First Steps towards the understanding of the Tsallis-Pareto Distributions in Heavy-Ion Collisons

G.G. Barnaföldi & T.S. Biró & K. Ürmössy & P. Ván

Wigner RCP of the Hungarian Academy of Sciences

Eur. Phys. J. A49 (2013) 110, Physica A 392 (2013) 3132

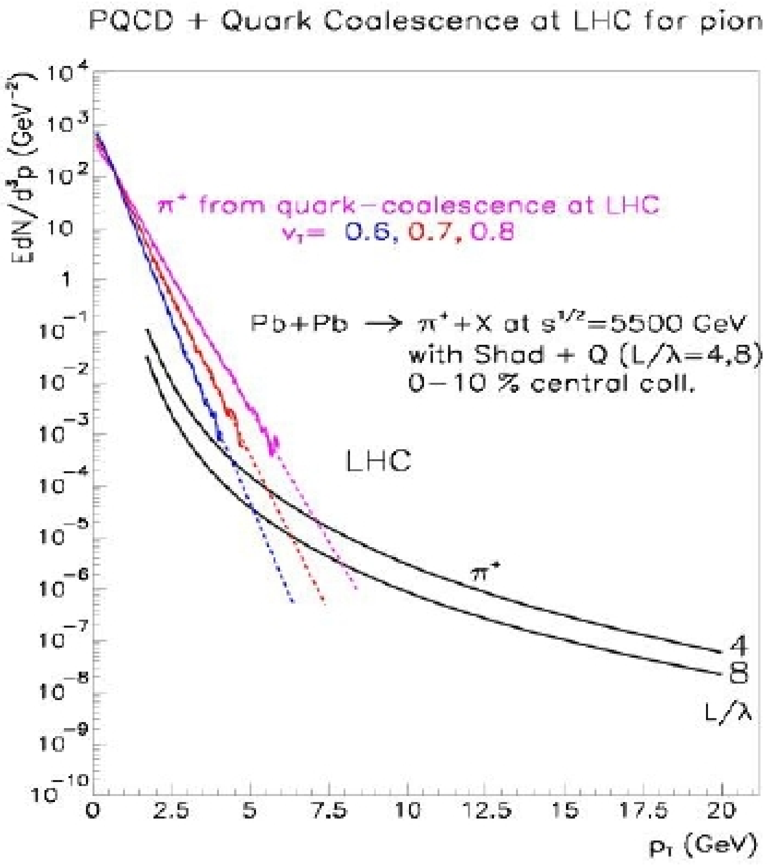
9th International Workshop on High-pT Physics at LHC,
LPSC, Grenoble, France, 24th-28th September 2013

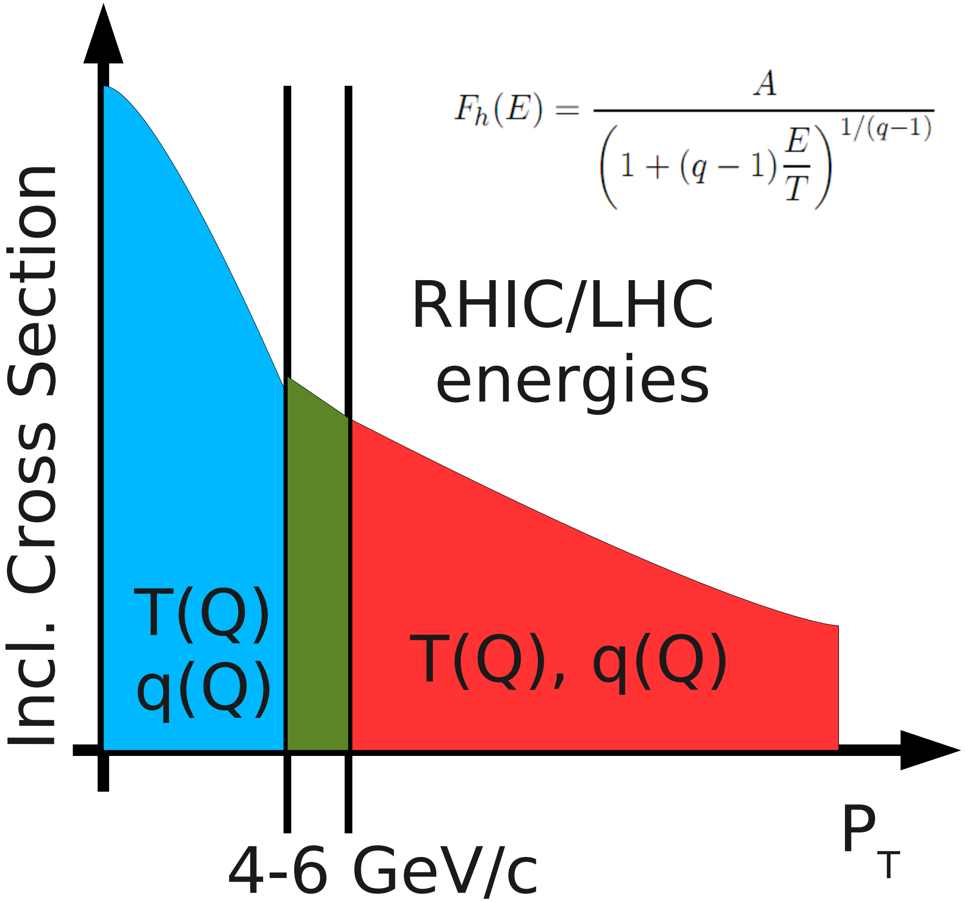
O U T L I N E

- 1/3 Motivation...
  - This is a talk on high-pT, BUT without QCD now.
  - But we might understand an OLD experimental parameter, T with the recent physical knowledge.
- 1/3 Derivation...
  - of the Tsallis/Rényi entropies from the first principles
  - Providing phyiscal meaning of the 'mysterious q'
- 1/3 Application
  - for Bag model
  - QGP temperature

