Integrable structures in gauge theory scattering amplitudes

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Outline

- ✔ Gauge theory and scattering amplitudes at tree-level.
- ✔ Dual superconformal symmetry and Yangian symmetry.
- ✔ Loop amplitudes and Wilson loops
- ✔ Analytic structure and bootstraps

Gauge theory

The Yang-Mills action:

$$S = \int d^4x \ \mathrm{Tr}\Big(-\tfrac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x)\Big)$$

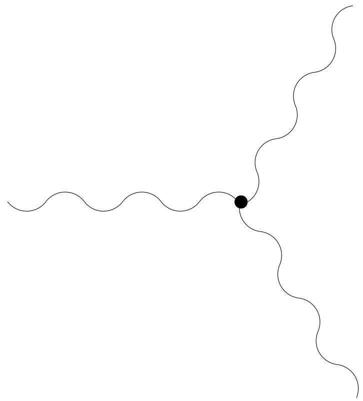
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

- ✔ It is local.
- ✔ It has a gauge symmetry.
- ✔ One must fix a gauge to compute amplitudes, e.g. $\partial_\mu A^\mu = 0$.
- ✔ Tree-level amplitude is given by a rational function with local poles,

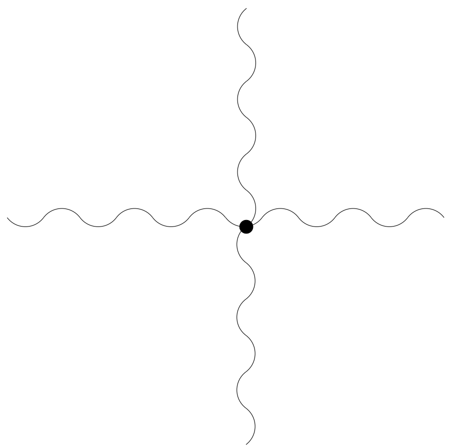
$$\frac{1}{(p_i + p_{i+1} + \dots p_j)^2} \,.$$

Feynman rules

$$\frac{1}{p^2}$$



$$f^{abc}\, p^{\mu}$$



$$f^{abc} f^{cde}$$

Tree-level amplitudes

An on shell momentum $p^2 = 0$ can be written as $p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$.

The free Yang-Mills field equations and Bianchi identity $\partial^{\mu}F_{\mu\nu} = 0, \quad \partial_{[\mu}F_{\nu\rho]} = 0$

become $$\partial^{\alpha\dot{\alpha}}F_{\alpha\beta} = 0, \quad \partial^{\alpha\dot{\alpha}}F_{\dot{\alpha}\dot{\beta}} = 0$$

with solution $$F_{\alpha\beta} = \lambda_{\alpha}\lambda_{\beta}G_{+}, \quad F_{\dot{\alpha}\dot{\beta}} = \tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}G_{-}$$

We are interested in ordered scattering amplitudes of these on-shell states.

Examples:

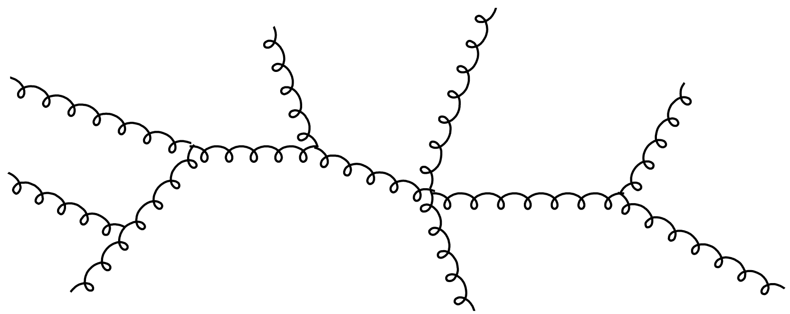
- ✔ $\mathcal{A}(- - + + \ldots +)$ (MHV),
- ✔ $\mathcal{A}(- - - + + \ldots +)$ (NMHV),
- ✔ ...

Tree-level recursion relations

Imagine calculating tree-level scattering amplitudes.

