Recent developments on Noncommutative Quantum Mechanics, Quantum Cosmology and Black Holes

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C. Bastos, O. B., N. Dias and J. Prata,

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Class. Quant. Grav. 28, 25007 (2011); Phys. Rev. D 86, 105030 (2012); D 88, 085013 (2013)*;
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- **Noncommutative Geometry**
- **Phase-Space Noncommutative Quantum Cosmology**
- **Phase-Space Noncommutative Black Holes**
- **Noncommutative Quantum Mechanics**
- **Gravitational Quantum Well (GQW)**
- **Noncommutative Gravitational Quantum Well (NCGQW)**
- **The NCGQW Berry Phase and the Seiberg-Witten map**
- **NC Dirac Equation, NC Graphene**
- **NC Entropic Gravity**
- **Violation of the Robertson-Schrödinger Uncertainty Principle**
- **NC Quantum Beating, NC Entanglement**

Motivation

- String Theory / M-Theory (configuration space NC);
- Gravitational Quantum Well (Phase-Space NC):
  - Measurement of the quantum states for ultra cold neutrons
- Noncommutativity: Quantum gravity feature?
  - Relevant effects at very high energy scales
  - Importance of momentum NC
- Black Holes radiate $\Rightarrow$ Thermodynamics
  - Quantum Gravity
  - Minisuperspace approximation (Quantum Cosmology)

Phase-Space Noncommutative Extension of Quantum Mechanics

$$[\hat{q}_i, \hat{q}_j] = i\theta_{ij} \quad , \quad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \quad , \quad [\hat{p}_i, \hat{p}_j] = i\eta_{ij} \quad , \quad i, j = 1, ..., d$$

- θ_{ij} , η_{ij} antisymmetric real constant (*dxd*) matrices
- Seiberg-Witten map: class of linear non-canonical transformations
  - Relates standard Heisenberg-Weyl algebra with noncommutative algebra
- States of system:
  - Wave functions of the ordinary Hilbert space
- Schrödinger equation:
  - Modified η,θ-dependent Hamiltonian
  - Dynamics of the system

The Cosmological Kantowski-Sachs Model (I)

$$ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\beta} dr^2 + e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} (d\theta^2 + \sin^2\theta d\varphi^2)$$

- ADM Formalism → Hamiltonian for KS metric

$$H = N\mathcal{H} = N e^{\sqrt{3}\beta + 2\sqrt{3}\Omega} \left[-\frac{P_\Omega^2}{24} + \frac{P_\beta^2}{24} - 2e^{-2\sqrt{3}\Omega} \right]$$

- Lapse function (gauge choice):

$$N = 24 e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega}$$

- WDW Equation (H≈0):

$$\exp\left(\sqrt{3}\hat{\beta} + 2\sqrt{3}\hat{\Omega}\right) \left[-\hat{P}_\Omega^2 + \hat{P}_\beta^2 - 48 e^{-2\sqrt{3}\hat{\Omega}} \right] \psi(\Omega, \beta) = 0$$

- Classical model:

$$\dot{P}_\beta = -\eta(-2P_\Omega) = -\eta\dot{\Omega} \quad \longrightarrow \quad P_\beta + \eta\Omega = C$$

PR D 78, 023516 (2008)

The Cosmological Kantowski-Sachs Model (II)

$c = \hbar = G = 1$

θ~ L_P^2 ~ 1

η~L_P^{-2}~1

- Quantum model:

$$\left[\hat{\Omega}, \hat{\beta}\right] = i\theta \;, \quad \left[\hat{P}_\Omega, \hat{P}_\beta\right] = i\eta \;, \quad \left[\hat{\Omega}, \hat{P}_\Omega\right] = \left[\hat{\beta}, \hat{P}_\beta\right] = i$$

  - Non-unitary linear transformation, SW map:

$$\hat{\Omega} = \lambda\hat{\Omega}_c - \frac{\theta}{2\lambda}\hat{P}_{\beta_c} \qquad \hat{\beta} = \lambda\hat{\beta}_c + \frac{\theta}{2\lambda}\hat{P}_{\Omega_c}$$

$$\hat{P}_\Omega = \mu\hat{P}_{\Omega_c} + \frac{\eta}{2\mu}\hat{\beta}_c \qquad \hat{P}_\beta = \mu\hat{P}_{\beta_c} - \frac{\eta}{2\mu}\hat{\Omega}_c$$

$$(\lambda\mu)^2 - \lambda\mu + \frac{\xi}{4} = 0 \Leftrightarrow \lambda\mu = \frac{1+\sqrt{1-\xi}}{2}$$

$$\xi \equiv \theta\eta < 1$$

- Noncommutative WDW Equation:

$$\left[-\left(-i\mu\frac{\partial}{\partial\Omega_c} + \frac{\eta}{2\mu}\beta_c\right)^2 + \left(-i\mu\frac{\partial}{\partial\beta_c} - \frac{\eta}{2\mu}\Omega_c\right)^2 - 48\exp\left[-2\sqrt{3}\left(\lambda\Omega_c + i\frac{\theta}{2\lambda}\frac{\partial}{\partial\beta_c}\right)\right]\right]\psi(\Omega_c,\beta_c) = 0$$

  - Constraint (NC version):

$$\hat{C} = \hat{P}_\beta + \eta\hat{\Omega} = \sqrt{1-\xi}\left(\mu\hat{P}_{\beta_c} + \frac{\eta}{2\mu}\hat{\Omega}_c\right)$$

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Solutions – Noncommutative WDW Equation (I)

From the constraint: $\hat{A} = \frac{\hat{C}}{\sqrt{1-\xi}}$ ➡ $\mu \hat{P}_{\beta_c} + \frac{\eta}{2\mu}\hat{\Omega}_c = \hat{A}$

$$\left[\hat{P}_\beta + \eta\hat{\Omega}, \hat{H}\right] = \left[\hat{P}_\beta + \eta\hat{\Omega}, -\hat{P}_\Omega^2 + \hat{P}_\beta^2 - 48e^{-2\sqrt{3}\hat{\Omega}}\right] = 0$$

- Solutions are simultaneous eigenstates of the Hamiltonian and of the above constraint

- If $\Psi_a(\Omega_c,\beta_c)$ is an eigenstate of the $\hat{A}$ operator with real eigenvalue a:

$$\left(-i\mu\frac{\partial}{\partial\beta_c} + \frac{\eta}{2\mu}\Omega_c\right)\psi_a(\Omega_c,\beta_c) = a\psi_a(\Omega_c,\beta_c) \quad \Rightarrow \quad \psi_a(\Omega_c,\beta_c) = \Re(\Omega_c)\exp\left[\frac{i}{\mu}\left(a - \frac{\eta}{2\mu}\Omega_c\right)\beta_c\right]$$

- After some algebraic manipulation and a change of variables

$$\phi''(z) + (\eta z - a)^2\phi(z) - 48\exp[-2\sqrt{3}z + \frac{\sqrt{3}\theta}{\lambda\mu}a]\phi(z) = 0$$