M O T I V A T I O N

- Hadron spectra low-p_T vs. high-p_T

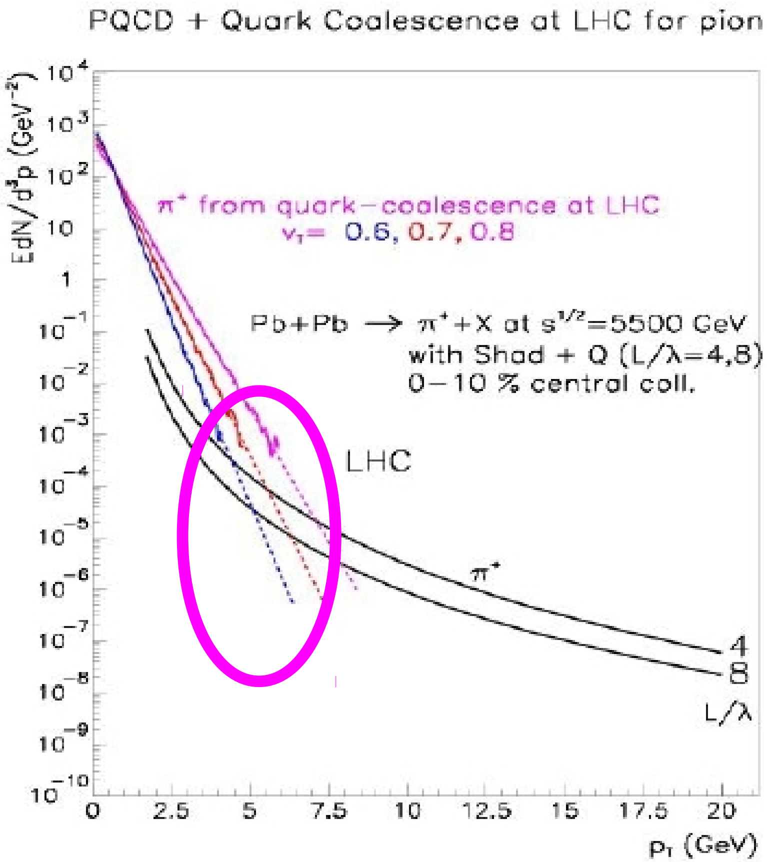


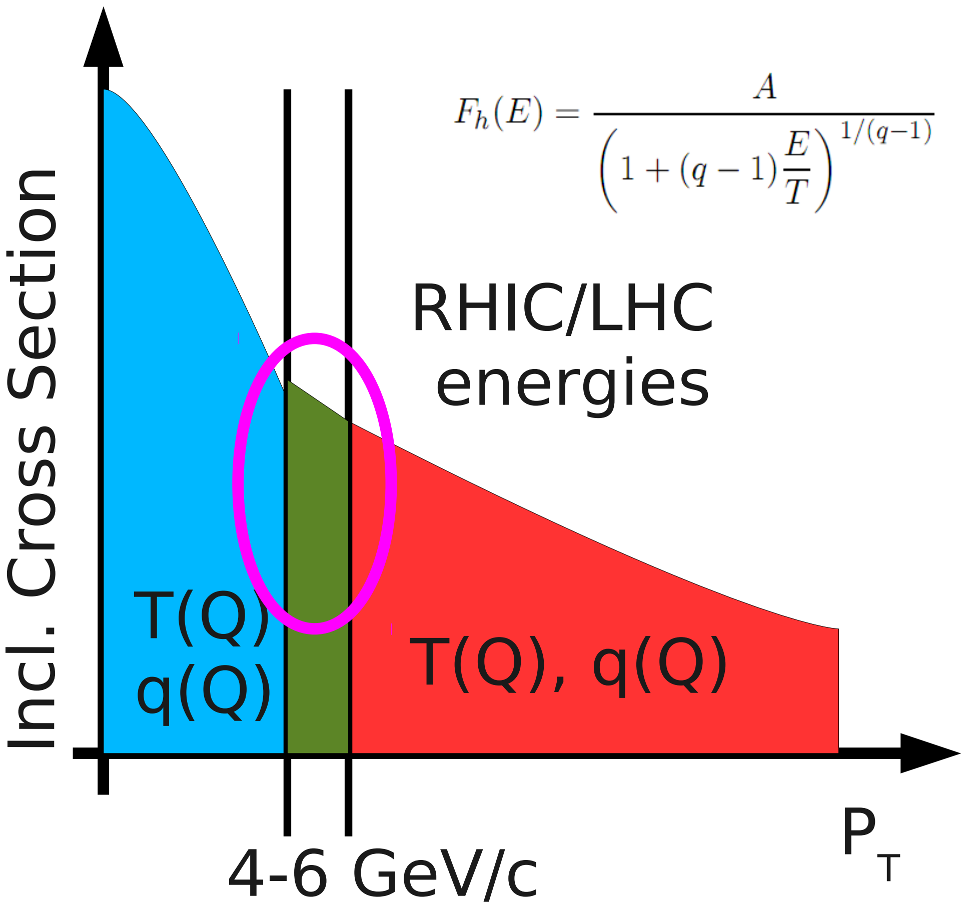


P. Lévai, GGB, G. Fai: JPG35, 104111 (2008)

MOTIVATION

- Hadron spectra low-p_T vs. high-p_T





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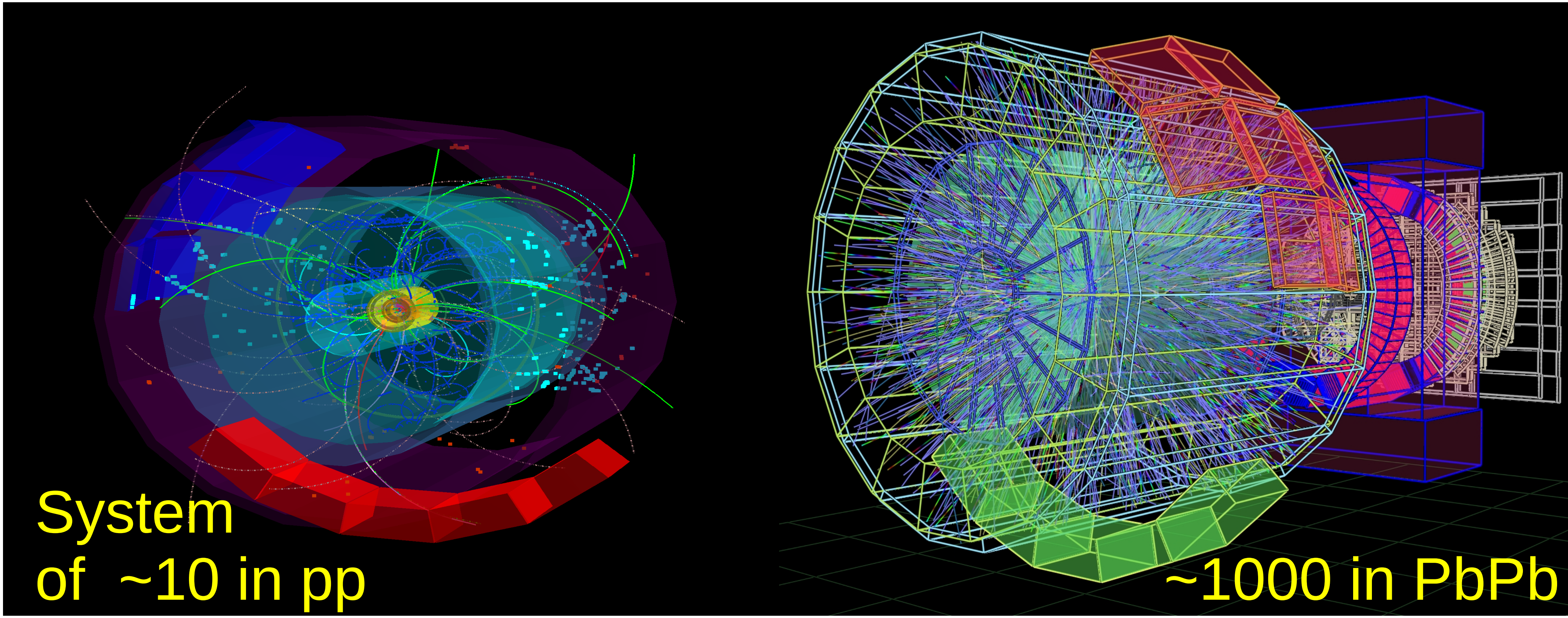
MOTIVATION

Formulated questions from the theory...

- What is responsible for the power low tail measured at high-p_T?
- Can we assume thermodynamical equilibrium for high-p_T particles?
- What is the origin of the ’collectivity’? Is it coming from ’quark level’ or ’hadron level’?
- Is there difference between baryon and meson formation? What is the statistical origin of this (e.g coalescence, fragmentation, etc.)?

The VHMPID LoI (2013) arXiv:1309.5880

De-M O T I V A T I O N



How can we measure the temperature of what?

This is not a system of 10^{23} particles, but 10^{3}

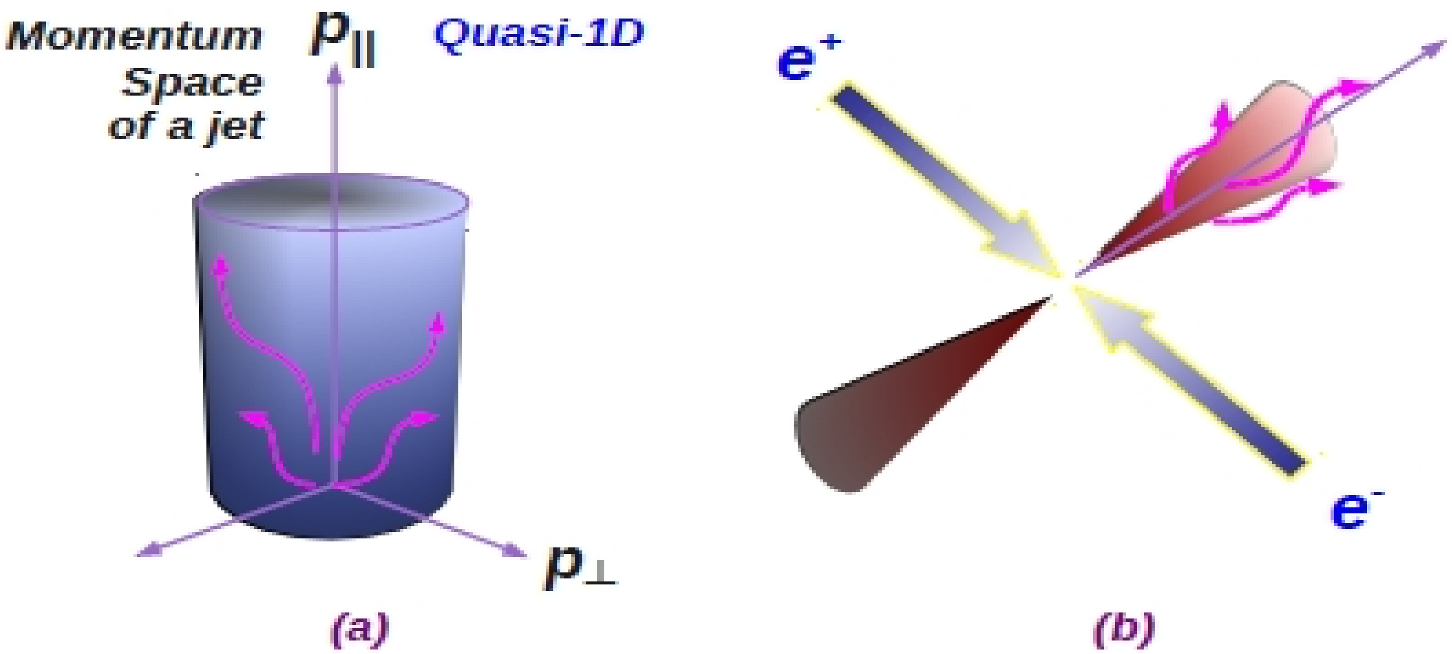
Can we handle this???

