An introduction to integrable systems

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What are integrable systems?

An elementary definition : Systems for which we can compute exactly (hence in a non-perturbative way) all observable (measurable) quantities.

They constitute a paradox as they are both exceptional (rare) and somehow ubiquitous systems : If we consider an arbitrary system it will hardly be integrable; however numerous "classical" examples of important (textbooks) physical systems are integrable!

- In classical and quantum mechanics : harmonic oscillators, Kepler problem, various tops, ...
- In continuous systems : integrable non-linear equations like KdV, Non-linear Shrodinger, sine-Gordon, ...
- In classical 2-d statistical mechanics : Ising, 6 and 8-vertex lattices, ...
- In quantum 1-d systems : Heisenberg spin chains, Bose gas, ...
- In 1+1 dimensional quantum field theories : CFT, sine-Gordon, Thirring model, σ-models, ...

A short historical overview

- classical mechanics : Liouville, Hamilton, Jacobi, ...
- continuous classical systems : non-linear partial differential equations, Lax pairs, classical inverse problem method, ...
- classical and quantum statistical mechanics : transfer matrix methods, Bethe ansatz, ...
- synthesis of these two lines in the 80' : quantum inverse scattering method, algebraic Bethe ansatz, Yang-Baxter equation, ...
- links to mathematics : Riemann-Hilbert methods, quantum groups and their representations, knot theory, ...
- many applications from string theory to condensed matter systems

Integrable systems in classical mechanics (I)

We consider Hamiltonian systems $H(p_i, q_i)$ with n canonical conjugate variables p_i and q_i, $i = 1, \dots n$ and equations of motion :

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

and Poisson bracket structure for two functions f and g of the canonical variables :

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right)$$

hence with the property $\frac{df}{dt} = \{H, f\}$

Definition : This Hamiltonian system is said to be Liouville integrable if it possesses n independent conserved quantities F_i in involution, namely $\{H, F_i\} = 0$ and $\{F_i, F_j\} = 0$ with $i, j = 1, \dots n$.

Liouville Theorem :The solution of the equations of motion of a Liouville integrable system is obtained by quadrature.

Integrable systems in classical mechanics (II)

Conserved quantities $F_i \rightarrow$ Poisson generators of corresponding symmetries and reductions of the phase space to the sub-variety M_f defined by $F_i = f_i$ for given constants f_i.

$\rightarrow$ separation of variables (Hamilton-Jacobi) and action-angles variables : canonical transformation $(p_i, q_i) \rightarrow (\Phi_i, \omega_i)$ with $H = H(\{\Phi_i\})$ and trivial equations of motion :

$$\{H, \Phi_i\} = 0 \rightarrow \Phi_i(t) = cte$$

$$\{H, \omega_i\} = \frac{\partial H}{\partial \Phi_i} = cte \rightarrow \omega_i(t) = t\alpha_i + \omega_i(0)$$

Construct inverse map $(\Phi_i, \omega_i) \rightarrow (p_i, q_i)$ to get $p_i(t)$ and $q_i(t)$.

Algebraic tools : classical systems

Main question : How to construct and solve classical integrable systems?

$\rightarrow$ Lax pair $N \times N$ matrices L and M which are functions on the phase space such that the equations of motion are equivalent to the N^2 equations :

$$\frac{d}{dt} L = [L, M]$$

which for any integer p leads to a conserved quantity since $\frac{d}{dt} tr(L^p) = 0$.

Integrable canonical structure (commutation of the invariants of the matrix L) equivalent to the existence of an r-matrix such that :

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]$$

Important (simple) cases : r_{12} is a constant matrix with $r_{21} = -r_{12}$ and satisfies (Jacobi identity) the classical Yang-Baxter relation,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

$\rightarrow$ reconstruction of M in terms of L and r (Lie algebras and Lie groups representation theory) and resolution of the equations of motion (algebraic factorization problem).