$$z = \frac{\Omega_c}{\mu}$$

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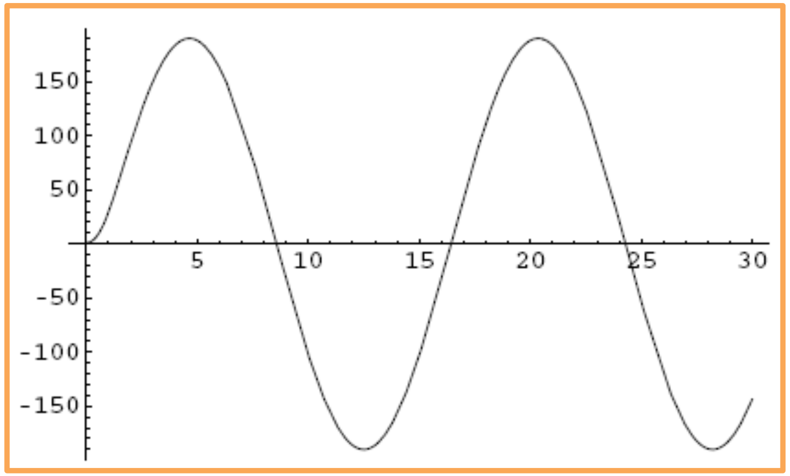
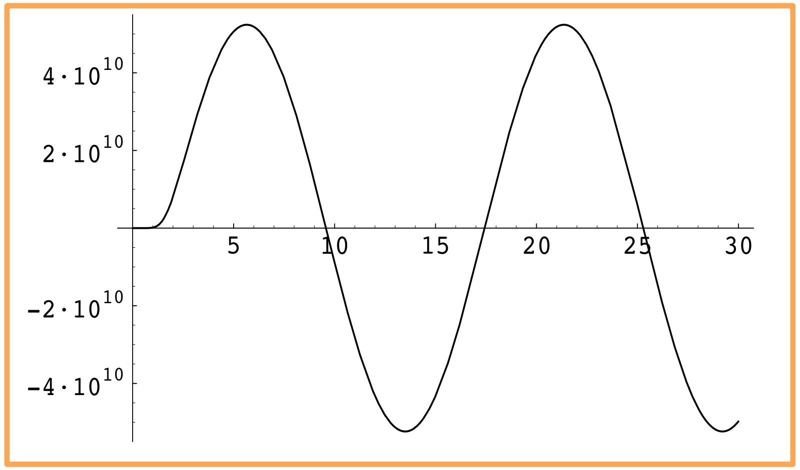
Solutions – Noncommutative WDW Equation (II)

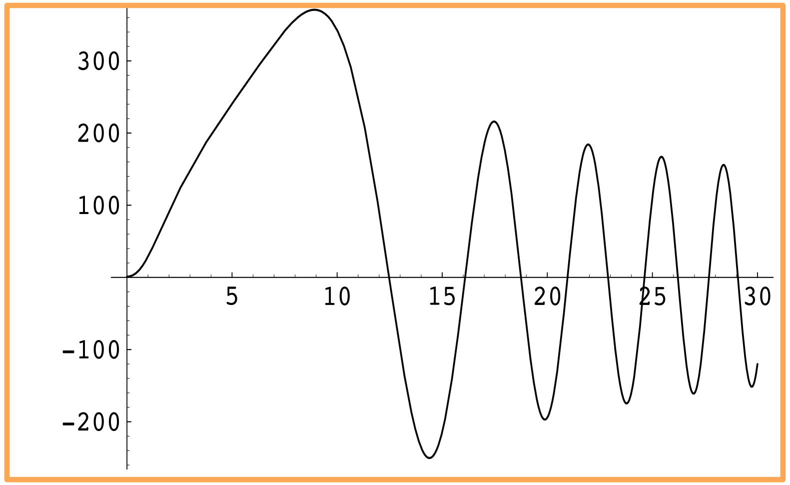
$P_{\Omega}(0)=0$, $P_{\beta}(0)=0.4$, $\Omega(0)=1.65$, $\beta(0)=10$



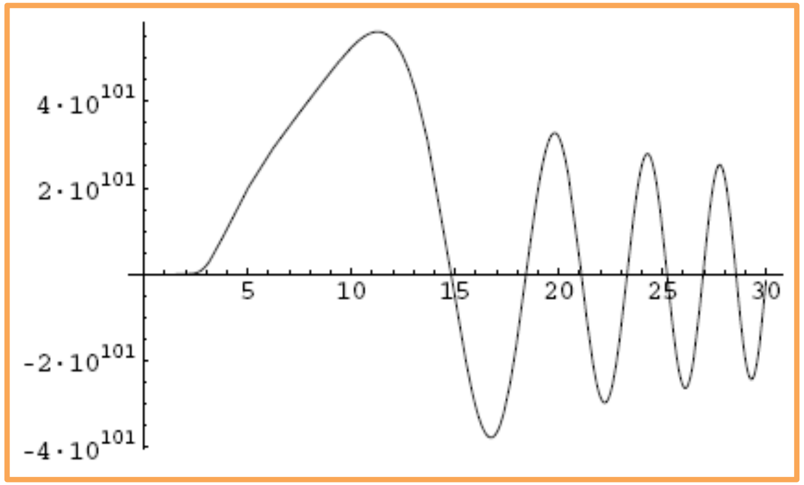
(a) $\theta=\eta=0$, a=0.4



(b) $\theta=5$, $\eta=0$, a=0.4



(c) $\theta=0$, $\eta=0.1$, a=0.565



(d) $\theta=5$, $\eta=0.1$, a=0.799

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Analysis of Solutions – Noncommutative WDW Equation (III)

$\boxed{\xi \equiv \theta\eta < 1}$

- For typical θ=5, wave function with damping: $0.05<\eta<0.12$
  - The wave function blows up for $\eta_c>0.12$
- For $\theta>\eta$, varying θ affects numerical values of $\phi(z)$ but its qualitative features remain unchanged
  - The range for possible values of η where the damping occurs is slightly different
  - The lower limit for η is about 0.05 for all possible values of θ
- For $\eta>\theta$, the damping behaviour of the wave function is more difficult to observe, only for certain values of θ that the wave function does not blow up ($\eta\in[1,2]$)
- For large z, the qualitative behaviour of the wave function is analogous to the one depicted in Figures

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Noncommutative Black Holes (I)

General Relativity → causal structure changes at different regions of space-time

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

- r<2M, time and radial coordinates interchange

$$ds^2 = -\left(\frac{2M}{t}-1\right)^{-1}dt^2 + \left(\frac{2M}{t}-1\right)dr^2 + t^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

  - Isotropic metric turns into an anisotropic one
  - Mapped into the Kantowski-Sachs metric

$$ds^2 = -N^2dt^2 + e^{2\sqrt{3}\beta}dr^2 + e^{-2\sqrt{3}\beta}e^{-2\sqrt{3}\Omega}(d\theta^2 + \sin^2\theta d\varphi^2)$$

  - Away from the horizon t = r = 2M:

$$N^2 = \left(\frac{2M}{t}-1\right)^{-1} \quad , \quad e^{2\sqrt{3}\beta} = \left(\frac{2M}{t}-1\right) \quad , \quad e^{-2\sqrt{3}\beta}e^{-2\sqrt{3}\Omega} = t^2$$

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Potential Function – Canonical NC (II)

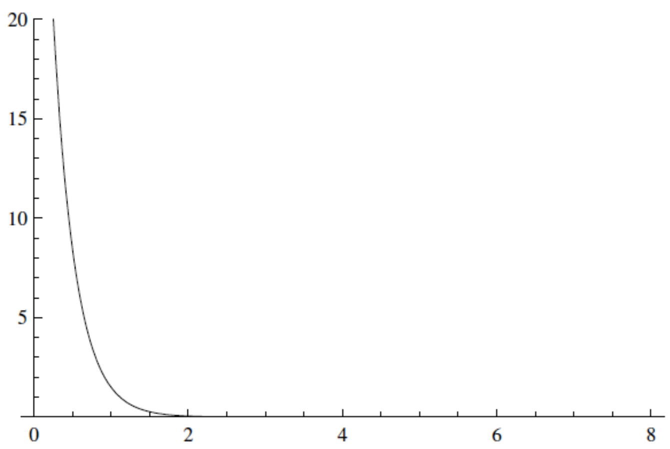
$$\hbar = c = k = G = 1$$

$$V(z) = 48 \exp\left[-2\sqrt{3}z + \frac{\sqrt{3}\theta}{\lambda\mu}a\right] - (\eta z - a)^2$$