You will have a sum of Feynman diagrams (many, many...)

$$\mathcal{A}(p_1, \ldots, p_n) = \sum$$



Now let us deform two of the momenta:

Britto, Cachazo, Feng
Britto, Cachazo, Feng, Witten

$$p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} \longrightarrow (\lambda_i^\alpha - z\lambda_j^\alpha)\tilde\lambda_i^{\dot\alpha} = p_i^{\alpha\dot\alpha}(z)$$
$$p_j^{\alpha\dot\alpha} = \lambda_j^\alpha \tilde\lambda_j^{\dot\alpha} \longrightarrow \lambda_j^\alpha(\tilde\lambda_j^{\dot\alpha} + z\tilde\lambda_i^{\dot\alpha}) = p_j^{\alpha\dot\alpha}(z)$$

The amplitude $\mathcal{A}(z)$ will have poles at values of z where an internal propagator goes on shell.

$$\frac{1}{P^2(z)} = \frac{1}{(p_k + \ldots + p_i(z) + \ldots + p_l)^2}$$

Can reconstruct amplitude from its residues - products of lower point amplitudes.

Maximal Supersymmetry

On-shell $\mathcal{N} = 4$ SYM is described by a PCT self-conjugate supermultiplet:

$$\Phi(\eta) = G^+ + \eta^A \Gamma_A + \tfrac{1}{2}\eta^A \eta^B S_{AB} + \tfrac{1}{3!}\eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D + \tfrac{1}{4!}(\eta)^4 G^-$$

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, \qquad q^{\alpha A} = \lambda^\alpha \eta^A, \qquad \bar{q}^{\dot{\alpha}}_A = \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \eta^A}.$$

Amplitudes:

$$\mathcal{A}(\Phi_1 \ldots \Phi_n) = (\eta_1)^4 (\eta_2)^4 \mathcal{A}(- - + + \ldots +) + \ldots$$

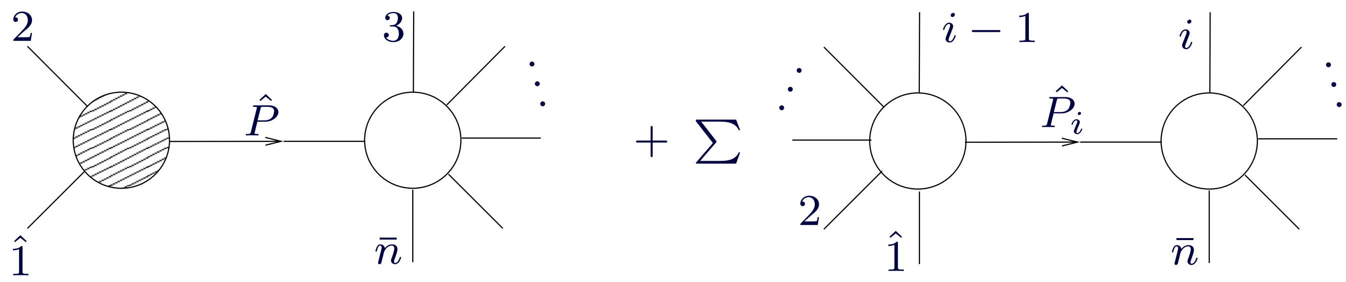
$$p\mathcal{A} = q\mathcal{A} = 0 \implies \mathcal{A}(\Phi_1, \ldots, \Phi_n) = \frac{\delta^4(p)\delta^8(q)}{\langle 12 \rangle \ldots \langle n1 \rangle} \mathcal{P}(\lambda, \tilde{\lambda}, \eta), \qquad \langle ij \rangle = \lambda_i^\alpha \lambda_{j\alpha}.$$

$$\mathcal{P} = \mathcal{P}^{\mathrm{MHV}} + \mathcal{P}^{\mathrm{NMHV}} + \ldots + \mathcal{P}^{\overline{\mathrm{MHV}}}.$$

Tree-level gluon amplitudes are identical in any gauge theory - e.g. QCD.

All tree-level amplitudes

Solve for all tree-level amplitudes. JMD, Henn



The full tree-level amplitude is fixed by its analytic structure.

NMHV example: JMD, Henn, Korchemsky Sokatchev

$$\mathcal{A}_n^{\mathrm{NMHV}} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n\,1\rangle} \sum_{2\leq s<t\leq n-1} R_{n;st}$$

$$R_{r;s,t} = \frac{\langle s\;\, s-1\rangle\langle t\;\, t-1\rangle\delta^4(\langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle)}{x_{st}^2\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rt}x_{ts}|s\rangle\langle r|x_{rt}x_{ts}|s-1\rangle}$$

$$p_i^{\alpha\dot\alpha} = \lambda_i^\alpha\tilde\lambda_i^{\dot\alpha} = x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha}, \qquad q_i^{\alpha A} = \lambda_i^\alpha\eta_i^A = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A}.$$

$R_{r;s,t}$ has non-local (spurious) poles - cancel in sum.

