Loop Quantum Gravity : state of the art

Karim NOUI

Laboratoire de Mathématiques et de Physique Théorique, TOURS
Fédération Denis Poisson

Overview

Quantum Gravity : Why and How ?

1. Classical framework: Ashtekar variables

- *Ashtekar Gravity looks like Yang-Mills*

2. Quantum Geometry

- *Polymer representation*
- *Kinematical States and Geometric Operators*

3. Quantum Dynamics?

- *From Wheeler-de Witt to Spin-Foams*

Successes and failures

Quantum Gravity : a general discussion
WHY ?

Physics of a relativistic system of mass m and length ℓ

- ▷ $\ell \sim \lambda_c = \frac{h}{mc}$: quantum physics
- ▷ $\ell \sim r_s = \frac{Gm}{c^2}$: general relativity
- ▷ $\ell \sim \sqrt{\lambda_c r_s} = \ell_p$: quantum gravity

Where is quantum gravity : at the GR singularities !

- ▷ Hawking-Penrose theorem
- ▷ Origin of the universe : quantum cosmology
- ▷ Black holes : "microscopic" explanation of entropy

But quantum physics and general relativity are not compatible

- ▷ Quantum Field theory based on a fixed background
- ▷ General Relativity is a non renormalisable theory
- ▷ Related problems : time, observables and diffeomorphisms etc...

One thinks that General Relativty fails at ℓ_p

▷ GR is the Fermi model of a Standard model ?

▷ Modify classical paradigms : String theory (extra-dimensions ...)

▷ GR appears as an effective theory with corrections at ℓ_p

One thinks that Quantum methods fail for GR

▷ New quantisation roads : Loop Quantum Gravity (polymer states)

▷ Problem of singularities similar of H atom : classical instability but existence of quantum ground state

▷ Quantisation resolves singularities : discretisation, minimal length...

Two or more roads... for one solution !

▷ Loops and Strings are orthogonal directions

▷ For loops : GR is fundamental $\Longrightarrow$ background independence

▷ For Strings : QFT with Fock spaces and so on...

Many classical actions for General Relativity

Lagrangian formulation : all actions lead to GR equations

▷ Einstein-Hilbert action : functional of the metric g

$$S_{EH}[g] = \int d^4x \sqrt{|g|} R$$

▷ Hilbert-Palatini action : functional of g and the connection Γ

$$S_{HP}[g] = \int d^4x \sqrt{|g|} R[g, \Gamma]$$

▷ Cartan formalism : $g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$ and $F = d\omega + \omega \wedge \omega$

$$S_C[e, \omega] = \int \langle e \wedge e \wedge \star F[\omega] \rangle = \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} \; e_\mu^I e_\nu^J F_{\rho\sigma}^{KL}$$

▷ Ashtekar-Barbero-Holst action : generalisation of Cartan

$$S_{ABH}[e, \omega] = \int \langle e \wedge e \wedge \star F[\omega] \rangle + \frac{1}{\gamma} \langle e \wedge e \wedge F[\omega] \rangle$$

Hamiltonian analysis : GR phase space

Hamiltonian formulation : $M = \Sigma \times \mathbb{R}$ ('61)

- ADM variables : $ds^2 = N^2 dt^2 - (N^a dt + h_{ab} dx^b)(N^a dt + h_{ac} dx^c)$
- ADM action : (h, π) canonical variables

$$S_{ADM}[h, \pi; N, N^a] = \int dt \int d^3x (\dot{h}\pi + N^a H_a[h, \pi] + NH[h, \pi])$$

- Equations, $H = 0 = H_a$, very complicated and highly non-linear

Ashtekar formulation : originally $\gamma = \pm i$ ('86)

- Partial gauge fixing (time gauge) : $SL(2, \mathbb{C}) \to SU(2)$
- Variables : A : $SU(2)$-connection and E : electric field
- Equations $H = 0 = H_a$ become polynomial !
- γ real : same structure but H not polynomial

A summary of the classical formulation

First order Lagrangian : variables are Cartan data

- A tetrad e_μ^I such that $g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$
- a $sl(2,C)$ spin-connection $\omega = \omega^i R_i + \omega^{0i} B_i$; $F(\omega)$ its curvature
- Classical action depends on the free parameter $\gamma \neq 0$

$$S_P[e,\omega] = \int e^I \wedge e^J \wedge (\star F_{IJ}(\omega) - \frac{1}{\gamma} F_{IJ}(\omega))$$