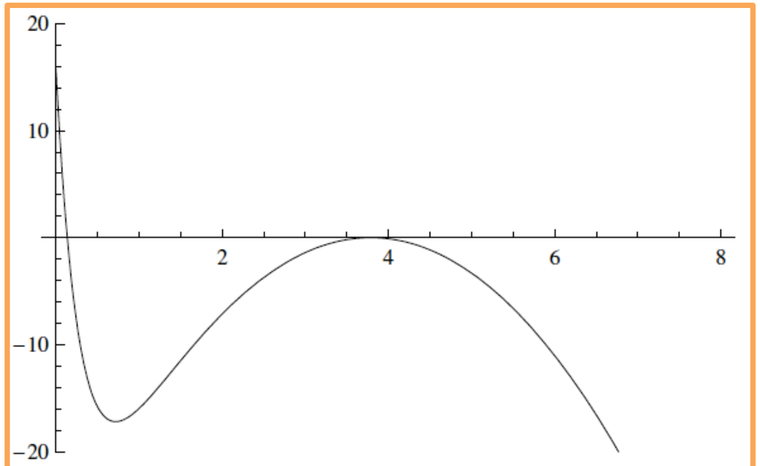
$$x = z - \frac{\theta}{2\lambda\mu}a$$

$$V(x) = 48 \exp\left(-2\sqrt{3}x\right) - (\eta x - c)^2$$

$$c = P_\beta(0) + \eta\Omega(0)$$



(a) $\eta = 0$ and $c = 0.01$



(b) $\eta = 1.5$ and $c = 5.68$

For non-vanishing and typical η values, the potential has a local minimum and maximum

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Potential Function – Canonical NC (III)

$\hbar = c = k = G = 1$

- Local minimum:

$$\frac{dV}{dx}\Big|_{x_0} = -96\sqrt{3}\exp\left(2\sqrt{3}x_0\right) - 2\eta^2 x_0 + 2\eta c = 0$$

- Solution (implicit):

$$\exp\left(-2\sqrt{3}x_0\right) = \zeta D - \frac{\zeta^2}{\sqrt{3}}x_0$$

$$\zeta = \eta/4\sqrt{3}$$

$$6\exp\left(-2\sqrt{3}x_0\right) - \zeta^2 > 0 \Leftrightarrow x_0 < -\frac{1}{\sqrt{3}}\ln\left(\frac{\zeta}{\sqrt{6}}\right)$$

$$D = c/12$$

- Potential function in second order in x-x_0:

$$V(x) = 48(6e^{-2\sqrt{3}x_0} - \zeta^2)(x - x_0)^2 + 48e^{-2\sqrt{3}x_0} - (\eta x_0 - c)^2$$

- NCWDW Equation:

$$-\frac{1}{2}\frac{d^2\phi}{dx^2} + 24(6e^{-2\sqrt{3}x_0} - \zeta^2)(x - x_0)^2\phi + \left[24e^{-2\sqrt{3}x_0} - \frac{1}{2}(\eta x_0 - c)^2\right]\phi = 0$$

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Feynman-Hibbs Procedure – Canonical NC (IV)

- Comparing with Schrödinger equation of the harmonic oscillator:

$$V_{NC}(y) = 24(6e^{-2\sqrt{3}x_0} - \zeta^2)y^2$$

$$y = x - x_0$$

  - Quantum correction:

$$\frac{\beta_{BH}}{24} V''_{NC}(y) = 2\beta_{BH}(6e^{-2\sqrt{3}x_0} - \zeta^2)$$

  - Potential function:

$$U_{NC}(y) = 24(6e^{-2\sqrt{3}x_0} - \zeta^2)\left(y^2 + \frac{\beta_{BH}}{12}\right)$$

- Partition Function:

$$Z_{NC} = \sqrt{\frac{1}{48(6e^{-2\sqrt{3}x_0} - \zeta^2)}\frac{1}{\beta_{BH}}} \exp\left[-2\beta_{BH}^2\left(6e^{-2\sqrt{3}x_0} - \zeta^2\right)\right]$$

- NC Internal Energy:

$$\bar{E}_{NC} = \frac{1}{\beta_{BH}} + 4(6e^{-2\sqrt{3}x_0} - \zeta^2)\beta_{BH}$$

- NC Temperature ($\bar{E}_{NC}=M$) for M>>1:

$$T_{BH} = \frac{4}{M}(6e^{-2\sqrt{3}x_0} - \zeta^2)$$

$\eta_0=0.025$

$$T_{BH} = \frac{1}{8\pi M},$$

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Feynman-Hibbs Procedure – Canonical NC (V)

- Noncommutative Entropy (neglecting terms prop. to η^2/M^2):

$$\boxed{S_{BH} \simeq \frac{M^2}{2b(\zeta)} + \ln \frac{\sqrt{b(\zeta)}}{M\sqrt{3}}}$$

- Thermodynamical quantities:
  - Non-trivial dependence on momentum noncommutativity
  - Matches Hawking temperature $T_{BH}=1/8\pi M$ for $\eta_0 = 0.025$

- $\eta=0$:
  - Thermodynamics of BH is ill-defined in what concerns the Feynman-Hibbs procedure
  - Potential function reduces to a monotonous exponential term with no local minima

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Singularity, t=r=0 - Canonical NC (VI)

- By the identification between the metrics:

$$t = 0,\ \Omega \to +\infty \text{ and } \beta \to +\infty$$

- Study the limit:

$$\psi(\Omega_c, \beta_c) = \int da\, C(a)\psi_a(\Omega_c, \beta_c)$$

$$\lim_{\Omega_c, \beta_c \to +\infty} \psi(\Omega_c, \beta_c)$$

$$\phi_a''(z) + (\eta z - a)^2\, \phi_a(z) = 0$$

- NCWDW equation in this limit:

$$\left\{-\frac{\partial^2}{\partial z^2} - (\eta z - a)^2\right\} \phi_a(z) = 0 \quad \Longleftrightarrow \quad \left\{-\frac{\partial^2}{\partial \tilde{z}^2} - \eta^2 \tilde{z}^2\right\} \tilde{\phi}_a(\tilde{z}) = 0$$

$$\tilde{z} = z - \tfrac{a}{\eta} \text{ and } \tilde{\phi}_a(x) = \phi_a(x + \tfrac{a}{\eta})$$

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Singularity, t=r=0 – Canonical NC (VII)

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- Solution to NCWDW equation at t=r=0:

$$\boxed{\tilde{\phi}_a(\tilde{z}) \sim \frac{1}{\tilde{z}^{1/2}} \exp\left[\pm i \frac{\eta}{2} \tilde{z}^2\right]}$$

- For all a:

$$\boxed{\lim_{z \to +\infty} \phi_a(z) = \lim_{z \to +\infty} \tilde{\phi}_a\left(z - \frac{a}{\eta}\right) = 0 \quad \Longrightarrow \quad \lim_{\Omega_c, \beta_c \to +\infty} \psi_a(\Omega_c, \beta_c) = 0}$$

- Thus, for a suitable, although fairly general, choice of C(a):

$$\boxed{\lim_{\Omega_c, \beta_c \to +\infty} \psi(\Omega_c, \beta_c) = 0}$$

Necessary condition to provide a quantum regularization of the BH singularity

Does the probability of reaching the BH singularity vanish? (VIII)

- Wave function oscillatory for β_C. Fix β_C-hypersurface:
  - Probability of reaching the BH singularity:

$$P(r=0,t=0) = \lim_{\Omega_c,\beta_c \to +\infty} \int_{\Omega_c}^{+\infty} |\psi(\Omega' \cdots d\Omega'_c \simeq \lim_{\Omega_c \to +\infty} \int_{\Omega_c}^{+\infty} |\phi_a(\frac{\Omega'_c}{\mu})|^2 d\Omega'_c$$

DIVERGES!

Inverted harmonic oscillator displays non-normalizable eigenstates!

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Noncommuativity of this form cannot be regarded as a solution for the singularity problem of the Schwarzschild BH!