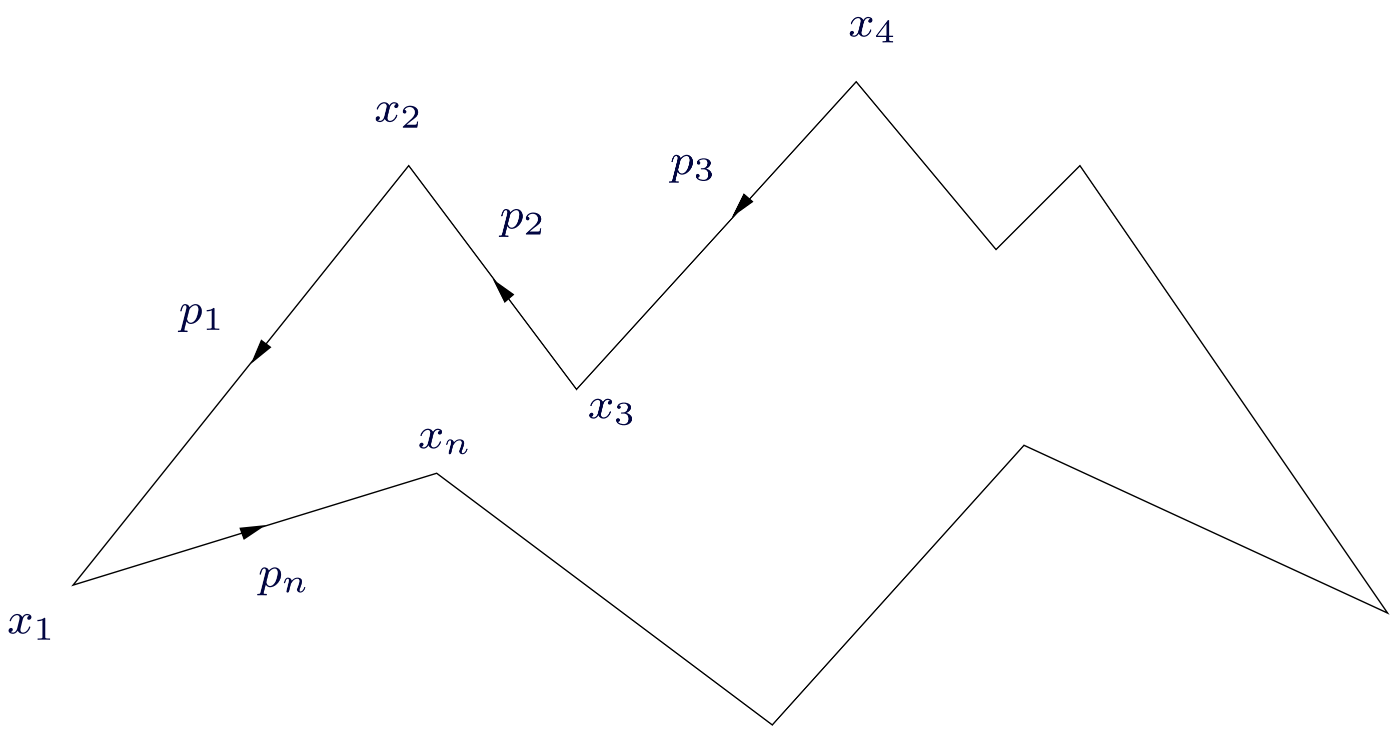
Dual Superconformal Symmetry

Solution for amplitudes also has a new symmetry. JMD, Henn, Korchemsky Sokatchev

The variables x and θ have a meaning as coordinates of a dual space,

The amplitudes are covariant under superconformal transformations of the dual variables x and θ.

Distinct from the original superconformal symmetry of the $\mathcal{N} = 4$ SYM Lagrangian.



Yangian symmetry JMD,Henn,Plefka

Think of dual superconformal symmetries as fundamental. JMD,Ferro

$$\mathcal{A}_n = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}\mathcal{P}_n, \qquad J_a \mathcal{P}_n = 0.$$

Momentum twistors $\mathcal{W}_i^{\mathcal{A}} = (\lambda_i^{\alpha} \quad \mu_i^{\dot\alpha} = x_i^{\alpha\dot\alpha}\lambda_{i\alpha} \quad | \quad \chi_i^A = \theta_i^{\alpha A}\lambda_{i\alpha})$: Hodges

$$J^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}$$

Original conformal symmetry $k_{\alpha\dot\alpha}\mathcal{A}_n = 0$ induces second order symmetry of $\mathcal{P}_n$,