Integrability for quantum systems

Quantum systems described by an Hamiltonian operator H acting on a given Hilbert space (the space of states) $\mathcal{H}$.

A definition of integrability : There exists a commuting generating operator of conserved quantities $\tau(\lambda)$, namely such that for arbitrary λ, μ

$$[H, \tau(\lambda)] = 0 \qquad [\tau(\lambda), \tau(\mu)] = 0$$

H is a function of $\tau(\lambda)$ and $\tau(\lambda)$ has simple spectrum (diagonalizable) $\longrightarrow$ complete characterization of the spectrum and eigenstates of H.

$\longrightarrow$ what we wish to compute in an algebraic way :

- spectrum and eigenstates of H and $\tau(\lambda)$ (energy levels and quantum numbers)
- matrix elements of any operator in this eigenstate basis (leads to measurable quantities like structure factors)

Algebraic tools : quantum systems

Yang-Baxter equation and algebras for the L and R matrices : quantum version of the corresponding classical structures for $L \in End(V \otimes \mathcal{A})$, $\mathcal{A}$ the quantum space of states, $R \in End(V \otimes V)$, $L_1 = L \otimes id$ and $L_2 = id \otimes L$,

$$R_{12}\ L_1 \cdot L_2 = L_2 \cdot L_1\ R_{12}$$

$$R_{12}\ R_{13}\ R_{23} = R_{23}\ R_{13}\ R_{12}$$

$\rightarrow$ Recover classical relations for $R = id + i\hbar r + O(\hbar^2)$. These equations and algebras define quantum group structures as quantization of the corresponding Lie algebras and Lie groups of the classical case, and appear in :

- 2-d integrable lattice models (vertex models, ...) : Boltzman weights
- 1-d quantum systems (spin chains, Bose gas, ...) : monodromy matrix
- 1+1-d quantum field theories : scattering matrices

In all these cases, L and R are depending on additional continuous parameters $L = L(\lambda)$ and $R = R(\lambda, \mu)$.

Our favorite example : the XXZ Heisenberg chain

The *XXZ* spin-1/2 Heisenberg chain **in a magnetic field** is a quantum interacting model defined on a one-dimensional lattice with M sites, with Hamiltonian,

$$H_{XXZ} = \sum_{m=1}^{M} \{\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1)\} - \mathbf{h} \sum_{m=1}^{M} \sigma_m^z$$

Quantum space of states : $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$, $\dim \mathcal{H} = 2^M$.

$\sigma_m^{x,y,z}$: local spin operators (in the spin-$\frac{1}{2}$ representation) at site m
They act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.

- periodic boundary conditions
- disordered regime, $|\Delta| < 1$ and $h < h_c$

The spin-1/2 XXZ Heisenberg chain : results

Spectrum :

- Bethe ansatz : Bethe, Hulthen, Orbach, Walker, Yang and Yang,...
- Algebraic Bethe ansatz : Faddeev, Sklyanin, Taktadjan,...

Correlation functions :

- Free fermion point $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa,...
- Starting 1985 Izergin, Korepin : first attempts using Bethe ansatz for general Δ
- General Δ : multiple integral representations in 1992 and 1996 Jimbo and Miwa $\longrightarrow$ from qKZ equation, in 1999 Kitanine, Maillet, Terras $\longrightarrow$ from Algebraic Bethe Ansatz.

Several developments since 2000: (Kitanine, Maillet, Slavnov, Terras; Boos, Jimbo, Miwa, Smirnov,Takeyama; Gohmann, Klumper,Seel; Caux, Hagemans, Maillet; ...)