Can we handle this???

Can we handle this???

… many-many times

The 'Thermodynamics of Jets'



K. Ürmössy, G.G. Barnaföldi, T.S. Bíró:

- Microcanonical Jet-Fragmentation in pp at LHC energies:
  Phys. Lett. B701 (2011) 111
- Generalized Tsallis distribution in e^+e^- collisons
  Phys. Lett. B718 (2012) 125

ee: Basic model assumptions for e^+e^-

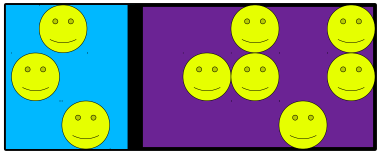
In case of a high-energy collisions (at high-p_T) we can expect:

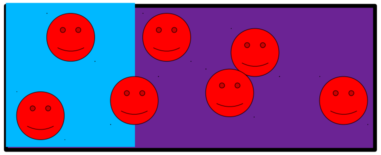
- Consider jets: narrow objects
- Narrow momentum distribution

} 1-dimensional object

- Events at $O(10^6)$ → Statistics
- At $\sqrt{s} \geq M_Z$ and ~90% of the events are 2-jet-events: $\sqrt{s}/2 = E$

- Energy-momentum conservation with $m_j = 0$ thus: $\epsilon_j = |\vec{p}|$
  - Micro-canonical: $\sum_j \epsilon_j = E$ (E conserv.)
  - Canonical case: $\sum_j \langle \epsilon_j \rangle = E$





Ref: K Ürmössy, GGB, TS Biró, PLB 710 (2011) 111, PLB 718 (2012) 125.

ee: Canonical & microcanonical ensambles

Canonical case for TP:

- One-particle distribution (with fix multiplicity *N*): $f_N(\epsilon) = A_c\, e^{-\beta_N \epsilon}$
- Gamma distribution for multiplicity: $p(N) = A_m\, N^{\alpha-1} e^{-\beta N},$

→ Momentum distribution (CTP): $\frac{\mathrm{d}\,\sigma}{\mathrm{d}^{\,D} p} = \sum p(N)\, N\, f_N(\epsilon) \approx \frac{\kappa_{D,E}}{\left(1 + \frac{D}{\beta} x\right)^{\alpha+D+1}}$

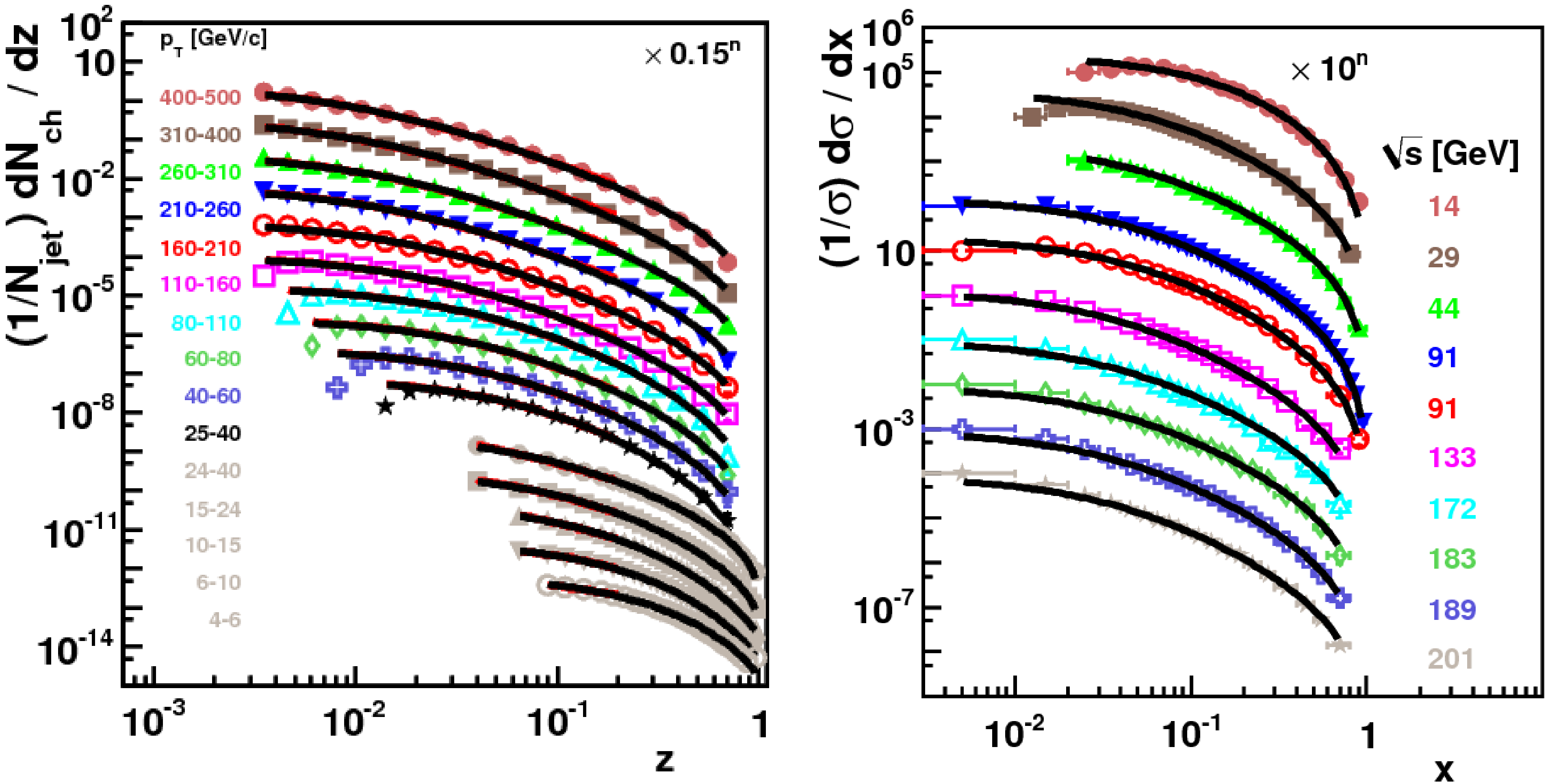
Microcanonical generalization of TP

- One-particle distrib. (with fix multiplicity *N*): $f_N(\epsilon) = A_{mc}\,(1-x)^{D(N-1)-1}$
- Shifted Gamma distribution for multiplicity (no to violate the KNO scaling, $N_0 = 1 + 2/D$): $p(N) = A_m\,(N - N_0)^{\alpha-1} e^{-\beta(N-N_0)}$

→ Momentum distribution (μCTP): $\frac{d\sigma}{d^D p} \propto \frac{1-x}{\left(1 - \frac{D}{\beta}\ln(1-x)\right)^{\alpha+D+1}}$

Ref: K Ürmössy, GGB, TS Biró, PLB 710 (2011) 111, PLB 718 (2012) 125.

