# Cosmological perturbation theory at three-loop order

Mathias Garny (CERN)

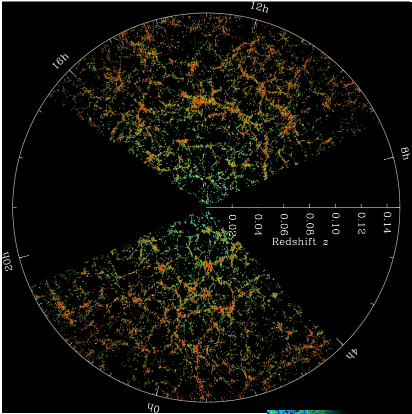
Grenoble, 04.12.13

based on 1304.1546 and 1309.3308
with Diego Blas, Thomas Konstandin

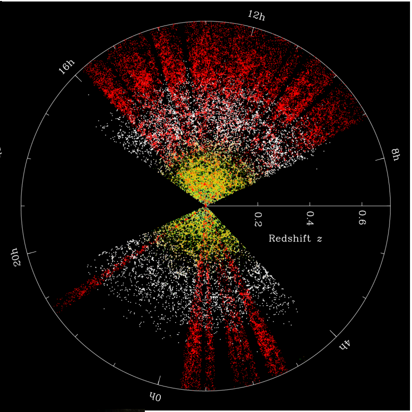
# Outline

- Large-scale structure
- Non-linear corrections
- Three loop result
- Padé resummation

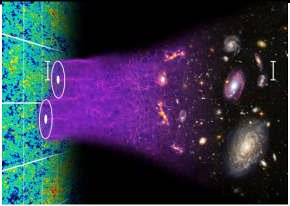
# Large-scale structure




*SDSS-III (sdss3.org)*



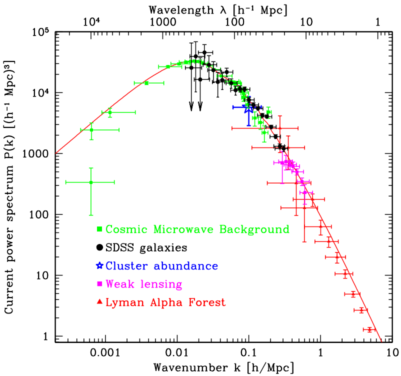

*BOSS (red), SDSS (white/yellow)*

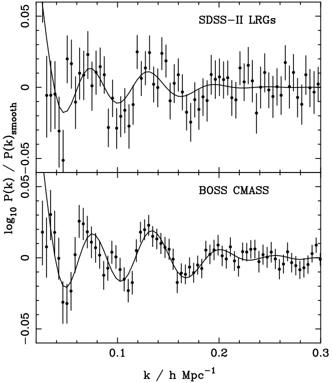


# Large-scale structure

Power spectrum of density contrast $\delta(\mathbf{x}, z) = \rho(\mathbf{x}, z)/\bar{\rho}(z) - 1$

$$\langle \delta(\mathbf{k}, z)\delta(\mathbf{k}', z)\rangle = \delta^{(3)}(\mathbf{k} + \mathbf{k}')P(k, z)$$



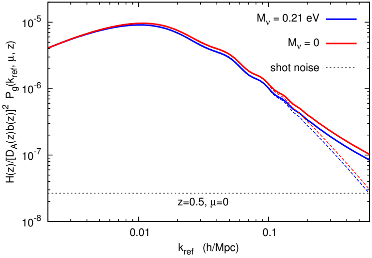

*SDSS 2003 (Tegmark et al) 0310725* *BOSS 1203.6594*

DES (2013-18); LSST, Euclid (∼2020): (sub-)percent at BAO scales

# Large-scale structure

- ⇒ cross-check prediction from CMB for $\Lambda CDM$
- ⇒ improve constraints for extended/non-standard models
- ⇒ extract information on late-time effects (expansion history, neutrino mass)

*Audren, Lesgourgues, Bird et. al. 1210.2194*

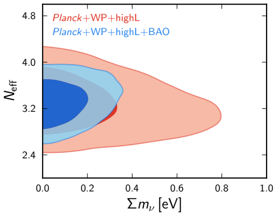



$$\rho_\nu \sim \begin{cases} a^{-4} & T > m_\nu \\ m_\nu a^{-3} & T < m_\nu \end{cases}$$

