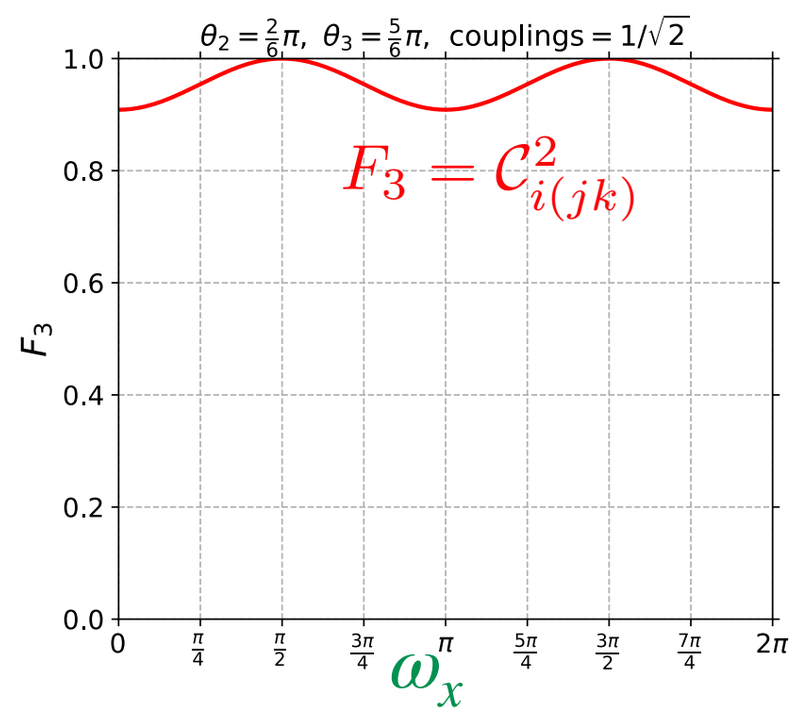


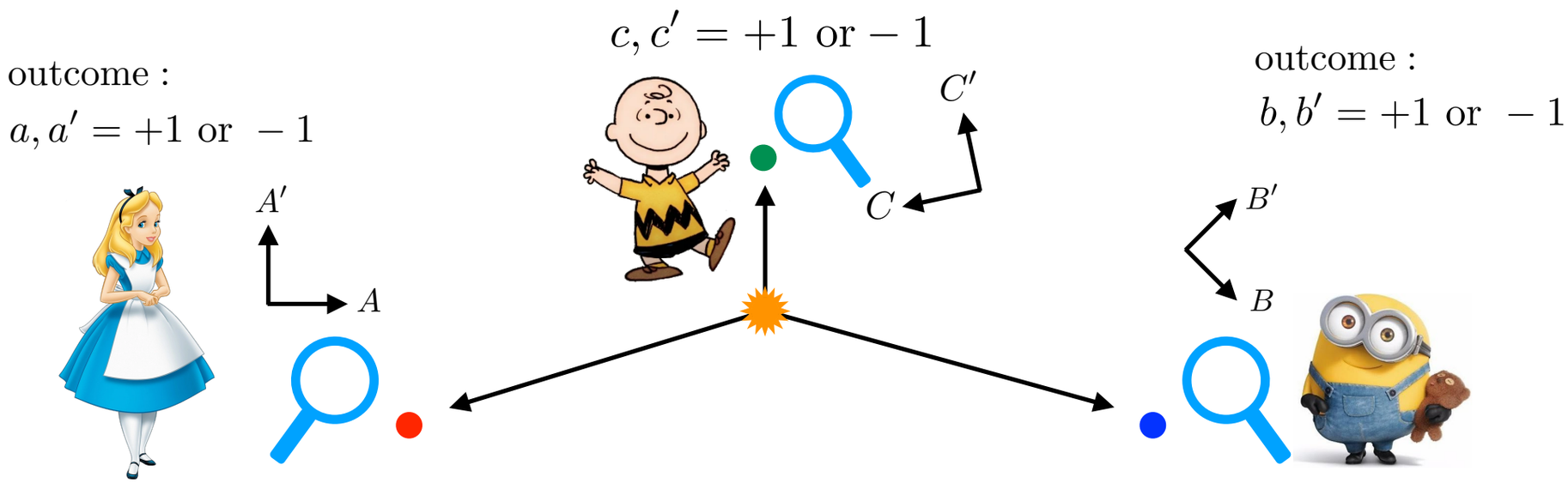




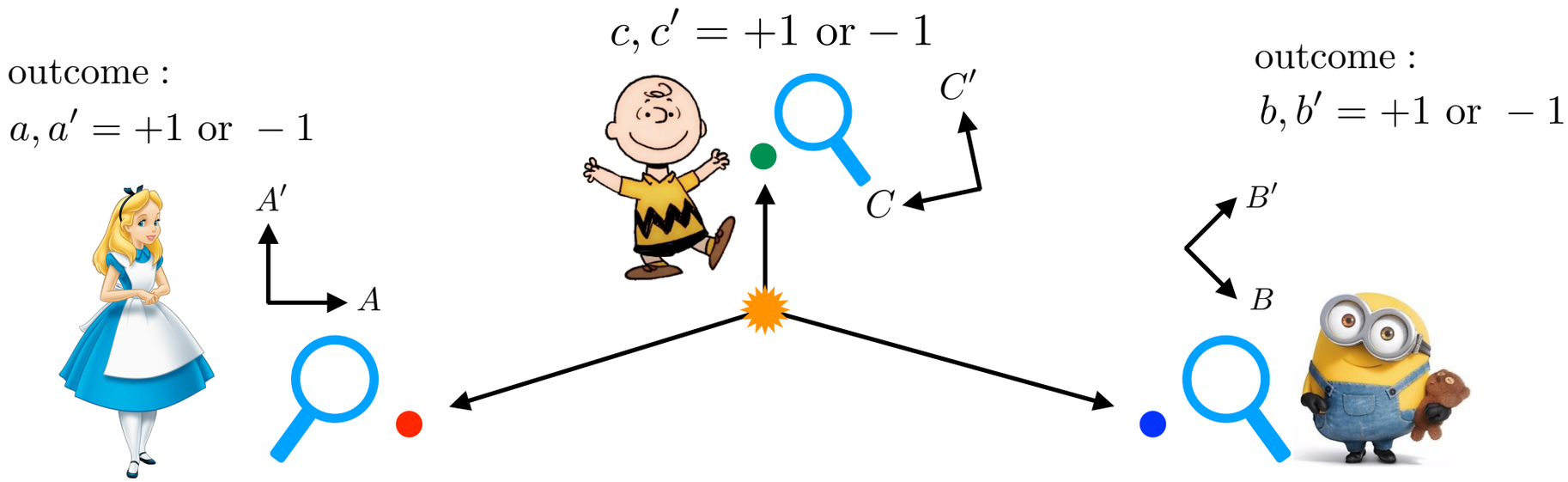








3-partite Bell inequality



- **Completely** Hidden Variable: $\langle ABC \rangle_{\rm CHV} = \int p(\lambda)[a(\lambda)b(\lambda)c(\lambda)]d\lambda$

- Mermin correlation: $\mathcal{B}_{\rm M} = ABC' + AB'C + A'BC - A'B'C'$

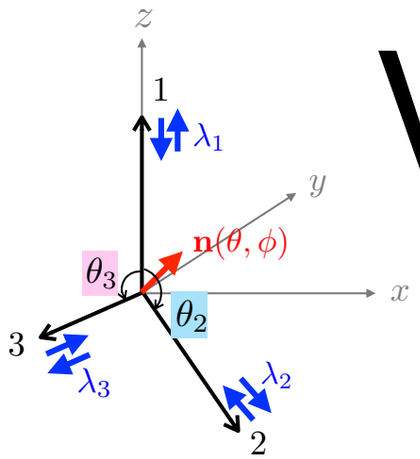
- **Mermin inequalities**: $\langle \mathcal{B}_{\rm M} \rangle_{\rm CHV} \leq 2 \qquad \langle \mathcal{B}_{\rm M} \rangle_{\rm QM} \leq 4$ [Mermin '90]