Fits for jet spectra in pp (left) and e^+e^- (right)



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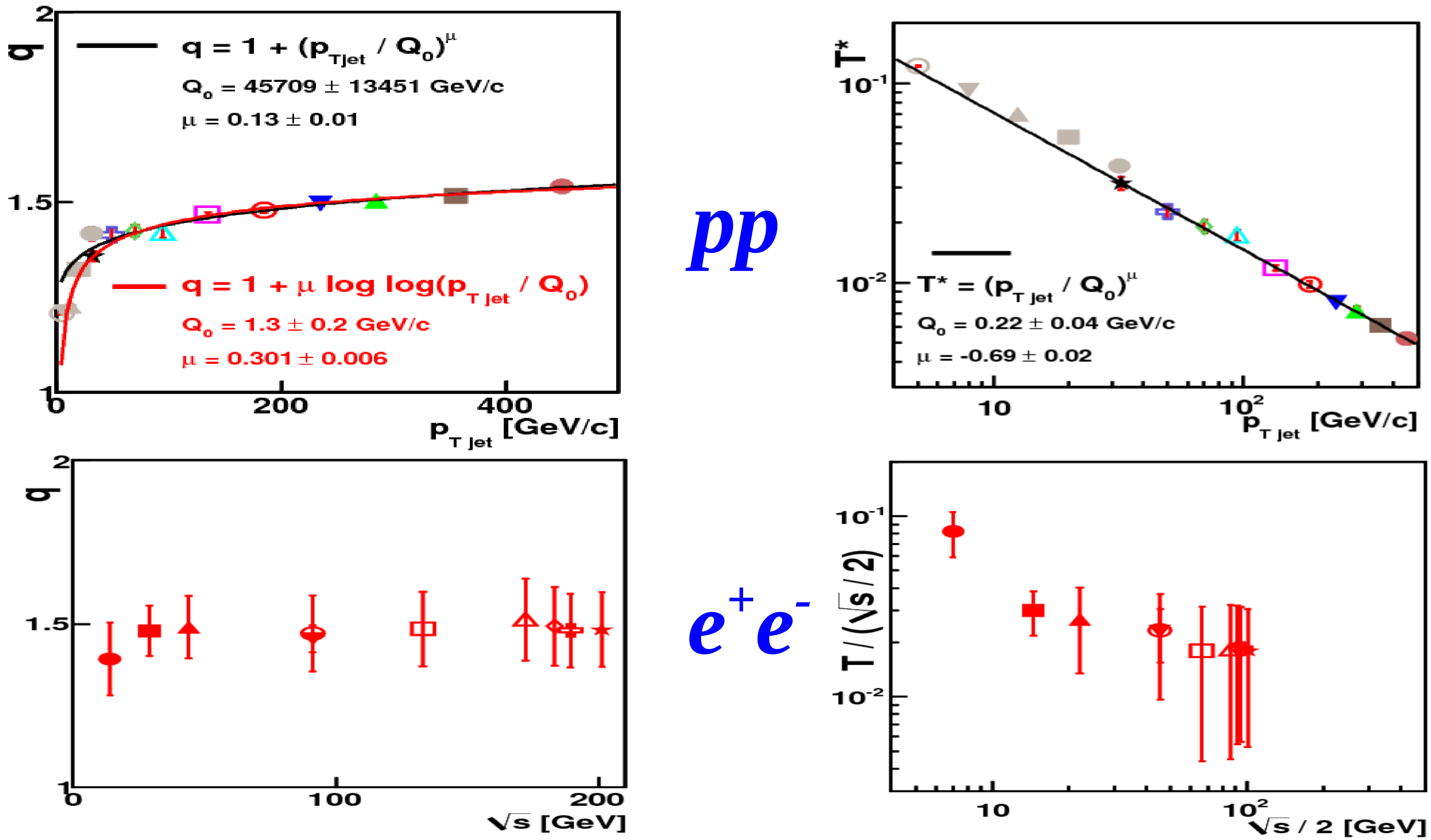
The evolution of q and T parameters



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Why to use Tsallis/Rényi entropy formula?

- It generalizes the Boltzmann-Gibbs-Shanon formula.
- It treats statistical entanglement between subsystem and reservoir (due to conservation).
- It claims to be universal: applicable for whatever material quantity of the reservoir.
- It leads to a cut power law energy distribution in the canonical treatment.

Why NOT to use Tsallis/Rényi formulas?

- They lack 300 years of classical thermo-dynamic foundation
- Tsallis is NOT additive, Rényi is NOT linear
- There is an extra parameter: the mysterious q
- How do different q systems equilibrated?
- Why this and not other?
- It looks pretty formal....

So here is some input to get rid of bad feelings...

The derivation of Tsallis/Rényi Entropy
and
the Physical Meaning of the q

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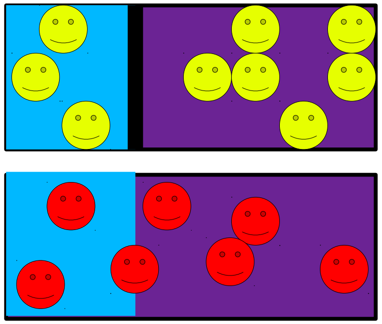
General derivation as inproved canonical

The story is about...

- Two body thermodynamics: 1 subsystem *(E_1)* + reservoir *(E-E_1)*
- Finite system, finite energy → microcanonical description
  - microcanonical $\sum_j \epsilon_j = E$
  - canonical $\sum_j < \epsilon_j > = E$



- Maximize a monotonic function of the Boltzmann-Gibbs entropy, *L(S)* (0^{th} law of thermodynamics)
- Taylor expansion of the *L(S)* = max, principle beyond *-βE*

Description of a system & reservoir

- For generalized entropy function $L(S_{12}) = L(S_1) + L(S_2)$
- In order to exist β of the system $L\left(S(E_1)\right) + L\left(S(E - E_1)\right) = \max$

  TS Biró P. Ván: Phys Rev. E84 19902 (2011)

- Thermal contact between system *(E_1)* & reservoir *(E-E_1)*, requires to eliminate E_1 :
$$\begin{aligned}\beta_1 &= L'\left(S(E_1)\right) \cdot S'(E_1) \\ &= L'\left(S(E - E_1)\right) \cdot S'(E - E_1)\end{aligned}$$

- This is usually handled in canonical limit, but now, we keep higher orders in the Taylor-expansion in E_1/E

$$\beta_1 = L'\left(S(E)\right) \cdot S'(E) - \left[S'(E)^2 L''(S(E)) + S''(E) L'(S(E))\right] E_1 + \ldots$$

Description of a system & reservoir

- Assuming $\beta_1=\beta$, the Lagrange multiplier become familiar for us:
$$\beta = L'\left(S(E)\right) \cdot S'(E) = L'(S) \cdot \frac{1}{T}$$
- To satisfy this, simply solve
$$\frac{L''(S)}{L'(S)} = -\frac{S''(E)}{S'(E)^2}$$
- Universal Thermostat Independence (UTI) Principle: l.h.s. must be as an *S*-independent constant for solving *L(S)*,
$$\frac{L''(S)}{L'(S)} = a$$
- Based on *L(S)* →*S* for small *S*, coming from 3rd law of the thermodynamics *L'(0)=1* and *L(0)=0*
$$L(S) = \frac{e^{aS}-1}{a}$$
- EoS derivatives do have physical meaning:
$$S'(E) = 1/T$$
$$S''(E) = -1/CT^2$$

Description of a system & reservoir

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$$L(S) = \frac{e^{aS} - 1}{a}$$

- Non-additivity parameter is simply the heat capacity of the reservoir:

$$\boxed{a = 1/C}$$

From two system to many...

- Analogue to Gibbs ensamble generalize

$$S = -\sum_i P_i \ln P_i \quad \rightarrow \quad L(S) = \sum_i P_i L(-\ln P_i)$$

- The *L*-additive form of a generally non-additive entropy,

  given by $$L(S(E_1)) - \beta E_1 = \frac{1}{a}\left(\mathrm{e}^{aS(E_1)} - 1\right) - \beta E_1 = \max.$$

- Introducing $a = 1/C(E) \quad \rightarrow \quad L(S(E_1)) = L(-\ln P_1) = \frac{1}{a}\left(P_1^{-a} - 1\right)$

- we need to maximize: $\frac{1}{a}\sum_i \left(P_i^{1-a} - P_i\right) - \beta \sum_i P_i E_i - \alpha \sum_i P_i = \max.$

which, results Tsallis: $$S_{\mathrm{Tsallis}} := L(S) = \frac{1}{q-1}\sum_i \left(P_i - P_i^q\right)$$

and its inverse Rényi: $$S_{\mathrm{Rényi}} := S = \frac{1}{1-q}\ln \sum_i P_i^q$$

The temperature slope

- Taking P_i weights of system, E_i , results cut power law:

$$P_i = \left(Z^{1-q} + (1-q)\frac{\beta}{q} E_i \right)^{\frac{1}{q-1}} = \frac{1}{Z}\left(1 + \frac{Z^{-1/C}\, \mathrm{e}^{S/C}}{C-1}\frac{E_i}{T} \right)^{-C}$$

- Partition sum is related to Tsallis entropy, $L(S_1)$ and E_1

$$\ln_q Z := C\left(Z^{1/C} - 1 \right) = L\left(S_1\right) - \frac{1}{1-1/C}\beta E_1$$

- In $C \to \infty$ limit, the inverse log slope of the energy distribution:

$$T_{\rm slope}(E_i) = \left(-\frac{d}{dE_i}\ln P_i \right)^{-1} = T_0 + E_i/C,$$ with $$T_0 = T\mathrm{e}^{-S/C} Z^{1/C}(1-1/C)$$

Application: Quark Gluon Plasma temperature

Application:
Quark Gluon Plasma temperature
or
What is the meaning of T?