Noncanonical Phase Space Noncommutativity (I)

$\hbar = c = k = G = 1$

$$\left[\hat{\Omega}, \hat{\beta}\right] = i\theta\left(1 + \epsilon\theta\hat{\Omega} + \frac{\epsilon\theta^2}{1+\sqrt{1-\xi}}\hat{P}_\beta\right)$$
$$\left[\hat{P}_\Omega, \hat{P}_\beta\right] = i\left(\eta + \epsilon(1+\sqrt{1-\xi})^2\hat{\Omega} + \epsilon\theta(1+\sqrt{1-\xi})\hat{P}_\beta\right)$$
$$\left[\hat{\Omega}, \hat{P}_\Omega\right] = \left[\hat{\beta}, \hat{P}_\beta\right] = i\left(1 + \epsilon\theta(1+\sqrt{1-\xi})\hat{\Omega} + \epsilon\theta^2\hat{P}_\beta\right),$$

PR D 82, 041502 (2010) [R.C.];

PR D 84, 024005 (2011)

- SW – Map:

$$\hat{\Omega} = \lambda\hat{\Omega}_c - \frac{\theta}{2\lambda}\hat{P}_{\beta_c} + E\hat{\Omega}_c^2 \ , \quad \hat{\beta} = \lambda\hat{\beta}_c + \frac{\theta}{2\lambda}\hat{P}_{\Omega_c}$$
$$\hat{P}_\Omega = \mu\hat{P}_{\Omega_c} + \frac{\eta}{2\mu}\hat{\beta}_c \ , \quad \hat{P}_\beta = \mu\hat{P}_{\beta_c} - \frac{\eta}{2\mu}\hat{\Omega}_c + F\hat{\Omega}_c^2.$$

$$E = -\frac{\theta}{1+\sqrt{1-\xi}}F$$
$$F = -\frac{\lambda}{\mu}\epsilon\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right).$$

$$\hat{A} = \mu\hat{P}_{\beta_c} + \frac{\eta}{2\mu}\hat{\Omega}_c.$$

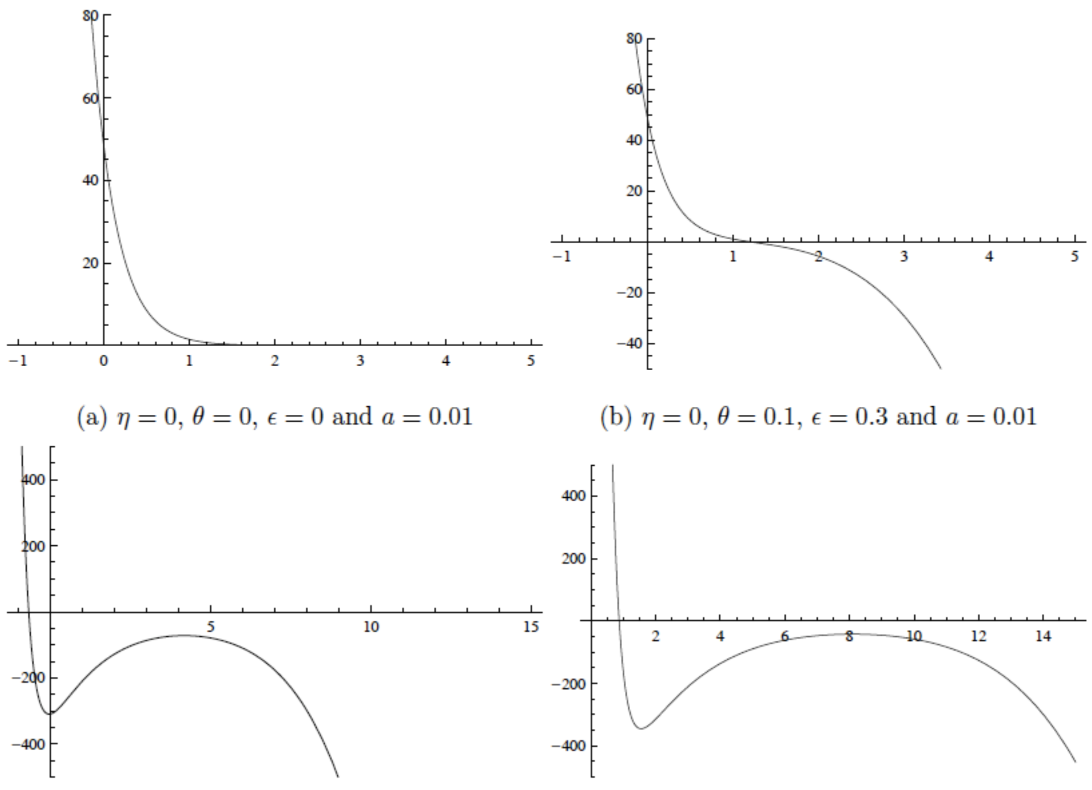
$$\left[\hat{P}_\beta + \eta\hat{\Omega}, \hat{H}\right] = \left[\hat{P}_\beta + \eta\hat{\Omega}, -\hat{P}_\Omega^2 + \hat{P}_\beta^2 - 48e^{-2\sqrt{3}\hat{\Omega}}\right] = 0$$

- Solutions of NCWDW Eq. are simultaneous eigenstates of the Hamiltonian and Â constraints:

$$-\phi_a''(z) + V(z)\phi_a(z) = 0\,,$$

$$V(z) = -\left(F\mu^2 z^2 + \eta z - a\right)^2 + 48\exp\left(-2\sqrt{3}z - 2\sqrt{3}\mu^2 E z^2 + \frac{\sqrt{3}\theta a}{\mu\lambda}\right).$$

Potential Function – Noncanonical NC (II)



(a) $\eta = 0,\ \theta = 0,\ \epsilon = 0$ and $a = 0.01$

(b) $\eta = 0,\ \theta = 0.1,\ \epsilon = 0.3$ and $a = 0.01$

(c) $\eta = 5,\ \theta = 0,\ \epsilon = 0.3$ and $a = 18.89$

(d) $\eta = 5,\ \theta = 0.1,\ \epsilon = 0.3$ and $a = 18.89$

For η≠0 potential has a local minimum and maximum.

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- Minimum for small z values:

$$V(z) = -(\eta z - a)^2 + 48 \exp\left[-2\sqrt{3}\left(z - \frac{\theta a}{2\mu\lambda} + \mu^2 E z^2\right)\right]$$

Potential Function – Noncanonical NC (III)

Local minimum:

$$\frac{dV}{dz}|_{z_0} = \eta(\eta z_0 - a) + 48\sqrt{3}(1 + 2\mu^2 E z_0) \exp\left[-2\sqrt{3}(z_0 - \frac{\theta a}{2\mu\lambda} + \mu^2 E z_0^2)\right] = 0$$

Solution (implicit):

$$l := \eta/4\sqrt{3},$$

$$\exp\left[-2\sqrt{3}(z_0 - \frac{\theta a}{2\mu\lambda} + \mu^2 E z_0^2)\right] = -\frac{\eta(\eta z_0 - a)}{48\sqrt{3}(1 + 2\mu^2 E z_0)} .$$

$$6\left((1 + 2\mu^2 E z_0)^2 - \frac{\sqrt{3}}{3}\mu^2 E\right) \exp\left[-2\sqrt{3}(z_0 - \frac{\theta a}{2\mu\lambda} + \mu^2 E z_0^2)\right] - l^2 > 0$$

Potential function in second order in x-x$_0$:

$$V(z) = 48B(z - z_0)^2 - (\eta z_0 - a)^2 + 48k ,$$

$$B := 6k[(1 + 2\mu^2 E z_0)^2 - (\sqrt{3}/3)\mu^2 E] - l^2$$

$$k := \exp\left[-2\sqrt{3}(z_0 - \tfrac{\theta a}{2\mu\lambda} + \mu^2 E z_0^2)\right]$$

NCWDW Equation:

$$-\frac{1}{2}\frac{d^2\phi}{dz^2} + 24\left[B - l^2\right](z - z_0)^2\phi = \left[\frac{1}{2}(\eta z_0 - a)^2 - 24k\right]\phi .$$

Feynman-Hibbs Procedure – Noncanonical NC (IV)