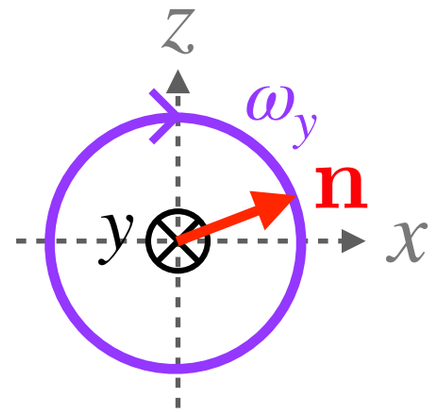
- **Partially** Hidden Variable: $\langle ABC \rangle_{\rm PHV} = \int p(\lambda)[ab(\lambda)c(\lambda)]d\lambda \ , \ \int p(\lambda)[a(\lambda)bc(\lambda)]d\lambda, \cdots$

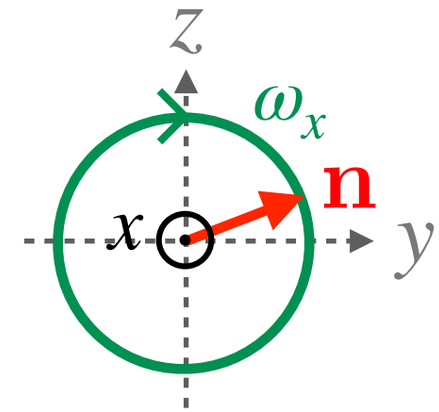
- Svetlichny correlation:
$$\mathcal{B}_{\rm S} = ABC + ABC' + AB'C + A'BC - A'B'C' - A'B'C - A'BC' - AB'C'$$

- **Svetlichny inequalities**: $\langle \mathcal{B}_{\rm S} \rangle_{\rm PHV} \leq 4 \qquad \langle \mathcal{B}_{\rm S} \rangle_{\rm QM} \leq 4\sqrt{2}$ [Svetlichny '87]

Vector


z
1
λ_1
y
$\mathbf{n}(\theta,\phi)$
θ_3
θ_2
x
3
λ_3
λ_2
2


z
ω_y
n
y
x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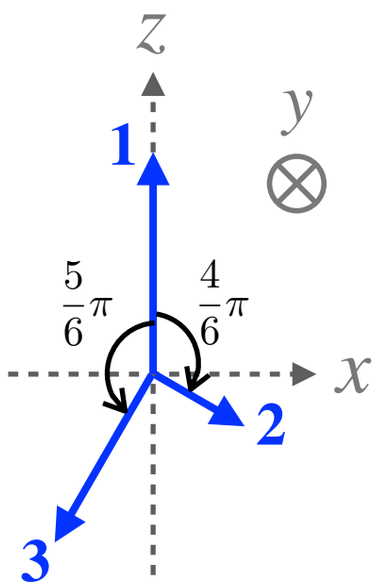
ω_x
n
x
y