Experimental data fits by $T_{slope}(E)$

- Taking the $T_{slope}(E)$ fit using $$T_{\rm slope}(E_i) = \left(-\frac{d}{dE_i}\ln P_i\right)^{-1} = T_0 + E_i/C,$$
- Fitted data
  - RHIC@200GeV AuAu: $T_0 = 48$ *MeV*, $C=4.5$

    T.S. Biró, K. Ürmössy, Zs. Schram: JPG36 064044 (2009)
    T.S. Biró, K. Ürmössy: JPG37, 0940027 (2010),
    K. Ürmössy, T.S. Bíró: PL B689 14 (2010)
  - ALICE@900GeV pp: $T_0 = 55$ *MeV*, $C=8$

    J. Cleymans, D. Worku: JPG39, 025006 (2012)

- Findings: $K=2$ (mesons) and $K=3$ (baryons)

$$P_{\rm hadron}(E) = P_i^K(E/K) \quad \text{and} \quad T^{\rm hadron}_{\rm slope}(E) = T^{\rm quark}_{\rm slope}(E/K)$$

The obtained values are surprizingly low!!! Why?

Thermal model to heavy-ion collisions

- Test of T_0 in physical models, in a finite termostats, small subsystem: $\lim_{C\to\infty} T_0 = T_1$ and $T_1 = 1/\beta_1 = T e^{-S/C}$

- Taking Stefan-Boltzmann in a bag, with a fix volume, *V* and bag constant, *B*

$E/V = \sigma T^4 + B$ $\quad p = \frac{1}{3}\sigma T^4 - B$ $\quad S = \frac{4}{3}\sigma V T^3$

- The heat capacity is: $C = \frac{dE}{dT} = 4\sigma V T^3 + \left(\sigma T^4 + B\right)\frac{dV}{dT}$

Thermal model to heavy-ion collisions

- Let's discuss some specific cases:

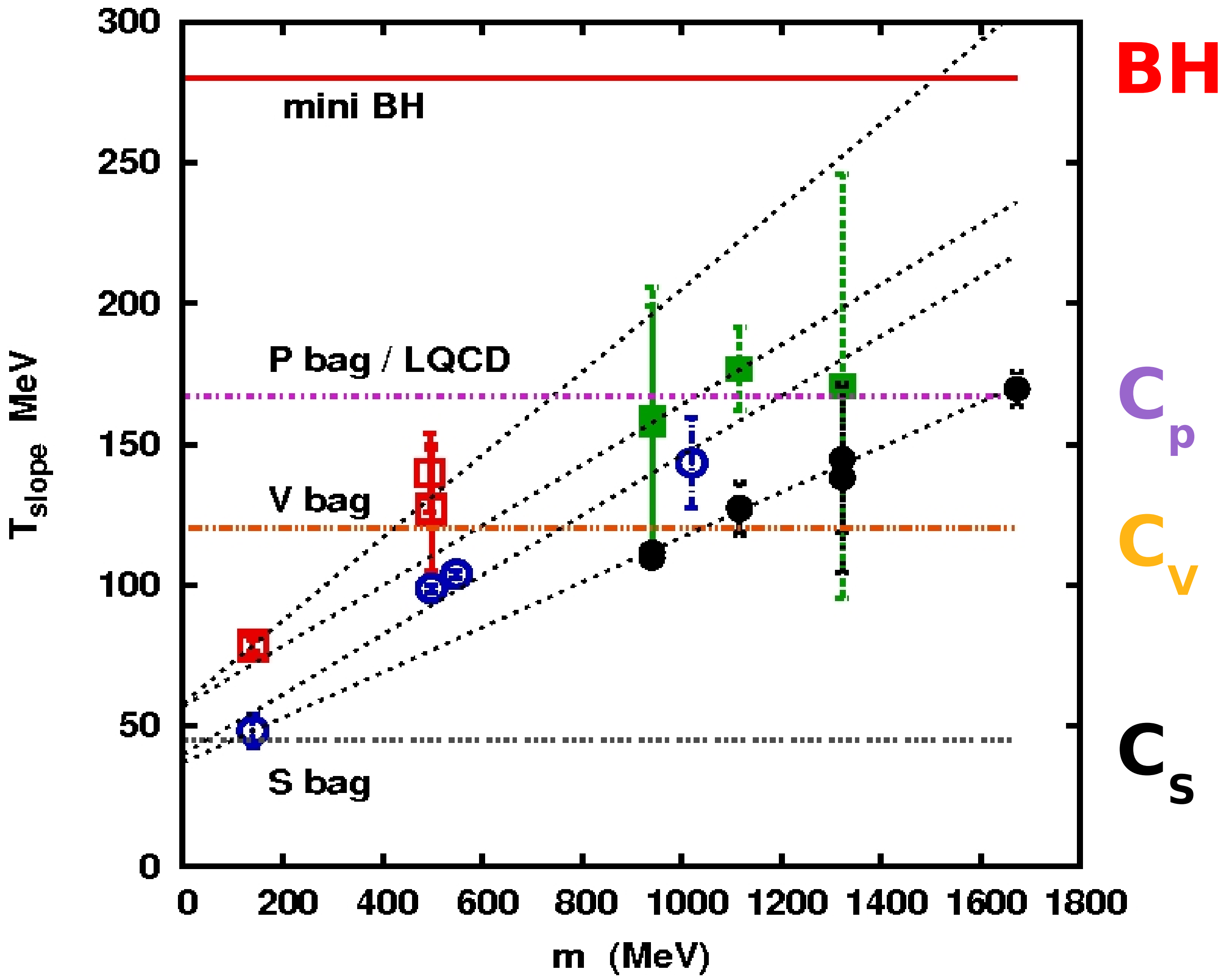
	Heat capacity	Subsystem's T	Note
C_V	$C_V = 4\sigma V T^3 = 3S$	$T_{1V} = T\mathrm{e}^{-1/3}$	
C_p	$C_p = \infty$	$T_{1P} = T$	
C_S	$C_S = 3S(1 - T_*^4/T^4)/4$	$T_{1S} \le T\mathrm{e}^{-4/3}$	$C_S \le 3S/4$
BH	$C = -2S$	$T_1 = T\mathrm{e}^{1/2}$	

- Taking the lattice QCD value T=167 MeV, T_{slopes} are:

$$T_{1P} = T = 167 \text{ MeV},\ T_{1V} = T\mathrm{e}^{-1/3} \approx 120 \text{ MeV and } T_{1S} \le T\mathrm{e}^{-4/3} \approx 45 \text{ MeV}$$