- Time gauge : partial gauge fix $SL(2,\mathbb{C})$ to $SU(2)$

Hamiltonian analysis : similarities with Yang-Mills

- New variables : $E_i^a = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e_b^j e_c^k$ and $A_a^i = \omega_a^i + \gamma\omega_a^{0i}$

$$\{A_a^i(x), E_j^b(y)\} = (8\pi\gamma G)\delta_a^b \delta_j^i \delta^3(x,y)$$

- The constraints are "almost" polynomial

$$H_a = F_{ab}^i E_i^b, \quad H = (F_{ab}^{ij} + (\gamma^2+1)K_{[a}^i K_{b]}^j) E_i^a E_j^b$$

- One more constraint : Gauss $\mathcal{G}_i = D_a E_i^a$

Quantisation of a point particle

Algebra of quantum operators

- ▷ Phase space : $\{P, Q\} = 1$
- ▷ Quantisation leads to Weyl algebra $[\hat{P}, \hat{Q}] = i\hbar$

Quantum states from representation theory

- ▷ Schrodinger representation : $\varphi \in L^2(\mathbb{R})$

$$(\hat{Q}\varphi)(q) = q\varphi(q) \quad , \quad (\hat{P}\varphi)(q) = -i\hbar\frac{\partial\varphi(q)}{\partial q}$$

- ▷ Fock like representation : $[a, a^\dagger] = 1$

$$|0\rangle \rightarrow |n\rangle \sim (a^\dagger)^n|0\rangle$$

- ▷ Stone-Von Neumann : unique representation !

Quantum Field Theory

- ▷ Representation is not unique
- ▷ The Fock representation is the good one

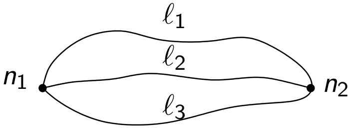
The Polymer representation

Schrodinger like representation

- ▷ States are functionnals of connection : $\varphi(A)$
- ▷ $\hat{A}$ acts by multiplication, $\hat{E}$ as a derivation
- ▷ Problem : no scalar product $\langle \varphi_1 | \varphi_2 \rangle \stackrel{?}{=} \int [\mathcal{D}A]\, \overline{\varphi_1(A)} \varphi_2(A)$

The polymer representation : states are "one-dimensional"

- ▷ Let Γ a graph : L links, V vertices



:ℓ_i are oriented links

:n_i are nodes

- ▷ Let f a function on $SU(2)^L$
- ▷ State : $\varphi_{\Gamma,f}(A) = f(U_{\ell_1}, \cdots, U_{\ell_L})$ where $U_\ell = P \exp(\int_\ell A) \in SU(2)$
- ▷ Ashtekar-Lewandowski measure :

$$\langle \varphi_{\Gamma,f} | \varphi_{\Gamma',f'} \rangle = \delta_{\Gamma,\Gamma'} \int (\prod_i d\mu(U_{\ell_i}))\, \overline{f(U_{\ell_i})} f'(U_{\ell_i})$$

Imposing the constraints

The Gauss constraint

- ▷ Gauge action : $A \mapsto A^g = g^{-1}Ag + g^{-1}dg \Longrightarrow U_\ell \mapsto g(s_\ell)^{-1}U_\ell g(t_\ell)$
- ▷ States are invariant under gauge action
- ▷ Orthonormal basis : ℓ_i with spins I_i and v_i with Clebsh-Gordan

The diffeomorphisms constraint

- ▷ Diffeomorphisms $Diff(\Sigma)$ on Γ and A
- ▷ States are now labelled by knots $[\Gamma]$
- ▷ Unique representation compatible with $Diff(\Sigma)$

The Hamiltonian constraint

- ▷ $\hat{H}\varphi = 0$ is Wheeler-de Witt
- ▷ Very few not interesting solutions
- ▷ Thiemann trick to define $\hat{H}$
- ▷ Spin-foams models from covariant quantisation

So far, no physical solutions...