Diagonalization of the Hamiltonian

Monodromy matrix:

$$T(\lambda) \equiv T_{a,1\ldots M}(\lambda) = L_{aM}(\lambda)\ldots L_{a2}(\lambda) L_{a1}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[a]}$$

with $\quad L_{an}(\lambda) = \begin{pmatrix} \sinh(\lambda + \eta\sigma_n^z) & \sinh\eta\,\sigma_n^- \\ \sinh\eta\,\sigma_n^+ & \sinh(\lambda - \eta\sigma_n^z) \end{pmatrix}_{[a]}$

$\hookrightarrow$ **Yang-Baxter algebra**: ∘ generators A, B, C, D

∘ commutation relations given by the R-matrix

$$R_{ab}(\lambda,\mu)\, T_a(\lambda) T_b(\mu) = T_b(\mu) T_a(\lambda)\, R_{ab}(\lambda,\mu)$$

$\rightarrow$ commuting conserved charges: $\mathcal{T}(\lambda) = A(\lambda) + D(\lambda)$

$\rightarrow$ construction of the space of states by action of B operators on a reference state $|\,0\,\rangle \equiv |\uparrow\uparrow\ldots\uparrow\rangle$

$\rightarrow$ eigenstates : $|\,\psi\,\rangle = \prod_k B(\lambda_k)|\,0\,\rangle$ with $\{\lambda_k\}$ solution of the Bethe equations.

Action of local operators on eigenstates

→ **Resolution of the quantum inverse scattering problem:** reconstruct local operators σ_j^α in terms of the generators $T_{\epsilon,\epsilon'}$ of the Yang-Baxter algebra:

$$\sigma_j^- = \{(A+D)(0)\}^{j-1} \cdot B(0) \cdot \{(A+D)(0)\}^{-j}$$
$$\sigma_j^+ = \{(A+D)(0)\}^{j-1} \cdot C(0) \cdot \{(A+D)(0)\}^{-j}$$
$$\sigma_j^z = \{(A+D)(0)\}^{j-1} \cdot (A-D)(0) \cdot \{(A+D)(0)\}^{-j}$$

→ use the Yang-Baxter commutation relations for A, B, C, D to get the action on arbitrary states

→ correlation functions = sums over scalar products that are computed as ratios of determinants.

Correlation functions of critical (integrable) models

- **Asymptotic results predictions**
  - Luttinger liquid approximation / C.F.T. and finite size effects
    Luther and Peschel, Haldane, Cardy, Affleck, ... Lukyanov, ...

- **Exact results (XXZ, NLS, ...)**
  - Free fermion point $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa ...
  - From 1984: Izergin, Korepin ... (first attempts using ABA)
  - General Δ: (form factors and building blocks)
    ⋆ 1992-96 Jimbo, Miwa ... $\longrightarrow$ for infinite chain from QG
    ⋆ 1999 Kitanine, M, Terras $\longrightarrow$ for finite and infinite chain from ABA
  - Several developments for the last twelve years: Temperature case, numerics and actual experiments, master equation representation, some asymptotics, fermionic structures, etc.

$\hookrightarrow$ **Compute explicitly relevant physical correlation functions?**

$\hookrightarrow$ **Connect to the CFT limit from the exact results on the lattice?**

Physical correlation function : general strategies

At zero temperature only the ground state $|\omega\rangle$ contributes :

$$g_{12} = \langle \omega | \theta_1 \theta_2 | \omega \rangle$$

Two main strategies to evaluate such a function:

(i) compute the action of local operators on the ground state $\theta_1\theta_2|\omega\rangle = |\tilde{\omega}\rangle$ and then calculate the resulting scalar product:

$$g_{12} = \langle \omega | \tilde{\omega} \rangle$$

(ii) insert a sum over a complete set of eigenstates $|\omega_i\rangle$ to obtain a sum over one-point matrix elements (form factor type expansion) :