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j}(-1)^{\mathcal{C}}\left[\mathcal{W}_i^{\mathcal{A}}\frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}}\mathcal{W}_j^{\mathcal{C}}\frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}} - (j,i)\right].$$

so we have

$$J_a \mathcal{P}_n = 0 \qquad J_a^{(1)}\mathcal{P}_n = 0.$$

Simplest invariant: Mason, Skinner

$$[a,b,c,d,e] = \frac{\delta^4(\chi_a(bcde) + \text{cyclic})}{(abcd)(bcde)(cdea)(deab)(eabc)} \qquad (abcd) = W_a^A W_b^B W_c^C W_d^D \epsilon_{ABCD}\,.$$

Beyond tree-level: amplitudes and Wilson loops

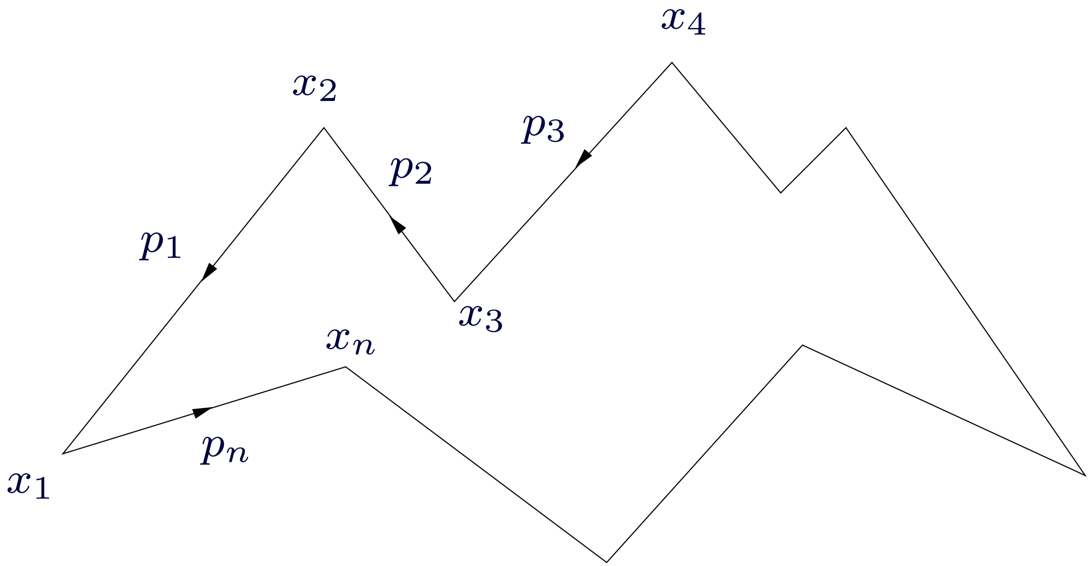
Beyond tree-level, amplitudes are infrared divergent.

$$\mathcal{A}_n^{\rm MHV} = \mathcal{A}_n^{\rm MHV tree} \exp\Big[{\rm IR\,div} + F_n^{\rm MHV}(p_1,\ldots,p_n) \Big].$$

For MHV: amplitude/Wilson loop duality - contour is the lightlike polygon.

Alday,Maldacena,
JMD,Korchemsky,Sokatchev,
Brandhuber,Heslop,Travaglini,
JMD,Henn,Korchemsky,Sokatchev.

$$W_n = \exp\Big[{\rm UV\,div} + F_n^{\rm WL}(x_1,\ldots,x_n) \Big].$$



Beyond MHV: appropriate supersymmetrisation.

Mason,Skinner,
Caron-Huot.

Wilson loops : conformal symmetry

Conformal Ward identity for WL $\Longrightarrow$ dual conformal symmetry for MHV amplitude.

JMD, Henn, Korchemsky Sokatchev

$$K^\mu F_n = \Gamma_{\rm cusp}(a) \sum_i (2x_i^\mu - x_{i-1}^\mu - x_{i+1}^\mu) \log x_{i-1,i+1}^2$$

Solution unique when there are no conformal invariants.

None for $n = 4, 5 \Longrightarrow$ unique solution: e.g. $F_4^{\rm MHV} = \Gamma_{\rm cusp}(a) \log^2(s/t)$

For six points and beyond there are invariants,

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad \Longrightarrow \qquad F_n = F_n^{\rm BDS} + R_n(u_1, \ldots u_m)\,.$$

Remainder function R_n finite and dual conformally invariant.