z
y
1
$\frac{5}{6}\pi$
$\frac{4}{6}\pi$
x
2
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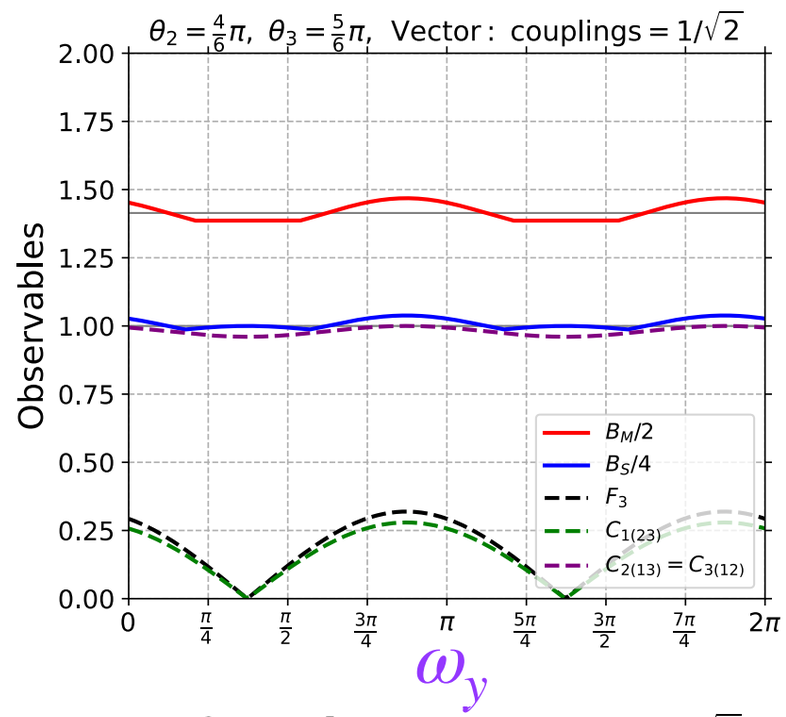
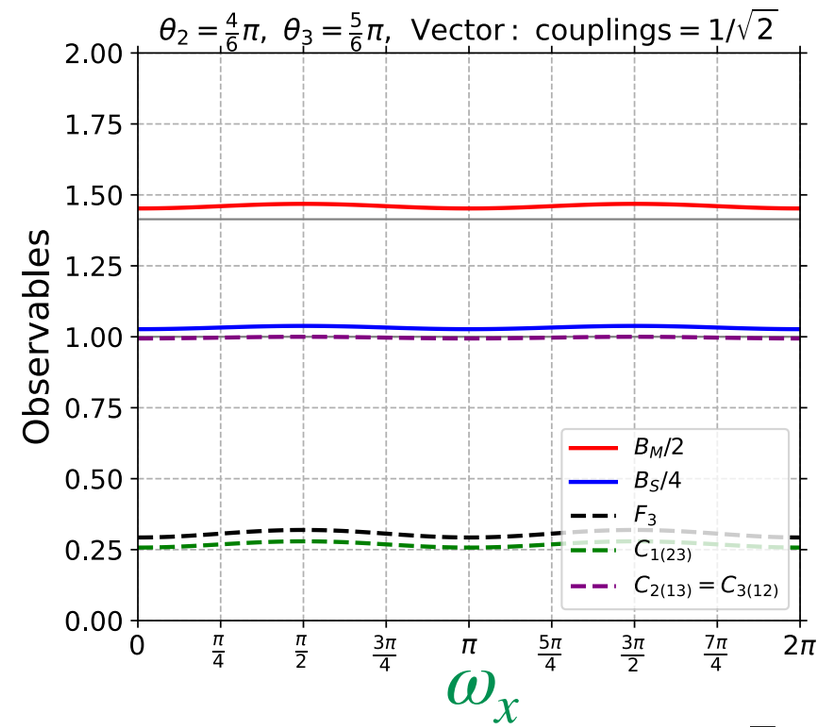