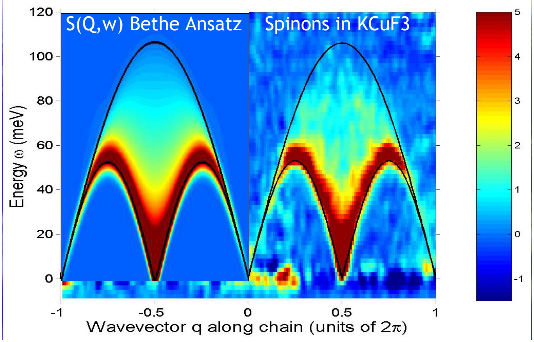
$$g_{12} = \sum_i \langle \omega | \theta_1 | \omega_i \rangle \cdot \langle \omega_i | \theta_2 | \omega \rangle$$

Correlation functions : ABA approach

1. **Diagonalise the Hamiltonian using ABA**
   $\longrightarrow$ key point : Yang-Baxter algebra $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$
   $\longrightarrow$ $|\,\psi_g\,\rangle = B(\lambda_1)\dots B(\lambda_N)|\,0\,\rangle$ with $\mathcal{Y}(\lambda_j;\{\lambda\}) = 0$ (Bethe eq.)
2. **Act with local operators on eigenstates**
   $\longrightarrow$ solve the quantum inverse problem (1999):
   $$\sigma_j^{(\alpha)} = (A+D)^{j-1}X^{(\alpha)}(A+D)^{-j} \text{ with } X^{(\alpha)} = A, B, C, D$$
   $\longrightarrow$ use Yang-Baxter commutation relations
3. **Compute the resulting scalar products** (determinant representation)
   $\longrightarrow$ determinant representation for form factors of the finite chain
   $\longrightarrow$ elementary building blocks of correlation functions as multiple integrals in the thermodynamic limit (2000)
4. **Two-point function:** sum up elementary blocks or form factors?
   $\longrightarrow$ master equation representation in finite volume
   $\longrightarrow$ numerical sum of form factors : dynamical structure factors
5. **Analysis of the two-point functions (2008-2011):**
   $\longrightarrow$ series expansion (multiple integrals) and large distance asymptotics
   $\longrightarrow$ analysis of correlation functions from form factor series

Numerical summation of form factor series (XXX)

- **Structure factors define the dynamics of the models**
- **They can be measured experimentally**



$S(Q, \omega)$ is the dynamical spin-spin structure factor. The Bethe ansatz curve is computed for a chain of 500 sites (with J.- S. Caux) compared to the experimental curve obtained by A. Tennant in Berlin by neutron scattering. Colors indicate the value of the function $S(Q, \omega)$.

Results from multiple integrals representations

Generating function

$$Q^{\kappa}_{1,m} = \prod_{n=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma^z_n \right) \quad \textit{with} \;\; \kappa = e^{\beta}$$

Asymptotic behavior (RH techniques applied to multiple integrals)

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \underbrace{G^{(0)}(\beta, m)[1 + o(1)]}_{\text{non-oscillating terms}} + \underbrace{\sum_{\sigma=\pm} G^{(0)}(\beta + 2i\pi\sigma, m)[1 + o(1)]}_{\text{oscillating terms}}$$

$$G^{(0)}(\beta, m) = C(\beta)\, e^{m\beta D}\, m^{\frac{\beta^2}{2\pi^2} Z(q)^2}$$

- $Z(\lambda)$ is the dressed charge $\quad Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda - \mu)\, Z(\mu) = 1$
- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu) \mathrm{d}\mu = \frac{1 - \langle \sigma^z \rangle}{2} = \frac{k_F}{\pi}$
- The coefficient $C(\beta)$ is given as the ratio of four Fredholm determinants.
- sub-leading oscillating terms restore the $2\pi i$-periodicity in β related to periodicity in Fredholm determinant of generalized sine kernel

2-point function asymptotic behavior

$$\langle \sigma^z_1 \sigma^z_{m+1} \rangle = (2D-1)^2 - \frac{2Z(q)^2}{\pi^2 m^2} + 2|F_{\sigma^z}|^2 \cdot \frac{\cos(2mk_F)}{m^{2Z(q)^2}} + o\left(\frac{1}{m^2}, \frac{1}{m^{2Z(q)^2}} \right)$$

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Form factors strike back!