- Comparing with Schrödinger eq. of harmonic oscillator:

$$V_{NC}(y) = 24(B - l^2)y^2 ,$$

  - Quantum correction to potential: $$\frac{\beta_{BH}}{24} V''_{NC}(y) = 2\beta_{BH}(B - l^2)$$
  - Potential function: $$\mathcal{V}_{NC}(y) = 24(B - l^2)\left(y^2 + \frac{\beta_{BH}}{12}\right)$$

- Partition Function: $$Z_{NC} = \sqrt{\frac{1}{48(B - l^2)}\frac{1}{\beta_{BH}}} \times \exp\left[-2\beta_{BH}^2(B - l^2)\right]$$

- NC Internal Energy: $$U_{NC} = \frac{1}{\beta_{BH}} + 4(B - l^2)\beta_{BH} ,$$

- NC Temperature (U_{NC}=M), M>> $$T_{BH} = \frac{4}{M}(B - l^2)$$ ➡ $$T_{BH} = \frac{1}{8\pi M} ;$$

$$z_0 = 2.26369 \qquad \eta_0 = 0.487.$$

- NC Entropy (neglecting terms proportional to η^2/M^2):

$$S_{BH} \simeq \frac{M^2}{8\left(6k[(1+2\mu^2 E z_0)^2 - \frac{\sqrt{3}}{3}\mu^2 E] - l^2\right)} - \frac{1}{2}\ln\left(\frac{3M^2}{6k[(1+2\mu^2 E z_0)^2 - \frac{\sqrt{3}}{3}\mu^2 E] - l^2}\right).$$

Singularity, t=r=0 - Noncanonical NC (V)

- Asymptotically: $$-\phi_a''(z) - Kz^4\phi_a(z) = 0, \quad z \to +\infty \,,$$
- Solution: square integrable

$$\phi_a(z) \simeq \frac{C}{z}\exp\left(\pm i\frac{K^{1/2}}{3}z^3\right), \quad z \to +\infty \,, \; a \in \mathbb{R} \,,$$

- Probability of system reaching the Schwarzschild singularity:

$$P(r=0,t=0) = \lim_{\tilde{\Omega}_c,\beta_c \to +\infty} \int_{\tilde{\Omega}_c}^{+\infty} \int_{-\infty}^{+\infty} \left|\psi(\Omega_c,\beta)\right|^2 d\xi =$$

$$= \lim_{\tilde{\Omega}_c,\beta_c \to +\infty} \int_{\tilde{\Omega}_c}^{+\infty} \left|\psi(\Omega_c,\beta_c)\right|^2 d\Omega_c = 0$$

Regularization of the singularity!

Singularity, t=r=0 - Noncanonical NC (VI)

$\beta_c \to +\infty,$

- Compute probabilities of the type:

$$P\left(\Omega_c \in I, \beta_c \to +\infty\right) = \lim_{\beta_c \to +\infty} \int_I \int_{\mathbb{R}} |\psi(\Omega_c, \beta)|^2 d\xi = \lim_{\beta_c \to +\infty} \int_I \left|\psi(\Omega_c, \beta_c)\right|^2 d\Omega_c \,,$$

  - I is some compact subset of IR.

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- Simple calculations:

$$\left|\psi(\Omega_c, \beta_c)\right|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} C(a)\overline{C(a')}\phi_a\left(\frac{\Omega_c}{\mu}\right)\overline{\phi_{a'}\left(\frac{\Omega_c}{\mu}\right)} \exp\left(\frac{i\beta_c}{\mu}a - \frac{i\beta_c}{\mu}a'\right) da da' =$$

$$= \left|\int_{\mathbb{R}} C(a)\phi_a\left(\frac{\Omega_c}{\mu}\right) \exp\left(\frac{i\beta_c}{\mu}a\right) da\right|^2 \,.$$

- Thus:

$$P\left(\Omega_c \in I, \beta_c \to +\infty\right) = \lim_{\beta_c \to +\infty} \int_I \left|\int_{\mathbb{R}} C(a)\phi_a\left(\frac{\Omega_c}{\mu}\right) \exp\left(\frac{i\beta_c}{\mu}a\right) da\right|^2 d\Omega_c$$

Singularity, t=r=0 - Noncanonical NC (VII)

PR D 82, 041502 (2010) [R.C.];

PR D 84, 024005 (2011)

- Potential varies smoothly with respect to a

- Regularity conditions:

$$1)\ C(a) \in L^1(\mathbb{R})$$
$$2)\ C(a)M(a) \in L^1(\mathbb{R})$$

$$M(a) \equiv ||\phi_a||_{L^\infty(\mathbb{R})} = sup_{x\in\mathbb{R}}|\phi_a(x)| < \infty, \quad \text{for each } a \in \mathbb{R}\,.$$

$$\int_I \left|\int_{\mathbb{R}} C(a)\phi_a\left(\frac{\Omega_c}{\mu}\right)\exp\left(\frac{i\beta_c}{\mu}a\right)da\right|^2 d\Omega_c \leq |I|\left[\int_{\mathbb{R}}|C(a)|M(a)da\right]^2 < \infty\,,$$

- Uniform convergence on every compact interval I and all βc:

$$P\left(\Omega_c \in I, \beta_c \to +\infty\right) = \int_I \lim_{\beta_c\to+\infty}\left|\int_{\mathbb{R}} C(a)\phi_a\left(\frac{\Omega_c}{\mu}\right)\exp\left(\frac{i\beta_c}{\mu}a\right)da\right|^2 d\Omega_c\,.$$

Singularity, t=r=0 - Noncanonical NC (VIII)

PR D 82, 041502 (2010) [R.C.];

PR D 84, 024005 (2011)

- Indeed, $|C(a)\phi_a\left(\frac{\Omega_c}{\mu}\right)| \leq |C(a)M(a)|,$

- Riemann-Lesbegue Lemma: every L1 function has a continuous Fourier transform, which vanishes at infinity:

$$\lim_{|\beta_c|\to\infty}\int_{\mathbb{R}} f(a)e^{\frac{i\beta_c}{\mu}a}da = 0 \quad \Longrightarrow \quad P\left(\Omega_c \in I, \beta_c \to +\infty\right) = 0$$

Regularization of the Kantowski-Sachs singularity!

Conclusions

- Kantowski-Sachs metric to study interior of a Schwarzschild BH ($r<2M$)
  - Thermodynamical quantities and singularity analyzed
  - Momentum NC is crucial:
    - Quadratic term in potential that allows for the Feynman-Hibbs procedure
    - NC Temperature and NC Entropy
- Singularity $t=r=0$:
  - Inverted harmonic oscillator
  - Wave function vanishes, but is not square integrable
  - Noncanonical NC yields a squared integrable wave function and the probability of reaching the singularities (Schwarzschild and Kantowski-Sachs) vanish!

Noncommutative Quantum Mechanics (I)

[Chaichian, Sheikh-Jabbari, Tureanu 2000]

[Gamboa, Loewe, Rojas 2001; Nair, Polychronakos 2001]

[Ho, Kao 2002; Zhang 2004]

- **Non-relativistic limit of NC quantum field theory (one-particle sector) corresponds to the Schrödinger equation:**

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V * \psi$$

where one can write the noncommutative part $V * \psi$ **as**

$$V\left(q'_i - \frac{p'_j \theta \varepsilon_{ij}}{2\hbar} \right)\psi(q')$$

through the "Seiberg-Witten" map.