$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ Vector : couplings $= 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$
ω_y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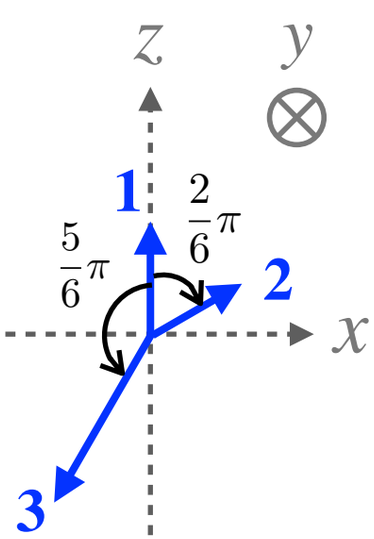
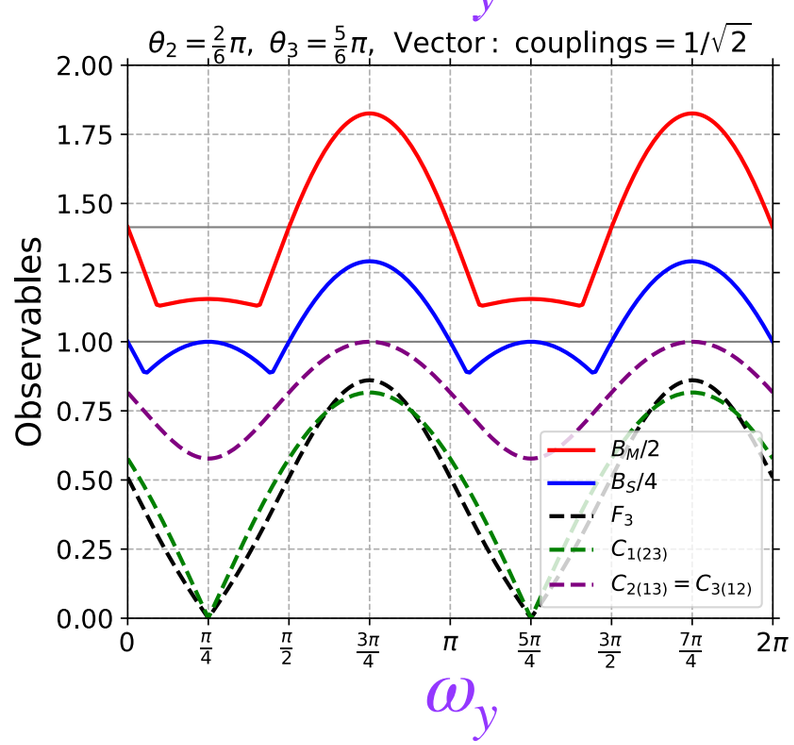
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$
ω_x


z
y
1
$\frac{2}{6}\pi$
$\frac{5}{6}\pi$
2
x
3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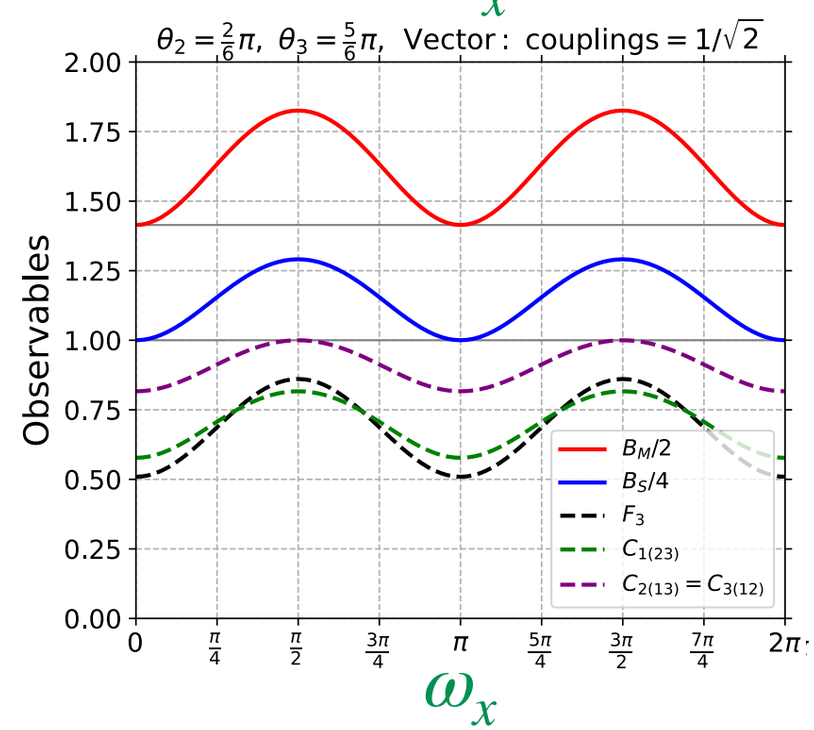
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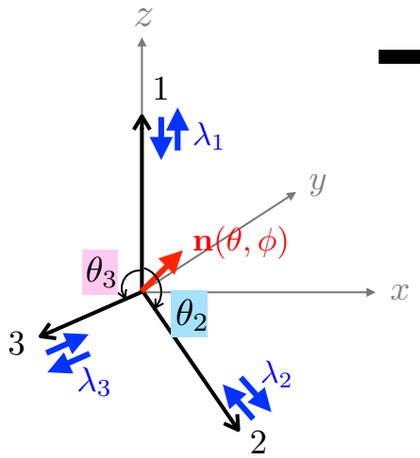
Observables
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$
ω_x