for Tsallis distribution of valence quarks

The temperature slope for different models



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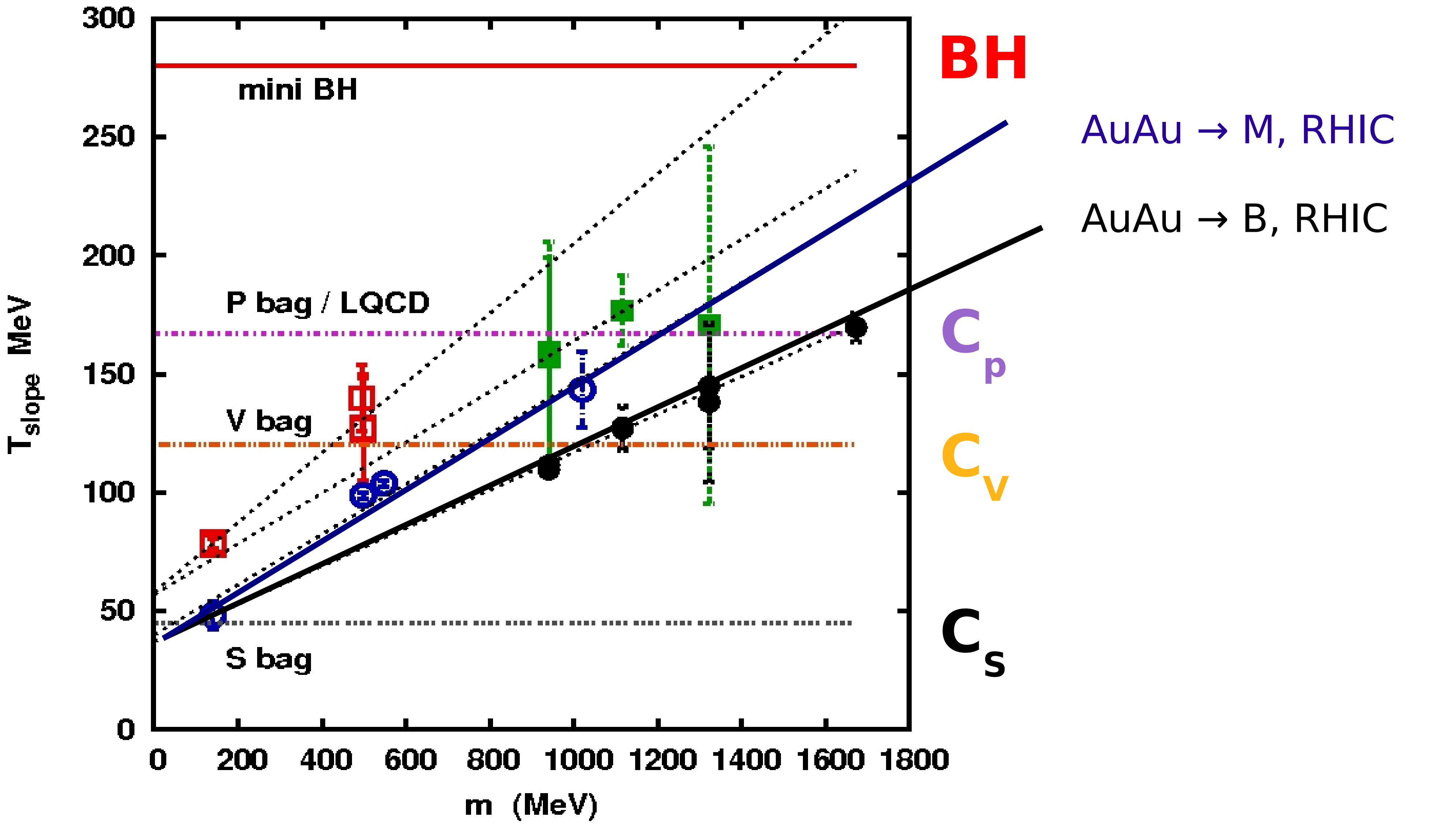

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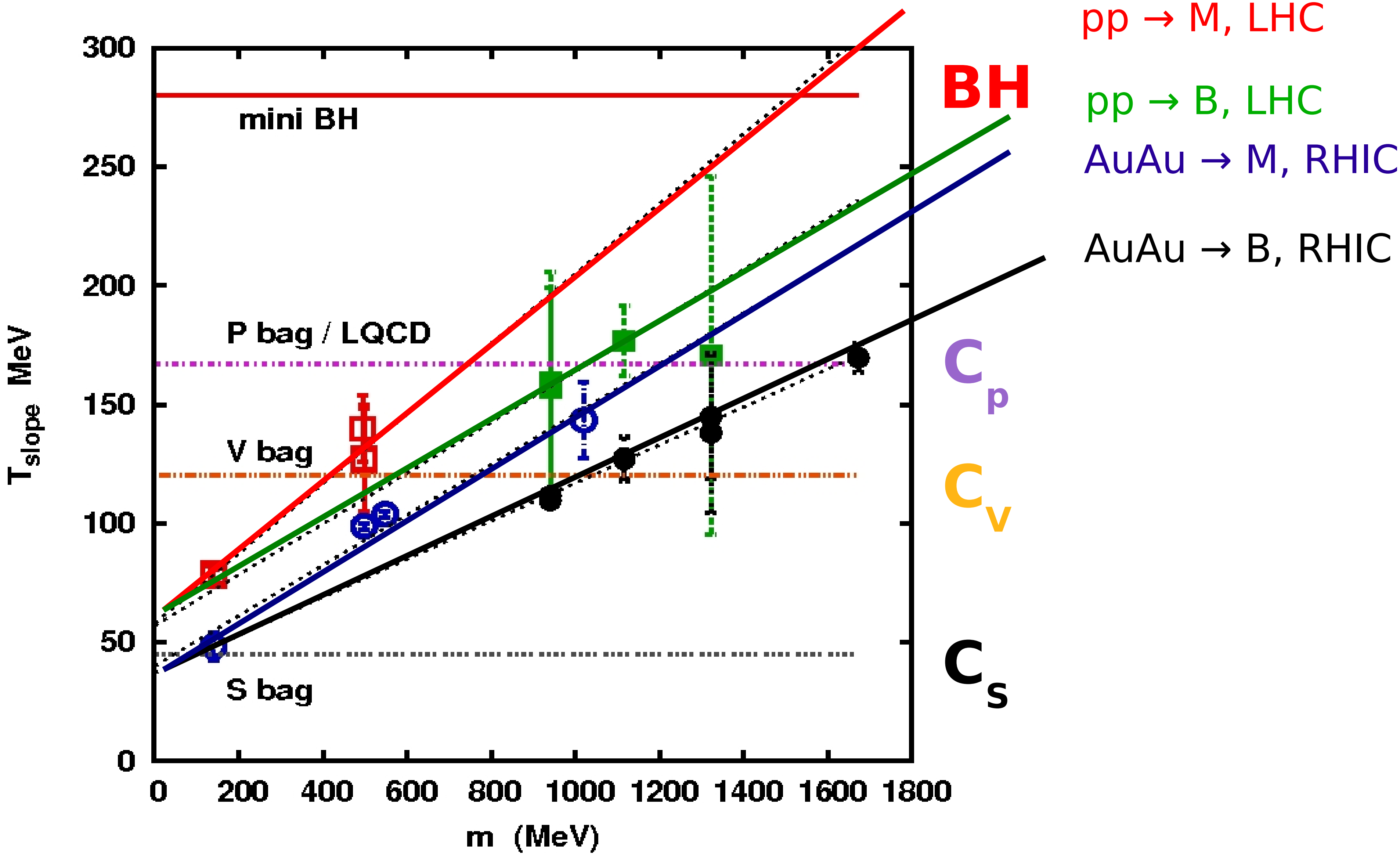

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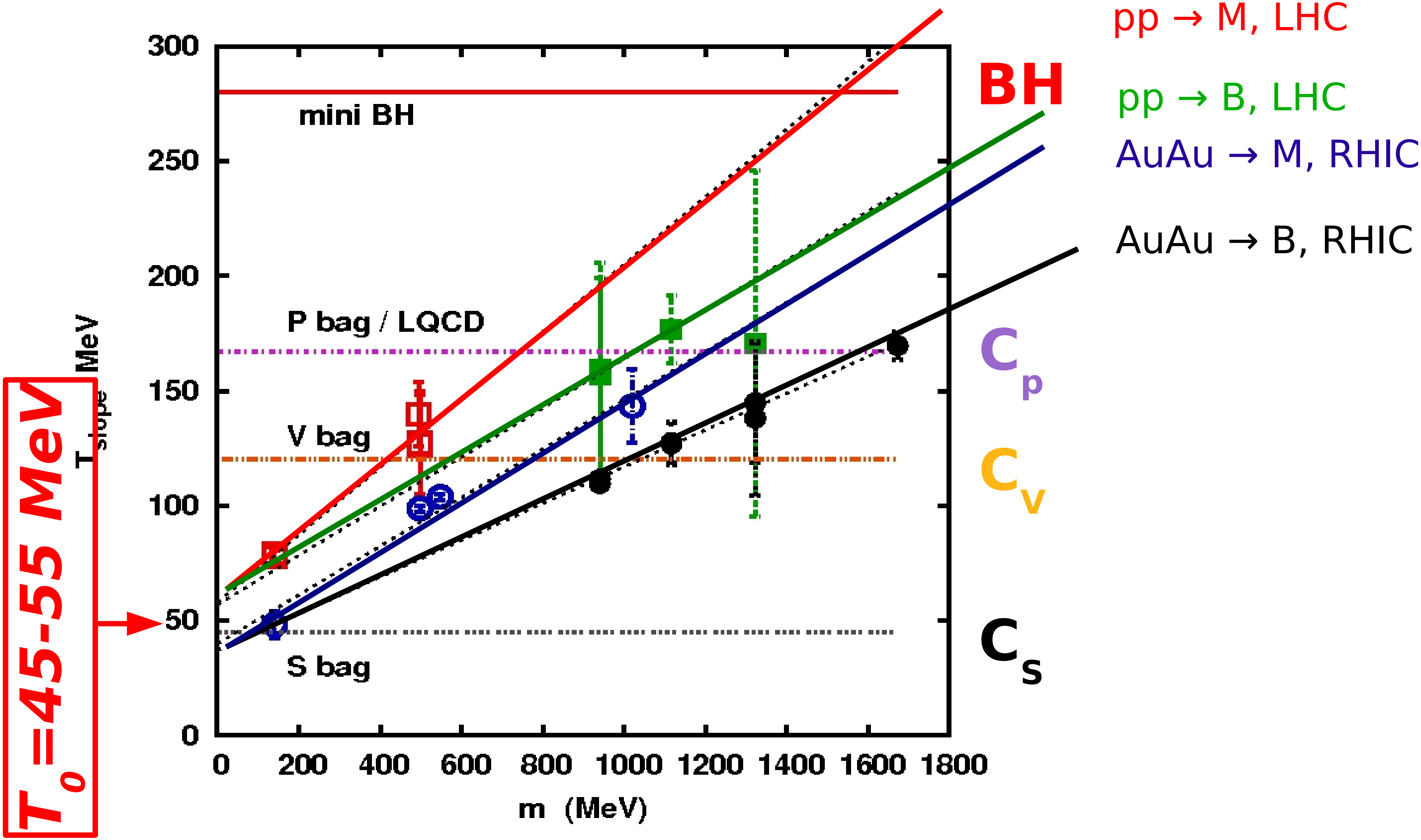

