Dynamical reflection algebras: examples from Calogero-Moser models

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Sponsored in part by ANR Project DIADEMS (ANR SIMI 1 2010-BLAN-0120-02)

References:

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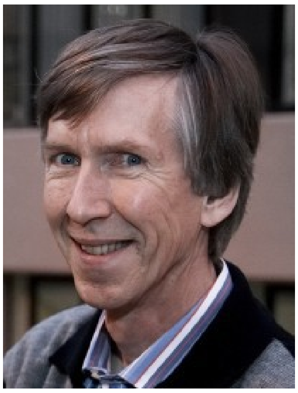
Zoltan Nagy, Jean Avan, Genevieve Rollet(Cergy-Pontoise U., LPTM)
Lett.Math.Phys. **67**, 1-11, 2004

I -1 INTRODUCTION: THE CALOGERO-MOSER MODEL(S)


F. Calogero


S.N.M. Ruijsenaars


J. Moser

A: n-body dynamical system, non-relativistic, with 2-body potential:

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j} v(q_i - q_j),$$

v (x) being *1/ (sn(x))²* or its limits : *1/ sin²(x)* and *1/x²*.

→ Initially built (Calogero '70) as nucleon-nucleon potential.
→ Also (Jevicki et al.) dynamics of collective variables in matrix models (discretization of strings)
→ Also (Airault-McKean-Moser) dynamics of Korteweg-de Vries solitons (shallow-water wave dynamics in 1+1 dimension)

Exist → external field generalizations (Inozemtsev; e.g. add one-body potential $w(q) = q^2$ to rational CM)

→ extensions with $q_i + q_k$ dependance (Olshanetski-Perelomov) ....

→ spin-dependent generalizations (Gibbons-Hermsen) : interaction $S_i S_j\, v(q_i - q_k))$

Remark: Quantum CM models → special polynomials as eigenfunctions (Jack polynomials).

B: Relativistic generalization : the Ruijsenaar-Schneider model

Relativistic version of the CM model with 2-body potentials combined as:

$$H = \sum_i (\exp (\beta\, p_i)\prod_{k\neq i} f(q_i - q_k))$$

where $f^2 (x) = 1 - g^2/x^2 \ldots$ with trigonometric and elliptic generalizations.

→ Dynamics of solitons for sine Gordon model (relativistic version of KdV) (Ruijsenaar-Schneider; Babelon-Bernard)

→ Possible connection to dynamics of Anti-de Sitter space configurations (``giant magnons") (Aniceto-Jevicki).

Both models and their generalizations : classically Liouville-integrable, with underlying structures of "dynamical reflection algebras" . What are they ?

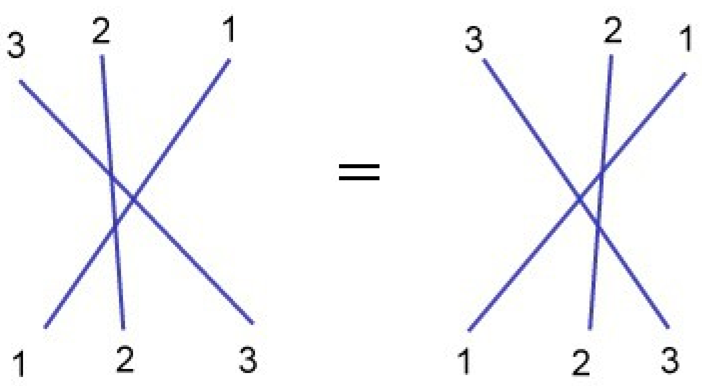
I -2 GENERAL FEATURES: QUANTUM YANG BAXTER AND QUANTUM REFLECTION EQUATIONS.

A: **Bulk factorizable amplitudes**

1+1 dimensional quantum integrable theories = infinite number of conserved quantities

Hence : factorization of amplitudes : $n \to n$ as combination of $2 \to 2$

Implies consistency conditions for $3 \to 3$ amplitudes (sufficient for $n \to n$):