Tensor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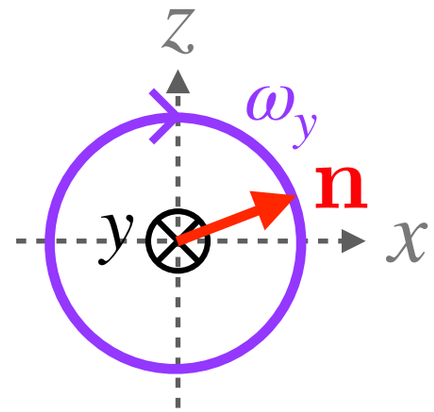
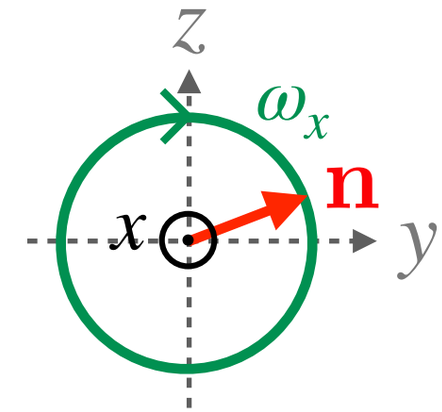
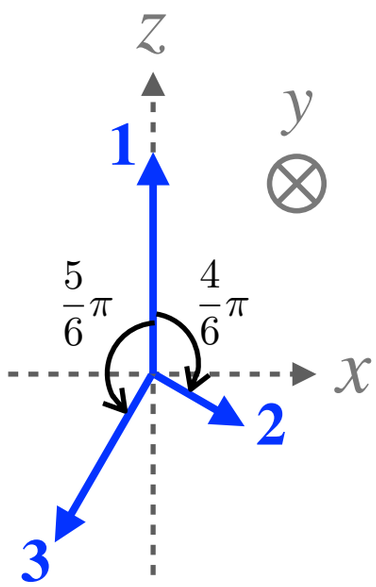
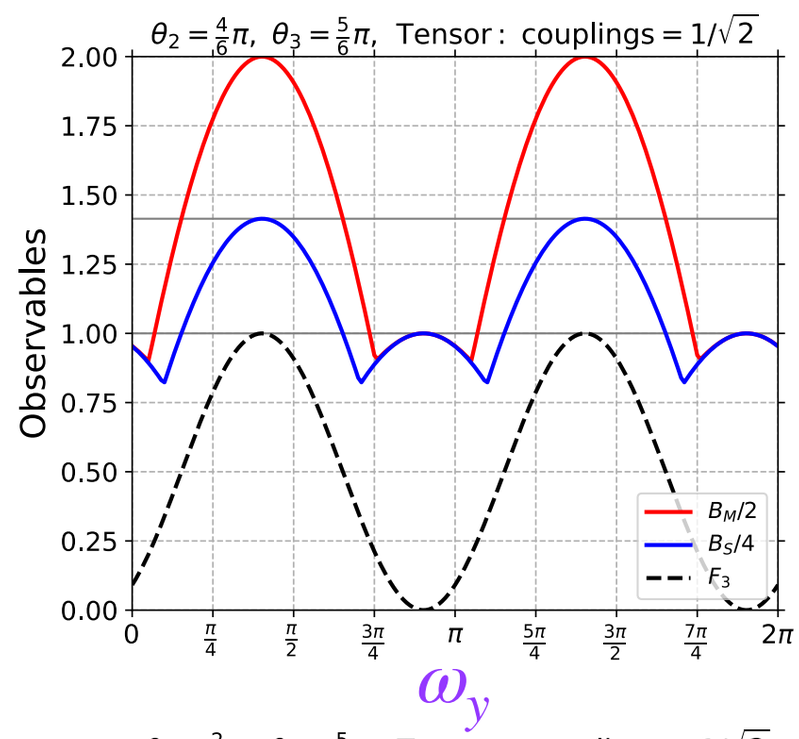
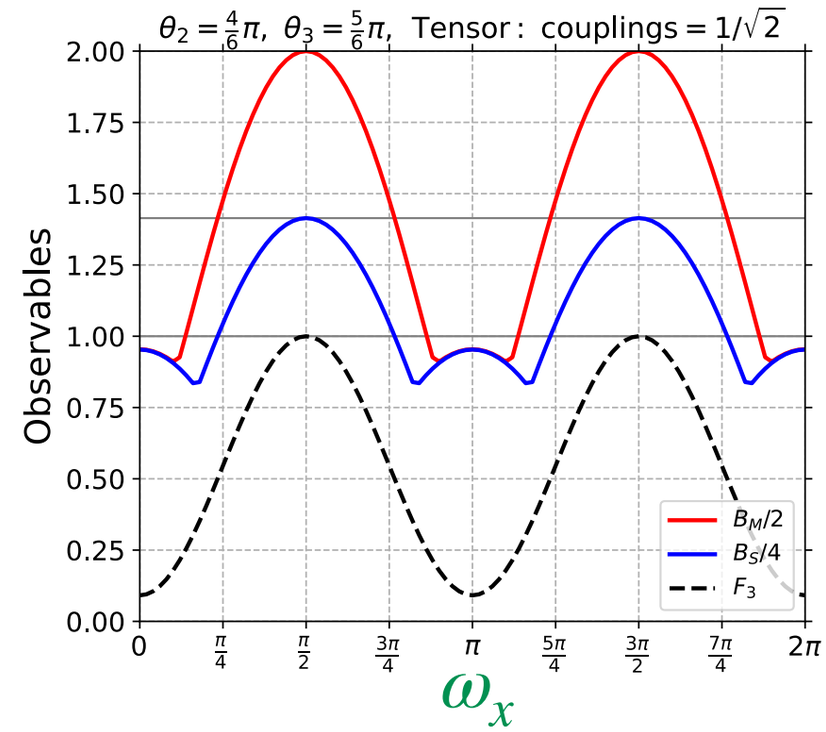
1
λ_1
y
$\mathbf{n}(\theta,\phi)$
θ_3
θ_2
x
3
λ_3
λ_2
2