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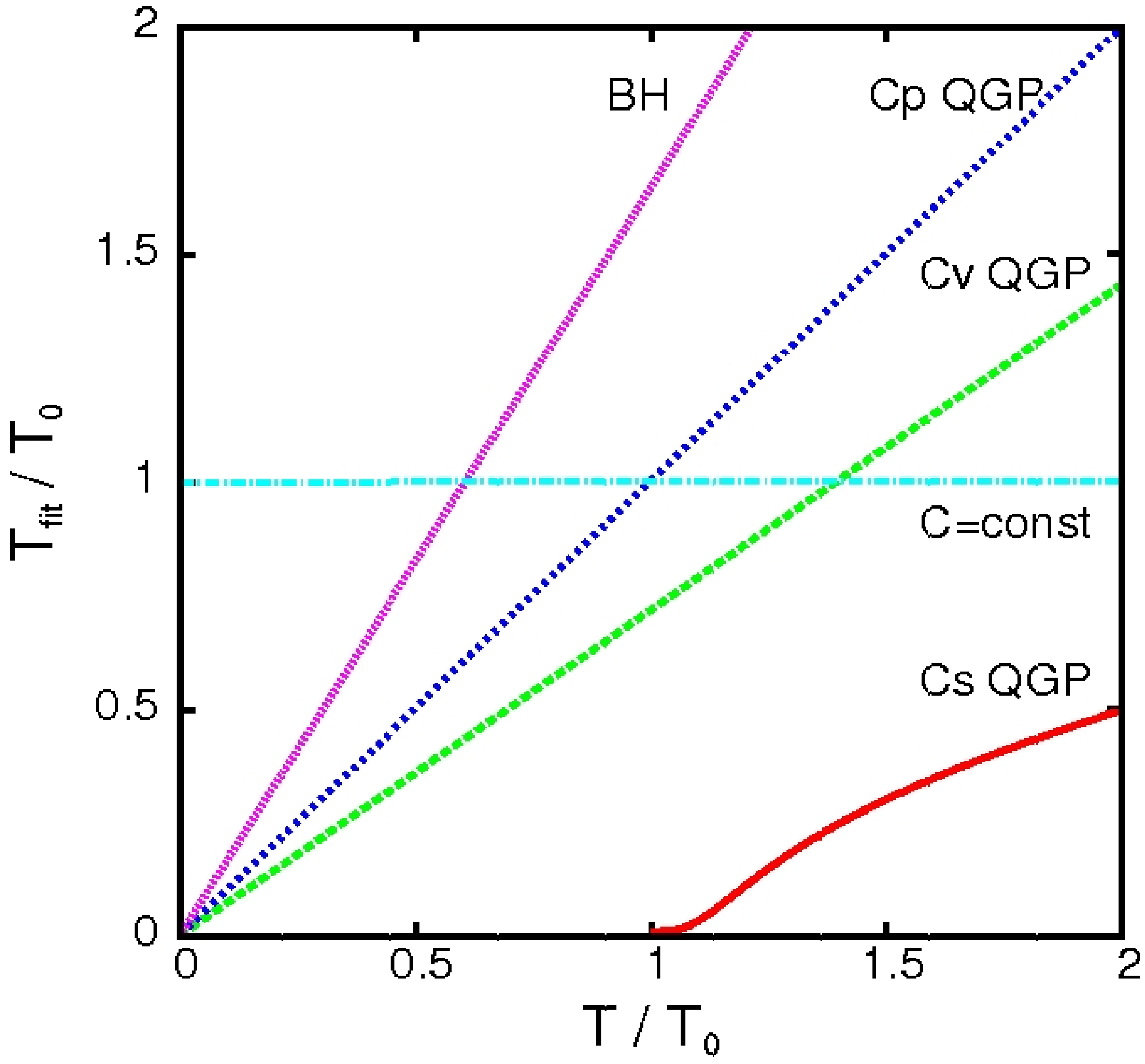
SUMMARY

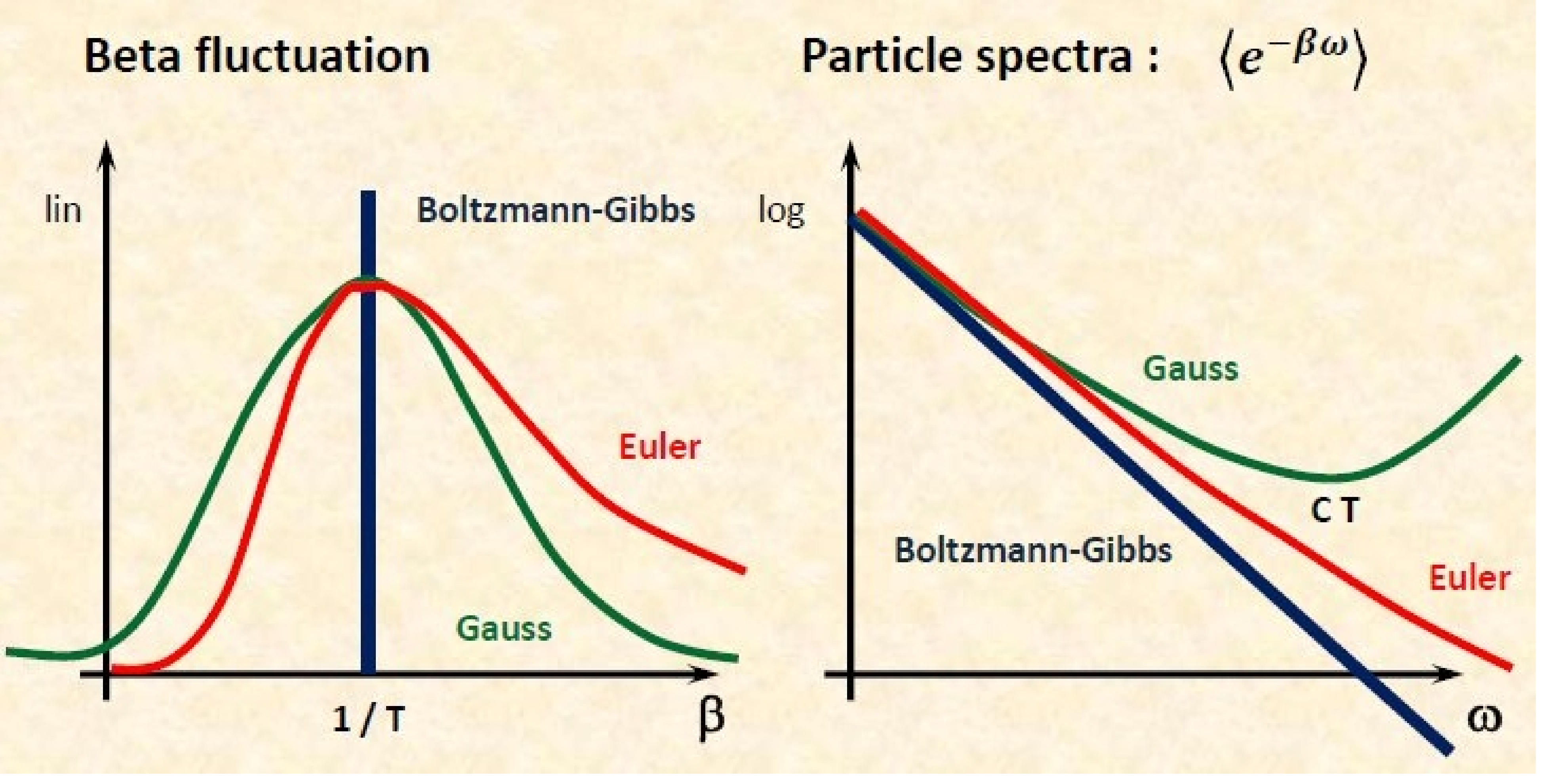
- Derivation
  - Microcanonical treatment
  - Obtained Tsallis/Rényi entropies from the first principles.
  - Not only assumption, but rather a recipe.
  - Providing phyiscal meaning of the 'mysterious q',
  - *$q=1-1/C=1-a$*
  - *Boltzmann Gibbs limit $C \to \infty$, $a \to 0$ ($q \to 1$), $L(S) \to S$*
- Application
  - for Bag model the QGP temperature
    TSB, GGB, PV: EPJ A49 (2013) 110
  - Ideal gas TSB Physica A392 (2013) 3132
- See more applications
  - TSB, GGB, K. Ürmössy: microcanonical Tsalls in ee/pp