Noncommutative Quantum Mechanics (II)

The "Seiberg-Witten" map is a linear set of non-canonical transformations

$$q_i = q'_i - \frac{\theta}{2\hbar}\varepsilon_{ij} p'_j \qquad\qquad p_i = p'_i$$

that relates the NC variables which satisfy the NC algebra

$$[q_i, q_j] = i\theta\varepsilon_{ij} \qquad [p_i, p_j] = 0 \qquad [q_i, p_j] = i\hbar\delta_{ij}$$

with the variables that satisfy the Heisenberg-Weyl algebra of Quantum Mechanics

$$[q'_i, q'_j] = 0 \qquad [p'_i, p'_j] = 0 \qquad [q'_i, p'_j] = i\hbar\delta_{ij}$$

Quantum Gravitational Well (QGW)

[Nesvizhevsky et al. 2002-2005]

- **Bound neutrons by gravity in the x-direction** $(\mathbf{g} = -g\mathbf{e_x})$

- **Hamiltonian:** $$H' = \frac{p_x'^2}{2m} + \frac{p_y'^2}{2m} + mgx'$$

- **Wave function ~ Airy function:** $$\psi_n(x') = A_n \phi(z)$$

- **Classical turning points:** $$x_n = E_n/mg$$

- **Energy spectrum:** $$E_n = -\left(\frac{mg^2\hbar^2}{2}\right)^{1/3} \alpha_n$$

- **Experimental results:** $$\begin{aligned} x_1^{exp} &= 12,2 \pm 1,8(syst.) \pm 0,7(stat.)\ (\mu\text{m})\ , \\ x_2^{exp} &= 21,6 \pm 2,2(syst.) \pm 0,7(stat.)\ (\mu\text{m})\ . \end{aligned}$$

- **Theoretical results:** $x_1 = 13,7\ \mu\text{m}$ $\quad$ $x_2 = 24,0\ \mu\text{m}$

Noncommutative Gravitational Quantum Well (NCGQW)

[O.B., Rosa, Aragão, Castorina, Zappalà 2005]

Consider the NC extension of QM in the phase space:

$$\begin{aligned} [x^\mu, x^\nu] &= i\theta^{\mu\nu} \;, \\ [p^\mu, p^\nu] &= i\eta^{\mu\nu} \;, \\ [x^\mu, p^\nu] &= i\hbar\delta^{\mu\nu} \end{aligned} \longrightarrow \begin{aligned} [x, y] &= i\theta \;, \\ [p_x, p_y] &= i\eta \;, \\ [x_i, p_j] &= i\hbar\delta_{ij} \end{aligned}$$

4-dimensions *2-dimensions*

$\sqrt{\theta}$ **and** $\sqrt{\eta}$ **set the fundamental scales of distance and momentum**

Phys. Rev. D 72, 025010 (2005)

Ways to implement a noncommutative version of QM:

$$\begin{aligned} x &= x' - \frac{\theta}{\hbar}p'_y \\ y &= y' \,, \\ p_x &= p'_x \,, \\ p_y &= p'_y - \frac{\eta}{\hbar}x' \end{aligned}$$

or

$$\begin{aligned} x &= x' \,, \\ y &= y' + \frac{\theta}{\hbar}p'_x \,, \\ p_x &= p'_x + \frac{\eta}{\hbar}y' \,, \\ p_y &= p'_y \,. \end{aligned}$$

To lift the ambiguity consider the combination of the above:

$$\begin{aligned} x &= x' - \frac{\theta}{2\hbar}p'_y \,, \\ y &= y' + \frac{\theta}{2\hbar}p'_x \,, \\ p_x &= p'_x + \frac{\eta}{2\hbar}y' \,, \\ p_y &= p'_y - \frac{\eta}{2\hbar}x' \,, \end{aligned} \longrightarrow \quad [x_i, p_j] = i\hbar \underbrace{\left(1 + \frac{\theta\eta}{4\hbar^2}\right)}_{\text{Effective Planck´s constant}} \delta_{ij}$$

Most general set of transformations:

[O.B., Rosa, Aragão, Castorina, Zappalà 2006]

$$x = \xi\left(x' - \frac{\theta}{2\hbar}p'_y\right), \qquad y = \xi\left(y' + \frac{\theta}{2\hbar}p'_x\right)$$

$$p_x = \xi\left(p'_x + \frac{\eta}{2\hbar}y'\right), \qquad p_y = \xi\left(p'_y - \frac{\eta}{2\hbar}x'\right)$$

That admits the 4-dimensional generalization:

$$x^\mu = \xi\left(x'^\mu - \frac{\theta^\mu{}_\nu}{2\hbar}p'^\nu\right), \qquad p^\mu = \xi\left(p'^\mu + \frac{\eta^\mu{}_\nu}{2\hbar}x'^\nu\right)$$

It then follows that:

$$[x, y] = i\xi^2\theta \equiv i\theta_{eff}, \qquad [p_x, p_y] = i\xi^2\eta \equiv i\eta_{eff}$$

$$[x_i, p_j] = i\xi^2\hbar\left(1 + \frac{\theta\eta}{4\hbar^2}\right)\delta_{ij} \equiv i\hbar_{eff}\delta_{ij}, \quad i = 1, 2 .$$

Mod. Phys. Lett. A 21, 795-802 (2006)

Hence, if one chooses $\xi = 1$:

$$\theta_{eff} = \theta, \ \eta_{eff} = \eta, \ \hbar_{eff} = \hbar\left(1 + \frac{\theta\eta}{4\hbar^2}\right)$$

If on the other hand one chooses $\xi = (1+\theta\eta/4\hbar^2)^{-1/2}$ then:

$$\theta_{eff} = \frac{\theta}{1 + \frac{\theta\eta}{4\hbar^2}}, \ \eta_{eff} = \frac{\eta}{1 + \frac{\theta\eta}{4\hbar^2}}, \ \hbar_{eff} = \hbar$$

That is, these models are equivalent.

Mod. Phys. Lett. A 21, 795-802 (2006)

Four-dimensional generalization

$$
\begin{aligned}
[x^\mu, x^\nu] &= i\theta^{\mu\nu} \ , \\
[p^\mu, p^\nu] &= i\eta^{\mu\nu} \ , \\
[x^\mu, p^\nu] &= i\hbar\left(\delta^{\mu\nu} + \frac{\theta^{\mu\alpha}\eta^{\nu}{}_{\alpha}}{4\hbar^2}\right)
\end{aligned}
$$

Non-diagonal

$$\hbar_{eff} = \hbar\left(1 + \frac{Tr[\theta\eta]}{4\hbar^2}\right)$$

* Noncommutative Hamiltonian ($C=[1-\xi_{NC}]^{-1}$, $\xi_{NC} = \eta\theta/4\hbar^2$):

$$H = \frac{\bar{p_x}^2}{2m} + \frac{\bar{p_y}^2}{2m} + mgx + \frac{C\eta}{2m\hbar}(x\bar{p_y} - y\bar{p_x}) + \frac{C^2}{8m\hbar^2}\eta^2(x^2+y^2)$$

$$(\overline{p_x} = Cp_x, \overline{p_y} = Cp_y + \frac{m^2 g\theta}{2\hbar})$$

At first order in the noncommutative parameters:

$$H = H' + \frac{\eta}{2m\hbar}(xp_y - yp_x)$$

GQW data allow setting bounds on the fundamental momentum scale:

$$|\sqrt{\eta}| \lesssim 0,90\,\mathrm{meV/c} \qquad (n=1)\,,$$
$$|\sqrt{\eta}| \lesssim 0,79\,\mathrm{meV/c} \qquad (n=2)\,.$$

Remarks:

Uncertainty Principle allows improving these bounds down to 1μeV/c

Second order terms are 6-7 orders of magnitude smaller than the first order ones

Bounds on the correction to Planck´s constant ($\hbar_{eff} = \hbar(1+\xi_{NC})$):

$$|\sqrt{\theta}| \lesssim 1\,\mathrm{fm} \longrightarrow \begin{array}{lll} |\xi_{NC}| \lesssim 5,2\times 10^{-24} & (n=1)\,, \\ |\xi_{NC}| \lesssim 4,0\times 10^{-24} & (n=2)\,. \end{array}$$

Phys. Rev. D 72, 025010 (2005)

Beyond the NCGQW

- **Higher-order corrections? Doubtful!**

$$H_{\text{eff}} = mc^2 - \frac{\hbar^2}{2m}\Delta - mU - \frac{\hbar^4}{8m^3c^2}\Delta^2 + \frac{3\hbar^2}{2mc^2}(U\Delta + \nabla U \cdot \nabla) + \frac{mU^2}{2c^2} + \frac{3\hbar^2 \Delta U}{4mc^2} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