$$R_{12}(u)\ R_{13}(u+v)\ R_{23}(v) = R_{23}(v)\ R_{13}(u+v)\ R_{12}(u),$$

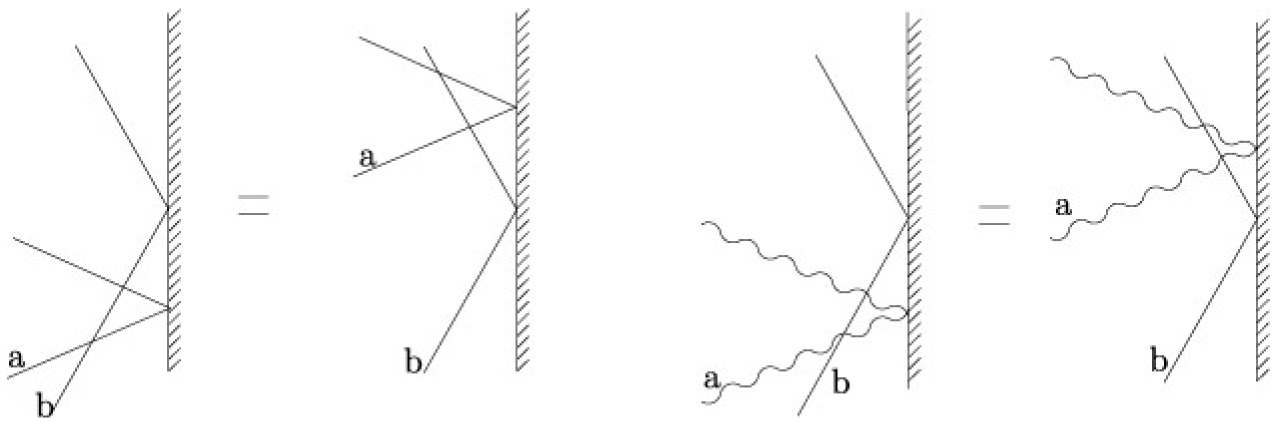
Also obtained as consistency condition for associativity of quantum group structure:

$$R_{23}(z-w)T_{12}(z)T_{13}(w) = T_{13}(w)T_{12}(z)R_{23}(z-w).$$

Considere here only bulk amplitudes (infinite space-time, no boundaries)

B: If boundaries occur:

factorizability requires extra condition including reflection matrix K on boundary:



simplest one: $R_{12}K_1R_{21}K_2 = K_2R_{12}K_1R_{21}$ (Cherednik, Sklyanin)

more general: $RKRK' = RK'RK$.

Generalization as quadratic (braid) algebra (Maillet-Freidel):

$$A_{12}K_1B_{21}K_2 = K_2C_{12}K_1D_{12}$$

with consistency conditions :

unitarity: $C_{12} = B_{21}$, $A_{12}A_{21} = 1\otimes 1$, $D_{12}D_{21} = 1\otimes 1$

associativity:

→ Yang Baxter equations for *A* and *D*;
→ Adjoint Yang Baxter equations for *A/C* and *D/B*, with form $ACC = CCA$ and $DBB = BBD$

I-3 A DYNAMICAL DEFORMATION OF YANG BAXTER

From:
→ quantum exchange relation of operators in Liouville theory (Gervais Neveu)
→ quantum Knizhnik Zamolodchikov equation in CFT (Felder)

Consider more general cubic equation (Gervais-Neveu-Felder equation) for exchange matrix:

$$D_{12}(\lambda+\gamma h_3)D_{13}D_{23}(\lambda+\gamma h_1)=D_{23}D_{13}(\lambda+\gamma h_2)D_{12} \qquad \textbf{(8)}$$

where $\{\lambda_i\}$ = coordinates on dual $\mathfrak{h}^*$ of (Cartan) abelian subalgebra $\mathfrak{h}$ in underlying Lie algebra (on example of Calogero-Moser model λ_i are position variables => « dynamical ») ; h_a = some suitable representation of $\mathfrak{h}$.

= abelian deformation of Yang Baxter equation.

→ Also holds for Boltzmann weight matrix of particular models in Statistical Mechanics (Interaction Round a Face or IRF).

→ Also occurs as consistency condition for associativity of *dynamical* deformation of quantum group structure :

$D_{12}(\lambda+h_q)T_1\,T_2(\lambda+h_1)=T_2\,T_1(\lambda+h_2)D_{12}$

Natural question: how to dynamically deform $R_{12}K_1R_{21}K_2=K_2R_{12}K_1R_{21}$? more precisely:

Introduce extra parameter λ in R and T such that R obey GNF equation ?

More generally how to deform similarly $A_{12}K_1B_{21}K_2=K_2C_{12}K_1D_{21}$?

II DYNAMICAL DEFORMATIONS OF REFLECTION ALGEBRAS

3 possibilities known at this time; general structure is :

$A_{12}(\lambda)\,K_1(\lambda+e_R h_2)\,B_{21}(\lambda)\;K_2(\lambda+e_L h_1)=K_2(\lambda+e_R h_1)\;C_{12}(\lambda)\,K_1(\lambda+e_L h_2)D_{21}(\lambda)$

plus zero-weight conditions :

$e_R\,[h_1+h_2,\,A_{12}]=e_L[h_1+h_2,\,D_{12}]=0;\quad [e_R h_1+e_L h_2,C_{12}]=[e_L h_1+e_R h_2,B_{12}]=0$

IIA: ``Dynamical boundary algebra'' : $e_R=e_L=+1$ up to scale γ .