B A C K U P

The temperature slope for different models



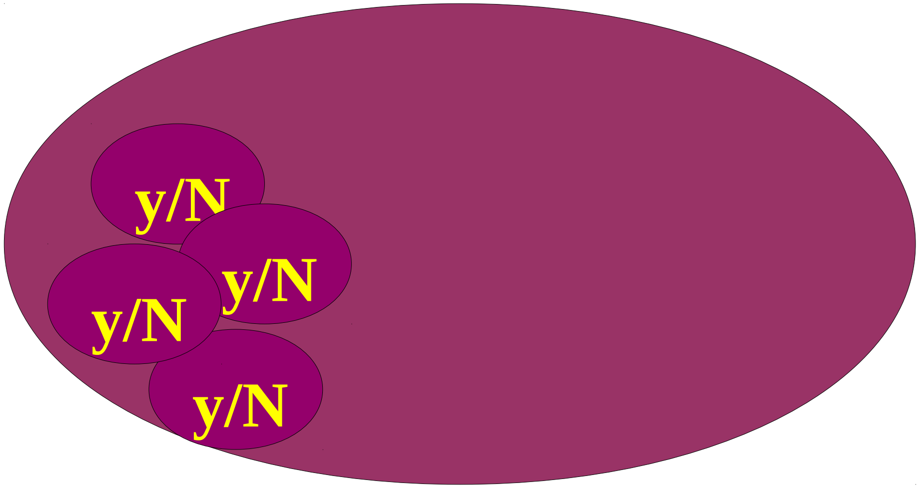
What do we measure as temperature?



Associative composition $\Rightarrow$ evolution eq.

Non-extensive Gibbs, generalised

logarithm: $f(x) = \frac{1}{Z} e^{-\beta X(x)}.$



Composition rule for sub-systems:

$$x_N(y) := \underbrace{h \circ \ldots \circ h}_{N-1} \left(\frac{y}{N}, \ldots, \frac{y}{N} \right)$$

Meanwhile satisfy: $\lim\limits_{N \to \infty} x_N(y) < \infty.$

Assimptotically, if $N_1, N_2 \to \infty.$:

$$x_{N_1+N_2} = \varphi(x_{N_1}, x_{N_2})$$

recursive equation can be given:

$$x_n = h\left(x_{n-1}, \frac{y}{N}\right) \text{, where } h(x,0) = x. \qquad x_n - x_{n-1} = h\left(x_{n-1}, \frac{y}{N}\right) - h(x_{n-1}, 0).$$

Evolution equation can carry out:

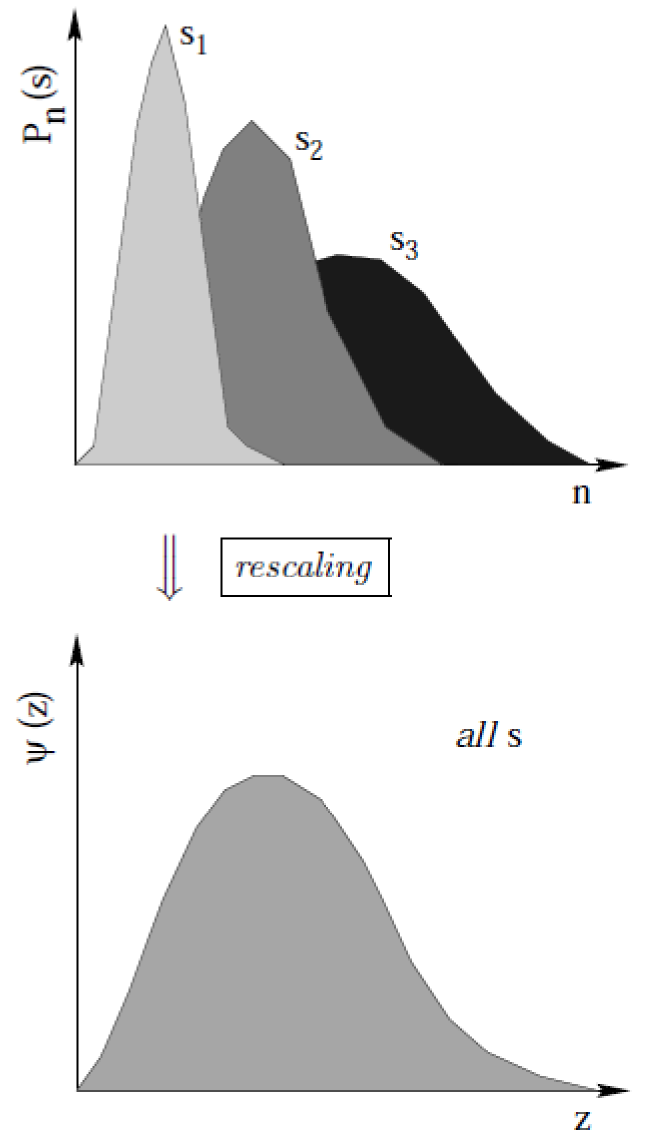
$$\frac{dx}{dt} = \frac{y}{t_f} h_2'(x, 0^+) \qquad L(x) = \int_0^x \frac{dz}{h_2'(z, 0^+)} = y \frac{t}{t_f}.$$

Koba–Nielsen–Olesen (KNO) scaling

Refs:S Hegyi: Nucl. Phys. B 40 (1972) 317, and arXiv:0011301

Hypothesis by Polyakov and Koba–Nielsen–Olesen at very high collision energies, the probability distributions $P_n(s)$ for detecting n final state particles exhibit a scaling (homogeneity) relation:

$$P_n(s) = \frac{1}{\langle n(s)\rangle}\,\psi\left(\frac{n}{\langle n(s)\rangle}\right)$$



As $s \to \infty$ with $\langle n(s) \rangle$ being the average multiplicity of secondaries measured at collision energy s.

KNO: Simple rescaled multiplicity distributions are only a copy of an universal one, $\Psi(z)$ depending on scale $z=n/\langle n(s)\rangle$ only,

Basics of non-extensive thermodynamics

Non-extensive thermodynamics (Based on: T.S. Biró: EPL84, 56003,2008)
associative composition rule, (non-additive) :

$$h(h(x,y),z) = h(x,h(y,z))$$

Then should exist a strict monotonic function, X(x) 'generalised logarithm' (an entropy-like quantity), for which:

$$h(x,y) = X^{-1}\left(X(x) + X(y)\right) \qquad X(h(x,y)) = X(x) + X(y).$$

Examples: (i) Classical Boltzmann–Gibbs thermodynamics:

$$f(E) = e^{-\beta E}/Z \qquad h(x,y) = x + y.$$

(ii) Tsallis–Pareto-like distribution with $a = q - 1$:

$$f(E) = \frac{1}{Z} e^{-\frac{\beta}{a}\ln(1+aE)} = \frac{1}{Z}\left(1 + aE\right)^{-\beta/a} \qquad h(x,y) = x + y + axy$$

$$S = \int f\,\frac{e^{-a\ln(f)} - 1}{a} = \frac{1}{a}\int\left(f^{1-a} - f\right).$$