- **Berry phase? Non-trivial topological phase acumulated on a adiabatic evolution of the system around a closed path in the parameter space**

$$\Psi_n(\mathbf{x}) \to e^{\mathrm{i}\gamma_n(C)}\Psi_n(\mathbf{x})$$

$$\gamma_n(C) = \mathrm{i}\oint_C \langle n(\mathbf{R})|\nabla_{\mathbf{R}} n(\mathbf{R})\rangle \cdot \mathrm{d}\mathbf{R}$$

$|n(\mathbf{R})\rangle$ -eigenstate

Berry Phase and the Seiberg-Witten Map

[Bastos, O.B. 2006]

- Commutative GQW

$$\gamma_n(C) = 0$$

- NCGQW

$$\Delta\gamma_n(C) \sim -i\left(\frac{2}{m^4 g\hbar^4}\right)^{2/3}\left(\frac{C\eta}{2}\right)^3 \Sigma_{n\neq m}\frac{1}{(\alpha_n-\alpha_m)^2}$$

$$C=[1-\xi_{NC}]^{-1},\ \xi_{NC}=\ \eta\theta/4\hbar^2$$

$$\gamma_n(C) = 0$$

PL A 372, 5556-5559 (2008)

Independently of the SW map

[Bastos, O.B., Dias, Prata, 2008]

J. Math. Phys. 49, 072101 (2008)

NC Dirac Equation for the Hydrogen Atom

[O.B. & Queiroz 2011]

$$H_{NC} = c\boldsymbol{\alpha}\cdot(\mathbf{p} - e\mathbf{A}) + \beta mc^2 + e\Phi - \frac{e}{2\hbar}[\nabla(c\boldsymbol{\alpha}\cdot\mathbf{A} - \Phi)\times\mathbf{p}]\cdot\boldsymbol{\theta} + \frac{c}{2\hbar}[\boldsymbol{\alpha}\times\mathbf{r}]\cdot\boldsymbol{\eta}$$

$$\theta = 0$$

$$\begin{aligned} H_{NC\eta} &= c\boldsymbol{\alpha}\cdot(\mathbf{p} - e\mathbf{A}) + \beta mc^2 + e\Phi + \frac{c}{2\hbar}[\boldsymbol{\alpha}\times\mathbf{r}]\cdot\boldsymbol{\eta} \\ &= c\boldsymbol{\alpha}\cdot(\mathbf{p} - \frac{e}{2}[\mathbf{B} + \frac{1}{e\hbar}\boldsymbol{\eta}]\times\mathbf{r}) + \beta mc^2 + e\Phi. \end{aligned}$$

Experimental bound on the hyperfine correction to the hydrogem atom spectrum

$$\eta \lesssim mh \times 0.0024 = 1.45\times 10^{-66}\ Kg^2m^2s^{-2}$$

$$\Rightarrow \sqrt{\eta} \lesssim 2.26\ \mu eV/c.$$

PL A 375, 4116 (2011)

NC Graphene

[Bastos, O.B., Costa Dias, Prata 2013]

Striking feature of the graphene, a two-dimensional of carbon atoms in a hexagonal honeycomb configuration, is that its low-energy excitations are relativistic and correspond to massless, quasi-free fermions described by the Dirac equation

$$H = v_F \vec{\sigma} \cdot \vec{P}$$

IR spectroscopy in the presence of a magnetic field allows to resolve the Landau spectrum for one single layer graphene and generalize it to the NC case:

$$E_K^{NC} = \pm 2\hbar v_F \sqrt{\left(\frac{eB}{2\hbar}\right)\left(1 + \frac{\eta}{eB\hbar}\right) n_e}$$

which leads to an experimental bound on the noncommutative parameter:

$$\eta < 2.1 \times 10^{-53}\ kg^2 m^2 s^{-2}$$

$$\Rightarrow \sqrt{\eta} < 8.6\ eV/c\ .$$

IJMP A 28, 1350064 (2013)

Entropic Gravity

[Bastos, O.B., Costa Dias, Prata 2011]

- Generalization of entropic derivation of ISL (Verlinde 2011) to include phase-space NC effects
  - Effective Planck constant:

$$\hbar_{eff} = \hbar \left(1 + \frac{\theta_{12}\eta_{12}}{\hbar^2}\right)^{\frac{1}{2}}$$

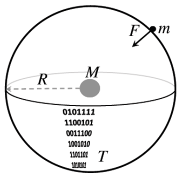
  - Considerations about a minimal size cell in phase-space:

$$\Delta x_1 \Delta p_1 \geq \hbar/2, \ \ \Delta x_1 \Delta x_2 \geq \theta_{12}/2 \,,$$

$$\Delta x_2 \Delta p_2 \geq \hbar/2, \ \ \Delta p_1 \Delta p_2 \geq \eta_{12}/2 \,.$$

$$V(\Delta x_1, \Delta p_1, \Delta x_2, \Delta p_2) \geq \frac{\theta_{12}\eta_{12}}{4}$$

  - Dependence on the anisotropy of NC parameters:



$$V(\Delta x_1, \Delta p_1, \Delta x_2, \Delta p_2) \geq \frac{\hbar^2}{4} + \frac{\theta_{12}\eta_{12}}{4} \equiv \frac{\hbar_{eff}^2}{4} \,,$$

$$F_{NC} = \frac{GMm}{r^2}\left(1 + \frac{\theta_{12}\eta_{12}}{2\hbar^2}\right)$$

- Putative anisotropy of the NC correction implies:
  - Violation of the Equivalence Principle
  - Allows to obtain bounds for the NC parameters

$$\frac{\Delta a}{a} \equiv 2\left(\frac{a_1 - a_2}{a_1 + a_2}\right) \simeq \frac{1}{4\hbar^2}(\theta_{12}\eta_{12} - \theta_{23}\eta_{23})$$

$$\frac{\theta\eta}{\hbar^2} \lesssim O(1) \times 10^{-13} \,.$$

CQG 28, 25007 (2011)

Violation of Robertson-Schrödinger Uncertainty Principle

- Violation of Robertson-Schrödinger Uncertainty Principle (RSUP) in Quantum Mechanics:

$$\mathbf{\Sigma} + \tfrac{i}{2}\mathbf{J} \geq 0.$$

$$\mathbf{J} = -\mathbf{J}^T = -\mathbf{J}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

  - Would signal a deformation of Heisenberg-Weyl algebra:

$$\left[\widehat{\Xi}_i, \widehat{\Xi}_j\right] = i\Omega_{ij}, \qquad i,j = 1,\cdots,2n,$$

$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{\Theta} & \mathbf{I} \\ -\mathbf{I} & \mathbf{\Upsilon} \end{pmatrix}$$

- NC Gaussian state:

$$\mathcal{G}_{\mathbf{\Sigma},\zeta}(z) = \frac{1}{(2\pi)^n\sqrt{\det\mathbf{\Sigma}}}\exp\left[-\tfrac{1}{2}(z-\zeta)\cdot\mathbf{\Sigma}^{-1}(z-\zeta)\right]$$

$$\mathbf{\Sigma} = \tfrac{1}{2}\mathbf{C}\mathbf{C}^T.$$

  - Violates RSUP
  - Is a quantum state of a NC extension of Quantum Mechanics

$$\mathbf{C} = \begin{pmatrix} \lambda\mathbf{I} & -\frac{\theta}{2\lambda}\mathbf{E} \\ \frac{\eta}{2\mu}\mathbf{E} & \mu\mathbf{I} \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

PR D 86, 105030 (2012)

Violation of Robertson-Schrödinger Uncertainty Principle

- Violation of the RSUP may signal the existence of a deformation of the Heisenberg-Weyl algebra: any Gaussian state in the phase-space is always a quantum state of an appropriate noncommutative extension of quantum mechanics

- Violation of the RSUP and Heisenberg's UP may not imply the breakdown of quantum mechanics, but rather the emergence of an underlying non-commutative structure of quantum mechanics

PR D 86, 105030 (2012)

- Violations of Heisenberg's formulation of the UP have already been detected experimentally and have confirmed Osawa's formulation!