First identified in IRF models with boundaries (Behrend-Pearce-O'Brien)
Studied extensively by Fan-Hou- Li-Shi and later by Nagy-Avan-Rollet

IIB: ``Semi-dynamical boundary algebra'': $e_R=0,\;e_L=1$ or $e_R=1,\;e_L=0$

First identified in quantum Ruijsenaar-Schneider model (see later) by Arutyunov-Chekov-Frolov
Extensive studies by Nagy-Avan-Rollet, Avan-Zambon, Avan-Rollet.

IIC: **``Second dynamical boundary algebra''** **:** $e_R = -1$, $e_L = 1$ or $e_R = 1$, $e_L = -1$

Identified (classical limit) in second Poisson structure of Calogero-Moser model (Avan-Ragoucy)
Studied extensively by Avan-Ragoucy.

IID: Associated Dynamical Yang Baxter equations:

$A_{12}(\lambda)A_{13}(\lambda+e_R h_2)A_{23}(\lambda) = A_{23}(\lambda+e_R h_1)A_{13}(\lambda)A_{12}(\lambda+e_R h_2)$ *(Dynamical YB eqn)*
$D_{12}(\lambda+e_L h_3)D_{13}(\lambda)D_{23}(\lambda+e_L h_1) = D_{23}(\lambda)D_{13}(\lambda+e_L h_2)D_{12}(\lambda)$ *(dual DYBE)*
$A_{12}(\lambda)C_{13}(\lambda+e_R h_2)C_{23}(\lambda) = C_{23}(\lambda+e_R h_1)C_{13}(\lambda)A_{12}(\lambda+e_L h_2)$ *(adjoint DYBE)*
$D_{12}(\lambda+e_R h_3)B_{13}(\lambda)B_{23}(\lambda+e_L h_1) = B_{23}(\lambda)B_{13}(\lambda+e_L h_2)D_{12}(\lambda)$ *(dual adjoint DYBE)*

III CONNECTIONS TO CALOGERO MOSER MODELS

IIIA: What is a classical r-matrix ?

2*n*-dimensional classical integrable system (canonical variables $\{p,q\}$) => *n* Poisson-commuting independent dynamical quantities including initial Hamiltonian.

Characterized by:

1) Lax representation $dL/dt = [L,M]$ where $L = L(p,q) =$ Lax matrix,
Lie-algebra (*G*) valued; $M = M(p,q)$ Lie algebra-valued.

2) Poisson structure of Lax matrix elements encapsulated into algebraic, *r*-matrix structure:

$\{L_1, L_2\} = [r_{12}, L_1] + [r_{21}, L_2]$ <=> conserved quantities $Tr\ L$^n Poisson commute.

r in *GxG* , depends on dynamical variables; Jacobi identity on Poisson bracket realized if
r obeys classical Yang Baxter equation. In general:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{r_{12}, L_3\} + \{r_{13}, L_2\} = 0$$

Complicated, semi-implicit non-algebraic equation.
Better understood if existence of algebraic form for $\{r_{12}, L_3\}$

examples: $\{r_{12}, L_3\} = (h_3 . d/dq)\, r_{12}$; $\{r_{12}, L_3\} = (e_R h_3 L_3 + e_L L_3 h_3) . d/dq\ r_{12}$

In particular when

→ degree zero expression available: $\{r_{12}, L_3\} = (h_3 . d/dq)\, r_{12}$
→ plus possibility of additional decomposition into equations for *d* and *s:*

=> dynamical cYB (Feher 1990 from WZNW models):

$$[d_{12}, d_{13}] + [d_{12}, d_{23}] + [d_{13}, d_{23}] + (h_1 . d/dq)\, d_{23} - (h_2 . d/dq)\, d_{13} + (h_3 . d/dq)\, d_{12} = 0$$

plus adjoint form for *s* and some other conditions … yields cYB for *d*+*s*.

= SEMI-CLASSICAL LIMIT OF GNF EQUATION ($D = 1 \otimes 1 + \hbar\, d + o(\hbar^2)$, $h \rightarrow \hbar h$: order 2 of expansion in powers of $\hbar$ is classical GNF equation).

Remark: if *r*-matrix has no dynamical dependance plus skew-symmetry: gets classical standard YB equation (classification by Belavin-Drinfel'd)

$[d_{12}, d_{13}] + [d_{12}, d_{23}] + [d_{13}, d_{23}] = 0$

IIIB Recall: The Calogero-Moser model is integrable

n-body dynamical system, non-relativistic, with 2-body potential:

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j} v(q_i - q_j),$$

$v(x)$ being $1/(sn(x))^2$ or its limits : $1/\sin^2(x)$ and $1/x^2$.