The umklapp form factor

$$\lim_{N,M\to\infty}\left(\frac{M}{2\pi}\right)^{2\mathcal{Z}^2}\frac{|\langle\psi(\{\mu\})|\sigma^z|\psi(\{\lambda\})\rangle|^2}{\|\psi(\{\mu\})\|^2\cdot\|\psi(\{\lambda\})\|^2}=|F_{\sigma^z}|^2.$$

with

$$2\mathcal{Z}^2=Z(q)^2+Z(-q)^2$$

- $\{\lambda\}$ are the Bethe parameters of the ground state
- $\{\mu\}$ are the Bethe parameters for the excited state with one particle and one hole on opposite sides of the Fermi boundary (umklapp type excitation).
- the critical exponents for the form factor behavior (in terms of size M) and for the correlation function (in terms of distance) are equal!

$\hookrightarrow$ Higher terms in the asymptotic expansion will involve particle/holes form factors corresponding to $2\ell k_F$ oscillations and properly normalized form factors will be related to the corresponding amplitudes

$\hookrightarrow$ Analyze the asymptotic behavior of the correlation function directly from the form factor series!

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Spin-spin correlation functions as sum over form factors

$$\langle \sigma_1^s \, \sigma_{m+1}^{s'} \rangle = \sum_{|\,\psi'\,\rangle} \mathcal{F}^{(s)}_{\psi_g \psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi' \psi_g}(m+1) \quad \text{with} \quad \mathcal{F}^{(s)}_{\psi\psi'}(m) = \frac{\langle\, \psi \,|\sigma_m^s|\, \psi' \,\rangle}{||\psi|| \cdot ||\psi'||}$$

$$\langle \sigma_1^s \, \sigma_{m+1}^{s'} \rangle_{\text{cr}} = \lim_{M\to\infty} \sum_{\ell=-\infty}^{\infty} \sum_{|\,\psi'\,\rangle \text{ in } \mathbf{P}_\ell \text{ class}} \mathcal{F}^{(s)}_{\psi_g \psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi' \psi_g}(m+1)$$

$$= \lim_{M\to\infty} \sum_{\ell=-\infty}^{\infty} e^{2im\ell k_F} \, M^{-\theta_\ell^{(ss')}} \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell} \, \mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\text{finite}} \prod_{\epsilon=\pm} \frac{G^2(1+\epsilon F_\epsilon)}{G^2(1+\epsilon\ell+\epsilon F_\epsilon)}$$

$$\times \underbrace{\sum_{\substack{\{p\},\{h\} \\ n_p^+ - n_h^+ = \ell}} e^{\frac{2\pi i m}{M}\mathcal{P}^{(d)}_{ex}} \prod_{\epsilon=\pm} R_{n_p^\epsilon, n_h^\epsilon}(\{p^\epsilon\},\{h^\epsilon\}|\epsilon F_\epsilon)}_{\substack{\text{sum over all possible configurations of integers} \\ \text{in the } \mathbf{P}_\ell \text{ class}}}$$

$$\sum_{\substack{n_p, n_h = 0 \\ n_p - n_h = \ell}}^{\infty} \sum_{\substack{p_1 < \cdots < p_{n_p} \\ p_a \in \mathbb{N}^*}} \sum_{\substack{h_1 < \cdots < h_{n_h} \\ h_a \in \mathbb{N}^*}} e^{\frac{2\pi i m}{M}\left[\sum_{j=1}^{n_p}(p_j - 1) + \sum_{k=1}^{n_h} h_k\right]} R_{n_p, n_h}(\{p\},\{h\}|F)$$

$$= \frac{G^2(1+\ell+F)}{G^2(1+F)} \, \frac{e^{\frac{i\pi m}{M}\ell(\ell-1)}}{\left(1 - e^{\frac{2i\pi m}{M}}\right)^{(F+\ell)^2}}$$