Beyond MHV:

$$\mathcal{A} = \mathcal{A}_{\rm tree}^{\rm MHV} \exp\Big[[\text{IR div}] + F^{\rm MHV}\Big]\mathcal{P}$$

Ratio function $\mathcal{P}$ IR finite and dual conformally invariant.

Ratio function

NMHV, 6 points:

$$[1,2,3,4,5] = (6)\ldots$$

$$\begin{aligned}\mathcal{P} = \quad & [(1)+(4)]V_1 + [(2)+(5)]V_2 + [(3)+(6)]V_3 \\ & +[(1)-(4)]\tilde{V}_1 - [(2)-(3)]\tilde{V}_2 + [(3)-(6)]\tilde{V}_3\,.\end{aligned}$$

$$\begin{aligned}V_1^{(1)} &= \frac{1}{2}\Big[-\log u \log w \log(uw) \log v + \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) - 2\zeta_2\Big]\,. \\ \tilde{V}_1^{(1)} &= 0\,.\end{aligned}$$

Dual superconformal symmetry:

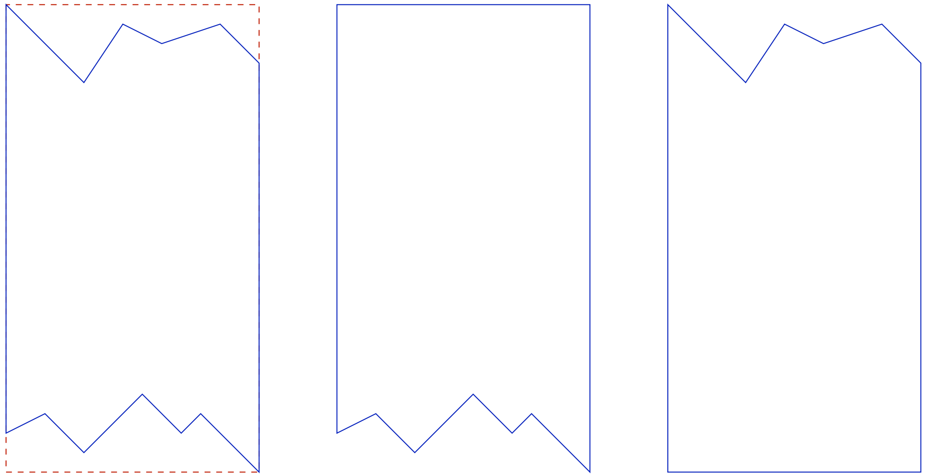
Rational functions $(1), (2), \ldots$ dual superconformal invariant.

Transcendental functions $V_1, \ldots$ only dual conformal invariant (like remainder function for MHV).

Non-zero variations $\bar{Q} e^R \mathcal{P}$ understood in terms of higher-point amplitudes. [Caron-Huot,He], [Bullimore,Skinner]

Analytic structure and bootstraps

Pick two sides of polygon and form a light-like square:



Can think of Wilson loop as flux tube propagating from bottom to top.

Perform an operator product expansion at bottom and top. Alday, Gaiotto, Maldacena, Sever, Vieira

Exchanged states are flux lines with excitations.

Can use this to predict discontinuities of Wilson loops.

Three-loop prediction for remainder function, two-loop ratio function. Dixon, JMD, Henn

Symbols

Technology from theory of iterated integrals very useful.

Consider ‘pure’ functions (iterated integrals, multi-dimensional (Goncharov) polylogarithms):

$$df^{(k)} = \sum_r f_r^{(k-1)} d\log\phi_r$$

Sum over r finite, ϕ_r algebraic functions. $f^{(1)} = \log\phi$.

Symbol:

$$S(f^{(k)}) = \sum_r S(f_r^{(k-1)}) \otimes \log\phi_r$$

Very useful in dramatic simplification of two-loop remainder function. Goncharov, Spradlin, Vergu, Volovich

Also in bootstrap procedure (for two-loop ratio function can even promote symbol to function).

Arises from Hopf algebra structure of iterated integrals.

Summary and Outlook

Symmetries of S-matrix in field theory can be much larger than Poincaré $\times$ Internal.

For tree-level amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory the symmetry is $Y(psl(4|4))$.

[Invariants of the symmetry $\longleftrightarrow$ leading singularities - Grassmannian integral - loop integrands]

Amplitudes can be represented as Wilson loops.

[At strong coupling - the integrability of the sigma model can also be used to calculate the Wilson loop (minimal surface).]

How to interpolate between weak and strong coupling? What is the best theory of the S-matrix?

What is the role of Hopf algebra structures of iterated integral functions?