Lax matrix is: $(v = \frac{1}{2} u^2)$

$$L_{jk} = p_j\, \delta_{jk} + \sqrt{-1}(1 - \delta_{jk})u(q_j - q_k),$$

r- matrix (for canonical structure $\{p_i, q_j\} = \delta_{ij}$ and rational potential $v(x) = 1/x^2$)

$r = \sum 1/(q_i - q_j)\,(e_{ij} \otimes e_{ji} + e_{ii} \otimes (e_{ij} - e_{ji}))$

IIIC Connection to semi-dynamical reflection algebra: $e_R = 0; e_L = 1$

$r = a - c = d - b$ with zero-weight conditions on *a,b,c,d* corresponding to semidynamical reflection algebra (Arutyunov-Chekov-Frolov): *d* skew-symmetric with sole elements $e_{ij} \otimes e_{ji}$; *b,c* semidiagonal ; *a* ``full" with both types of components.

→ *a* obeys non-dynamical cYB,
→ *d* obeys dynamical cYB a la Feher,
→ *b* and *c* obey semi-classical limit of semidynamical adjoint YB equations.

CLASSICAL 1ST POISSON STRUCTURE OF LAX CALOGERO-MOSER MATRIX
=
LINEAR CLASSICAL LIMIT OF SEMIDYNAMICAL REFLECTION ALGEBRA

IIID Second Poisson bracket of Calogero-Moser

1: What is ``second Poisson bracket'' ?

From works of Magri et al.:

classical integrability <=> hierarchy of Poisson-commuting Hamiltonians under one PB structure
OR
classical integrability <=> hierarchy of compatible Poisson structures for one Hamiltonian.

Hierarchies connected by dual time evolution : $\{H_n, X\}_m = \{H_m, X\}_n$

For skew-symmetric, non-dynamical *r*-matrix: easy formulation (Sklyanin; Li-Parmentier)

→ First Poisson bracket:

$\{L_1, L_2\} = [r_{12}, L_1] + [r_{21}, L_2]$

→ Second Poisson bracket:

$\{L_1, L_2\} = [r_{12}, L_1 L_2]$ generalized to non-skew-symmetric non-dynamical case
as $\{L_1, L_2\} = a_{12} L_1 L_2 + L_1 s_{12} L_2 + L_2 s_{12} L_1 + L_1 L_2 a_{12}$ (Li-Parmentier, see also Maillet-Freidel)

HERE: DYNAMICAL r-MATRIX, SKLYANIN FORMULATION UNAPPLICABLE

Second Poisson bracket developed for rational CM model by Magri; Bartocchi et al; Continuous limit by Aniceto et al: relevant in aspects of string theory and CFT.

Technically difficult to formulate in terms of (first) canonical variables, easy in terms of Lax observables *Tr L^n, Tr Q L^n*, Q = diag $(q_1, \dots q_n)$. Explicit form now available for 2 sites (Bartocchi et al.) and 3 sites (Avan-Ragoucy).

2: Connection to second DBA (Avan-Ragoucy)

2-site Lax matrix of rational CM model, with second Poisson bracket. PB structure reads:

$\{L_1, L_2\} = a_{12} L_1 L_2 + L_1 b_{12} L_2 + L_2 c_{12} L_1 + L_1 L_2 d_{12}$

where a,b,c,d obey semi-classical limit of 2nd dynamical YB equation.

CLASSICAL 2nd POISSON STRUCTURE OF LAX CALOGERO-MOSER MATRIX

=

QUADRATIC CLASSICAL LIMIT OF 2nd DYNAMICAL REFLECTION ALGEBRA

NOT TRUE FOR *n*>2 SITES: a,b,c,d matrices are not known but necessarily p,q dependent due to form of PB's .

3: Remark: Ruijsenaar-Schneider model

The Lax matrix of RS endowed with the canonical (first) Poisson structure $\{p_i , q_j\} = \delta_{ij}$
has quadratic r-matrix structure :

$$\{L_1 , L_2\} = a_{12} L_1 L_2 + L_1 b_{12} L_2 + L_2 c_{12} L_1 + L_1 L_2 d_{12}$$

BUT with a,b,c,d parametrizing the **first** Poisson structure of Calogero-Moser: = **SDRA**

CLASSICAL 1ST POISSON STRUCTURE OF LAX RUIJSENAAR-SCHNEIDER MATRIX

=

QUADRATIC CLASSICAL LIMIT OF SEMIDYNAMICAL REFLECTION ALGEBRA