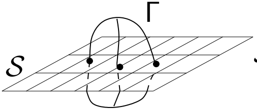
Physical interpretation and discretisation of the space

Area operator $\mathcal{A}(\mathcal{S})$ acting on $\mathcal{H}_0$

- Classical area of a surface $\mathcal{S}$: $\mathcal{A}(\mathcal{S}) = \int_{\mathcal{S}} \sqrt{n_a E_i^a n_b E_i^b}\, d^2\sigma$
- Quantum area operator : $\mathcal{S} = \cup_n^N \mathcal{S}_n$

$$\mathcal{A}(\mathcal{S}) = \lim_{N\to\infty} \sum_n \sqrt{E_i(\mathcal{S}_n)E_i(\mathcal{S}_n)} \quad \text{with} \quad E_i(\mathcal{S}_n) = \int_{\mathcal{S}_n} E_i$$

- Spectrum and Quanta of area



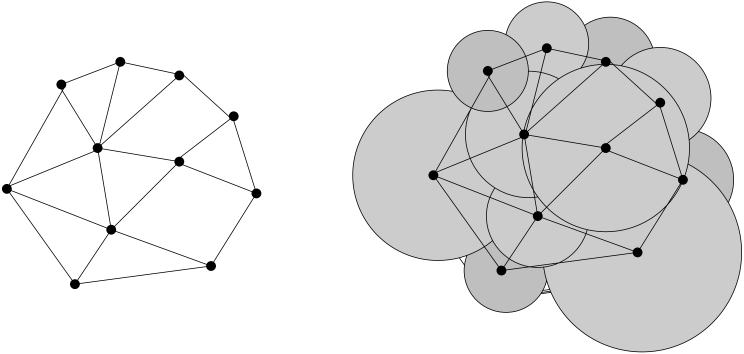
$$\mathcal{A}(\mathcal{S})|S\rangle = \frac{8\pi\gamma\hbar G}{c^3} \sum_{P\in\mathcal{S}\cap\Gamma} \sqrt{j_P(j_P+1)}|S\rangle$$

Volume operator $\mathcal{V}(\mathcal{R})$ acting on $\mathcal{H}_0$

- Classical volume on a domain $\mathcal{R}$: $\mathcal{V}(\mathcal{R}) = \int_{\mathcal{R}} d^3x \sqrt{\frac{|\epsilon_{abc}\epsilon_{ijk}E^{ai}E^{bj}E^{ck}|}{3!}}$
- It acts on the nodes of $|S\rangle$: discrete spectrum

Picture of space at the Planck scale

From the kinematics, Space is discrete...



... It is also non-commutative in 3 dimensions

Hamitonian constraint

Classical Hamiltonian constraint

- First part of Hamiltonian :

$$H(N) = \int_\Sigma d^3x\, N(x)\, \frac{Tr(F_{ab}E^a E^b)}{\sqrt{|det(E)|}}$$

  - Regularization using Thiemann trick :

$$H(N) = -\frac{1}{\kappa}\int_\Sigma N(x)\, Tr(F(x) \wedge \{A(x), V(R_x)\})$$

  - $V(R_x)$ is the volume of a region R_x around x

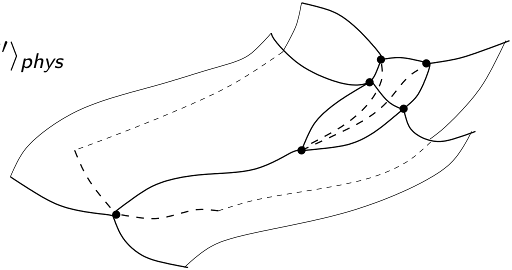
Quantization of the constraint :

- V is a well-defined positive self-adjoint operator on $\mathcal{H}$
- It creates new edges on Spin-network states
- Ultra-locality : action is confined around a vertex
- Ambiguities : ordering, representations etc...

An alternative solution : Spin-Foam models

Transition Amplitudes between states

$$\mathcal{A} = \langle S | S' \rangle_{phys}$$



- ▷ From Topological QFT
- ▷ Relation to LQG not clear
- ▷ Some promising models

Successes and failures

Very interesting program for Quantum Gravity

- ▷ A new quantisation scheme : polymer representation
- ▷ Uniqueness theorem of the representation
- ▷ Complete description of kinematical states
- ▷ Discrete spectrum of area : Black Hole entropy, Cosmology

But NO physical states

- ▷ No difference with a topological theory

Role of Immirzi parameter is unclear

- ▷ Value fixed by $S = A/4$

Compact vs. non compact gauge group ?