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Correlation function $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = -\frac{1}{2\pi^2} \, \partial_\alpha^2 \mathbf{D}_m^2 \langle e^{2\pi i \alpha \mathcal{Q}_m} \rangle \big|_{\alpha=0} - 2D + 1$$

where $\mathbf{D}_m^2$ is the second lattice derivative, D is the average density, and

$$\mathcal{Q}_m = \frac{1}{2} \sum_{k=1}^{m} (1 - \sigma_k^z)$$

$\rightsquigarrow$ study form factors $\langle \psi_\alpha(\{\mu\}) \, | e^{2\pi i \alpha \mathcal{Q}_m} | \, \psi_g \rangle$ where $| \, \psi_\alpha(\{\mu\}) \, \rangle$ is an α-deformed Bethe state, with $\{\mu\}$ solution of

$$M p_0(\mu_{\ell_j}) - \sum_{k=1}^{N} \theta(\mu_{\ell_j} - \mu_{\ell_k}) = 2\pi \Big(\ell_j + \alpha - \frac{N+1}{2} \Big)$$

For the $\mathbf{P}_\ell$ class:

- excitation momentum $2\alpha k_F + \mathcal{P}_{\text{ex}}$
- shift functions $F_\pm$: $F_- = F_+ = \alpha \mathcal{Z} + \ell(\mathcal{Z} - 1)$ with $\mathcal{Z} = Z(\pm q)$ where $Z(\lambda)$ is the dressed charge given by

$$Z(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} d\mu \, \frac{\sin 2\zeta}{\sinh(\lambda - \mu + i\zeta) \sinh(\lambda - \mu - i\zeta)} \, Z(\mu) = 1$$

- exponent $\theta_{\alpha+\ell}$: $\theta_{\alpha+\ell} = 2[(\alpha + \ell)\mathcal{Z}]^2$,

Correlation function $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$

$\rightsquigarrow$ leading asymptotic terms for all oscillating harmonics:

$$\langle e^{2\pi i \alpha \mathcal{Q}_m} \rangle_{cr} = \sum_{\ell=-\infty}^{\infty} |\mathcal{F}_{\alpha+\ell}|^2_{\text{finite}} \, \frac{e^{2im(\alpha+\ell)k_F}}{(2\pi m)^{\theta_{\alpha+\ell}}}$$

with $\quad \theta_{\alpha+\ell} = 2[(\alpha+\ell)\mathcal{Z}]^2,$

and $\quad |\mathcal{F}_{\alpha+\ell}|^2_{\text{finite}} = \lim_{M\to\infty} M^{\theta_{\alpha+\ell}} \frac{|\langle \psi_g | \psi_{\alpha+\ell} \rangle|^2}{||\psi_g||^2 \, ||\psi_{\alpha+\ell}||^2},$

where $|\, \psi_{\alpha+\ell} \,\rangle$ is the $(\alpha+\ell)$-shifted ground state

Rm: terms $\ell = 0, \pm 1$ coincide with results from multiple integrals analysis

$\rightsquigarrow$ leading asymptotic terms for the two-point function:

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{cr} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2 \sum_{\ell=1}^{\infty} |\mathcal{F}_\ell^z|^2_{\text{finite}} \, \frac{\cos(2m\ell k_F)}{(2\pi m)^{2\ell^2 \mathcal{Z}^2}}$$

with $\quad |\mathcal{F}_\ell^z|^2_{\text{finite}} = \lim_{M\to\infty} M^{2\ell^2\mathcal{Z}^2} \frac{|\langle\, \psi_g \,| \sigma_1^z |\, \psi_\ell \,\rangle|^2}{||\psi_g||^2 \, ||\psi_\ell||^2},$