  [Rozema, Darabi, Mahler, Hayat, Soudagar, Steinberg 2012]

  [Sulyok, Sponar, Erhart, Badurek, Osawa, Hasegawa 2013]

- RS version of Osawa's UP (arXiv: 1310.4762)

- NC version in preparation ...

Quantum beating, missing information and the thermodynamic limit
[Bernardini, O.B. 2013]

- Monitoring the time evolution of the components of the NC quantum harmonic oscillator allows one to track the transfer of information due to noncommutativity, as quantified by the linear quantum entropy and mutual information:

$$H^W_{HO}(\boldsymbol{Q},\boldsymbol{\Pi}) = \alpha^2\boldsymbol{Q}^2 + \beta^2\boldsymbol{\Pi}^2 + \gamma\sum_{i,j=1}^{2}\epsilon_{ij}\hat{\Pi}_i\hat{Q}_j$$

$$\alpha^2 \equiv \frac{\lambda^2 m\omega^2}{2} + \frac{\eta^2}{8m\mu^2\hbar^2}\,,$$
$$\beta^2 \equiv \frac{\mu^2}{2m} + \frac{m\omega^2\theta^2}{8\lambda^2\hbar^2}\,,$$
$$\gamma \equiv \frac{\theta}{2\hbar}m\omega^2 + \frac{\eta}{2m\hbar}.$$

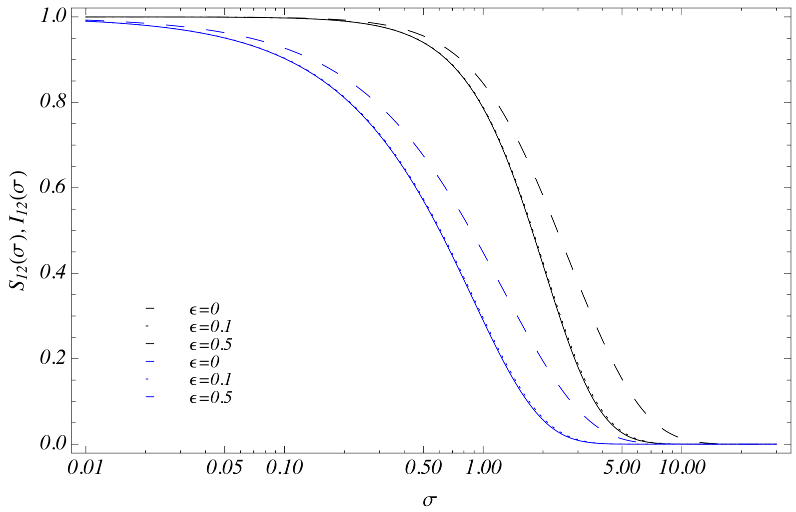
$$\hat{q}_i = \lambda\hat{Q}_i - \frac{\theta}{2\lambda\hbar}\epsilon_{ij}\Pi_j$$

$$\hat{p}_i = \mu\Pi_i + \frac{\eta}{2\mu\hbar}\epsilon_{ij}\hat{Q}_j$$

$$\epsilon = \gamma/\Omega$$

$$\sigma = \hbar\Omega/(k_B T)$$

$$\Omega = 2\alpha\beta$$



PR A 88, 012101 (2013)

Entanglement in Phase Space NC Quantum Mechanics (I)

Entanglement of Gaussian states may be exclusively induced by "switching on" the noncommutative deformation

- PPT criterium (Positive Partial Transpose):

$$\widetilde{\Sigma}' + \frac{i}{2}\mathbf{J} \geq 0$$

$$\mathbf{J} = -\mathbf{J}^T = -\mathbf{J}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

- SW map:

$$\Sigma' = \mathbf{S}\widetilde{\Sigma}'\mathbf{S}^T$$

- NCPPT criterium:

$$\Sigma' + \frac{i}{2}\Omega \geq 0 .$$

$$\Omega = \begin{pmatrix} \Theta & \mathbf{I} \\ -\mathbf{I} & \Upsilon \end{pmatrix}$$

- Invariance of SW map and symplectic eigenvalues:

$$\Sigma + \frac{i}{2}\Omega \geq 0 \Leftrightarrow \widetilde{\Sigma} + \frac{i}{2}\mathbf{J} \geq 0 \Leftrightarrow \tilde{\nu}_- \geq 1 \Leftrightarrow \nu_- \geq 1.$$

PR D 88 085013 (2013)

Entanglement in Phase Space NC Quantum Mechanics (II)

❑ 2-dimensional NC Gaussian State:

$$F(z) = \frac{1}{\pi^4 \sqrt{\det \mathbf{\Sigma}}} \exp(-z^T \mathbf{\Sigma}^{-1} z),$$

$$\mathbf{\Sigma} = \frac{b}{2} \begin{pmatrix} \mathbf{I}_4 & \gamma^{\mathbf{T}} \\ \gamma & \mathbf{I}_4 \end{pmatrix}, \text{ with } \gamma = \begin{pmatrix} n & 0 & m & 0 \\ 0 & n & 0 & -m \\ m & 0 & -n & 0 \\ 0 & -m & 0 & -n \end{pmatrix}.$$

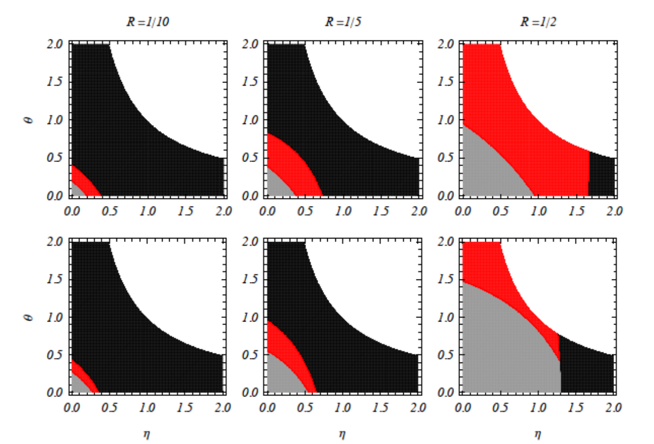
$$R = \sqrt{m^2 + n^2},$$

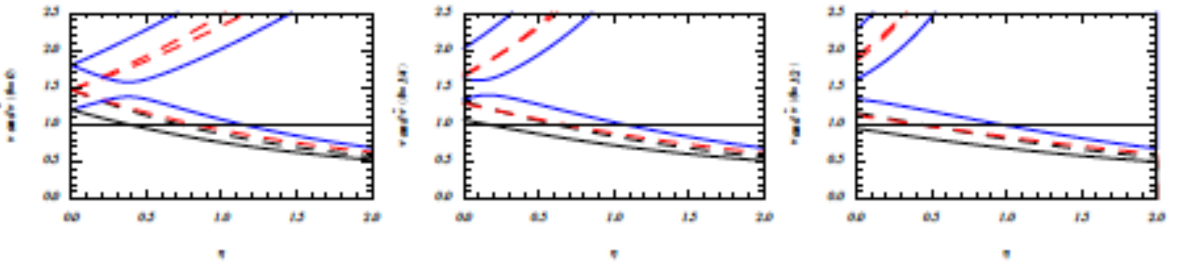
$$b = (1+R)/(1-R)$$

Smallest Williamson symplectic eigenvalues:

$$\nu_- = \frac{1}{1-\eta\theta} \frac{1+R}{1-R} \sqrt{\frac{\omega_-}{2} - \sqrt{\frac{\omega_-^2}{4} - (1-R^2)^2 (1-\eta\theta)^2}},$$

$$\nu'_- = \frac{1}{1-\eta\theta} \frac{1+R}{1-R} \sqrt{\frac{\omega_+}{2} - \sqrt{\frac{\omega_+^2}{4} - (1-R^2)^2 (1-\eta\theta)^2}},$$





PR D 88 , 085013 (2013)