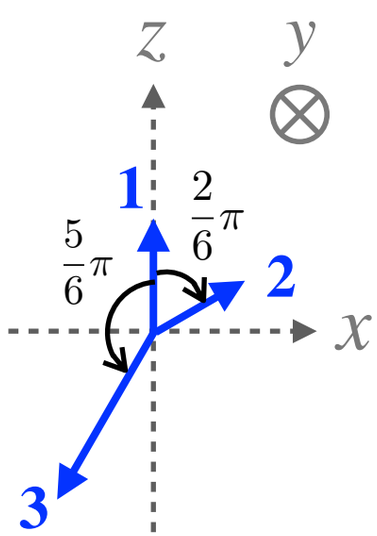

z
ω_y
n
y
x


z
ω_x
n
x
y


z
y
1
$\frac{5}{6}\pi$
$\frac{4}{6}\pi$
x
2
3


$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ Tensor : couplings $= 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_y


$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ Tensor : couplings $= 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_x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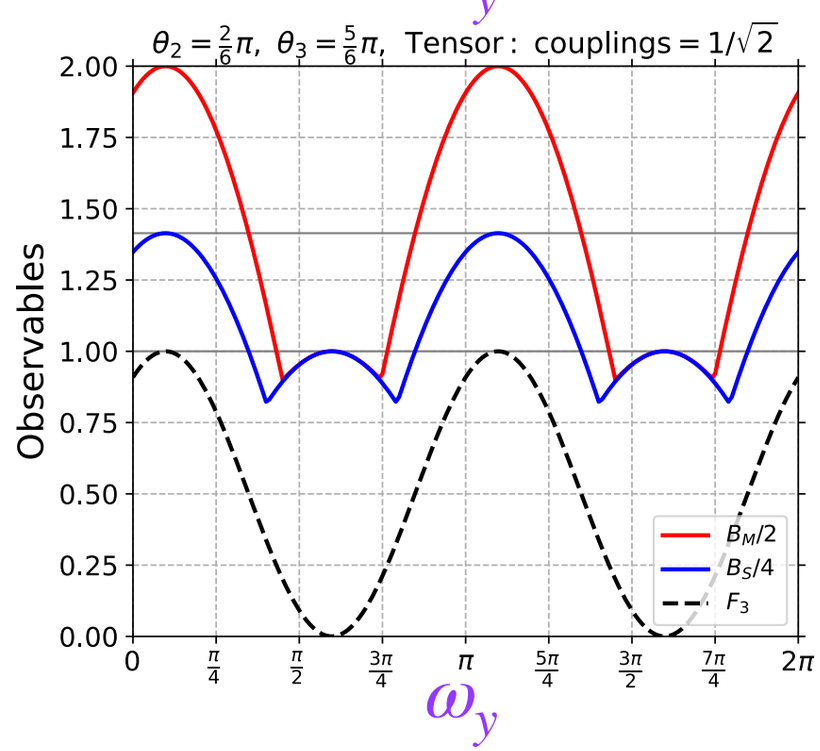
y
1
$\frac{2}{6}\pi$
$\frac{5}{6}\pi$
2
x
3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Observables
$B_M/2$
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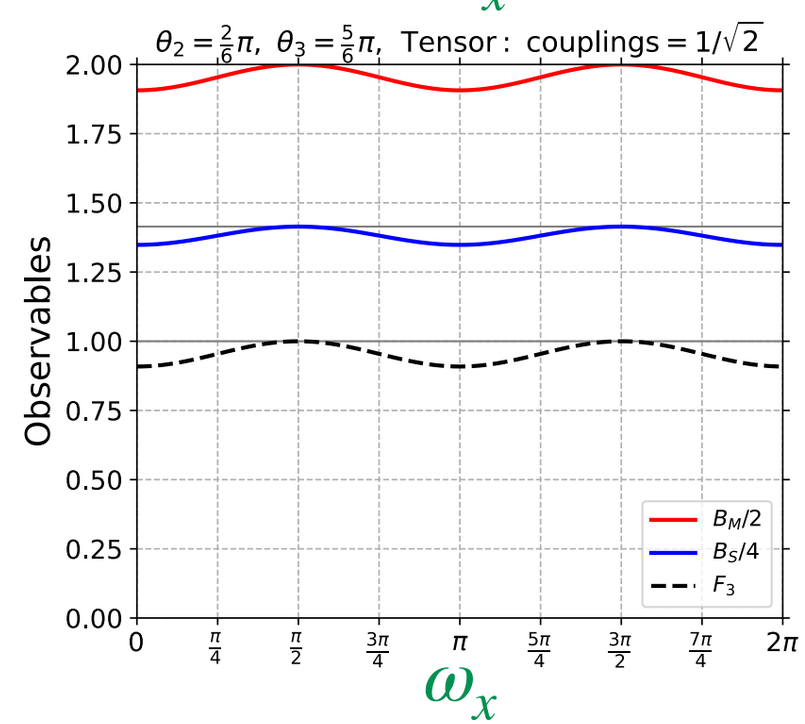
F_3
ω_y


$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ Tensor : couplings $= 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_x

Summary

- Quantum Theoretic quantities (e.g. entanglement and Bell non-locality) are useful in testing the SM and QM at high-energies and provides a new perspective to QFT.

- Entanglement in $pp \to t\bar{t}$ at the LHC has recently been measured. Entanglement and Bell violation may be measured also in the other channels, e.g. $pp \to WW, ZZ, H/Z \to \tau^+\tau^-$.

- We demonstrated that the entanglement and Bell violation in $H \to \tau^+\tau^-$ can be measured in good precision at future e^+e^- colliders.

- Entanglement and non-locality in the 3-qubit system from a 3-body decay are studied.

Future works and directions

- Effect of **masses** in the final particles

- More **spin structures**: $SFFV, VVFF, SFVF_{3/2}, SVVT \cdots$

- How entanglement is created and evolves in the particle physics interactions/measurements?

- Can we constrain new physics from the general properties of entanglement?

Thank you for listening!