where $|\, \psi_\ell \,\rangle$ is the ℓ-shifted ground state

Correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$

⇝ critical excited states of the $\mathbf{P}_\ell$ class in the (N_0+1)-sector

- critical values of the shift function in the $\mathbf{P}_\ell$ class:
  $$F_- = \ell(\mathcal{Z}-1) - \frac{1}{2\mathcal{Z}}, \qquad F_+ = \ell(\mathcal{Z}-1) + \frac{1}{2\mathcal{Z}}$$
- critical exponents: $\theta_\ell = 2\ell^2\mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}$
- simplest form factor in the $\mathbf{P}_\ell$ class:
  $$\left|\mathcal{F}_\ell^+\right|^2_{\text{finite}} = \lim_{M\to\infty} M^{(2\ell^2\mathcal{Z}^2+\frac{1}{2\mathcal{Z}^2})} \frac{|\langle\, \psi_g \,|\sigma_1^+|\, \psi_\ell \,\rangle|^2}{||\psi_g||^2\, ||\psi_\ell||^2}$$
  where $|\, \psi_\ell \,\rangle$ is the ℓ-shifted ground state in the (N_0+1)-sector

⇝ leading asymptotic terms for the two-point function:

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle_{cr} = \frac{(-1)^m}{(2\pi m)^{\frac{1}{2\mathcal{Z}^2}}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell\, |\mathcal{F}_\ell^+|^2_{\text{finite}}\, \frac{e^{2im\ell k_F}}{(2\pi m)^{2\ell^2\mathcal{Z}^2}}$$

Results for the XXZ chain

2-point functions

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{cr} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2\sum_{\ell=1}^{\infty} |\mathcal{F}_\ell^z|^2_{\text{finite}} \frac{\cos(2m\ell k_F)}{(2\pi m)^{2\ell^2\mathcal{Z}^2}}$$

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle_{cr} = \frac{(-1)^m}{(2\pi m)^{\frac{1}{2\mathcal{Z}^2}}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell |\mathcal{F}_\ell^+|^2_{\text{finite}} \frac{e^{2im\ell k_F}}{(2\pi m)^{2\ell^2\mathcal{Z}^2}}$$

- $\mathcal{Z} = Z(q)$ where $Z(\lambda)$ is the dressed charge
$$Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda-\mu)\, Z(\mu) = 1$$
- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu)\mathrm{d}\mu = \frac{1-\langle \sigma^z \rangle}{2} = \frac{k_F}{\pi}$
- $$|\mathcal{F}_\ell^z|^2_{\text{finite}} = \lim_{M\to\infty} M^{2\ell^2\mathcal{Z}^2} \frac{|\langle \psi_g | \sigma_1^z | \psi_\ell \rangle|^2}{\langle \psi_g | \psi_g \rangle \langle \psi_\ell | \psi_\ell \rangle}$$
- $$|\mathcal{F}_\ell^+|^2_{\text{finite}} = \lim_{M\to\infty} M^{\left(2\ell^2\mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}\right)} \frac{|\langle \psi_g | \sigma_1^+ | \psi_\ell \rangle|^2}{\langle \psi_g | \psi_g \rangle \langle \psi_\ell | \psi_\ell \rangle}$$

Further results and open questions

- **Further results**
  - Time dependent case for the Bose gas (simpler model: no bound-states) (to appear)
    $\rightsquigarrow$ contribution of a saddle point away from the Fermi surface
  - Asymptotics for large distances in the temperature case (contact with QTM method)
    $\rightsquigarrow$ see Kozlowski, Maillet, Slavnov *J. Stat. Mech. P12010 (2011)*
  - Arbitrary n-point correlation functions in the CFT limit (to appear)
  - In fact all the derivation applies to a large class of non integrable models as well

- **Some open problems...**
  - Sub-leading terms for each harmonics?
  - Time dependent case for XXZ : needs careful treatment of bound-states (complex roots)
  - Deeper links with TASEP, Z-measures, ...?