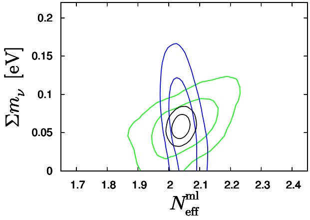
$$M_\nu = \sum m_\nu \geq \begin{cases} 0.056\text{eV (nh)} \\ 0.095\text{eV (ih)} \end{cases}$$

*Basse, Bjaelde, Hamann, Hannestad, Wong 1304.2321*

Planck 1303.5076

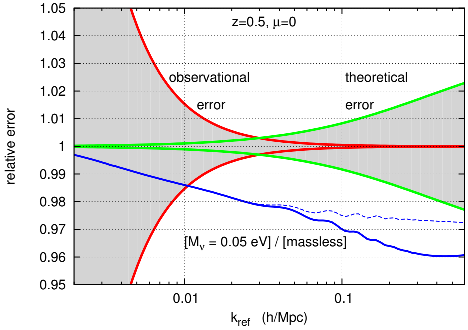



Euclid prospect



# Large-scale structure

Euclid forecast vs theoretical errors *Audren, Lesgourgues, Bird et. al. 1210.2194*

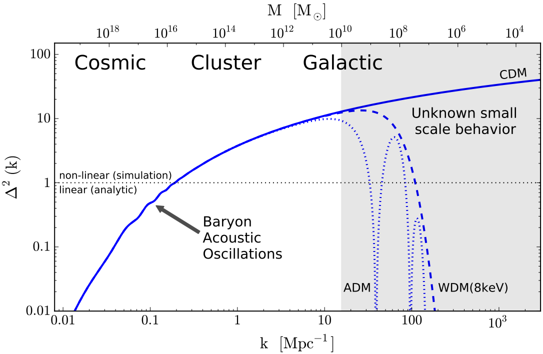



$$\sigma(M_\nu) \simeq \begin{cases} 25\text{meV} & \text{fiducial } (2\%\text{th. err. at } k = 0.4h/Mpc, z = 0.5) \\ 14\text{meV} & \text{th. err. } / = 10, k_{max} = 0.6h/Mpc \end{cases}$$

theoretical uncertainties from bias, non-linear evolution, ...

# Large-scale structure

- Desirable to develop a (fast but reliable) method for theoretical prediction of the power spectrum for a given set of parameters
- Understand onset of non-linearities
- Weakly non-linear regime $\Rightarrow$ perturbation theory



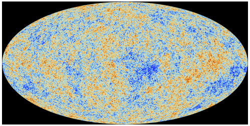

$$\Delta^2(k, z) = 4\pi k^3 P(k, z)$$

*Kuhlen, Vogelsberger, Angulo 1209.5745*

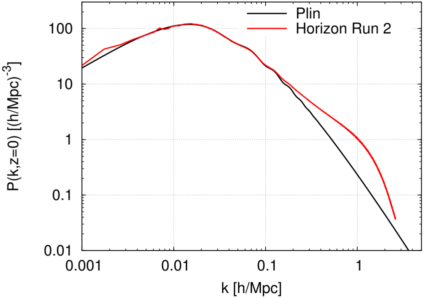
# Idea

- Initial conditions at $z_0 \sim 10^3$ set by linear dynamics of the photon-baryon-dm fluid as imprinted in the CMB (for a given model)

  *e.g. CAMB, CLASS*



$\Rightarrow$




- Goal: compute $P(k, z \sim 0)$ from $P(k, z_0)$ beyond linear approximation on scales $k_{horizon} \ll k \lesssim k_{NL}$ (governed by Newtonian dynamics)

# Basic formalism for large scale structure

- Particle picture (comoving coord., conf. time $d\tau = dt/a$)

$$\mathbf{x}_i(\tau), \mathbf{p}_i(\tau) = am\mathbf{u}_i(\tau), \quad d\mathbf{p}_i/d\tau = -am\nabla\Phi(\mathbf{x}_i)$$

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- Number of particles $dN = f(\tau, \mathbf{x}, \mathbf{p})d^3xd^3p$

$$f(\tau, \mathbf{x}, \mathbf{p}) = f(\tau + d\tau, \mathbf{x} + \frac{\mathbf{p}}{am}d\tau, \mathbf{p} - am\nabla\Phi d\tau)$$

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$$0 = \left(\frac{\partial}{\partial\tau} + \frac{\mathbf{p}}{am}\frac{\partial}{\partial\mathbf{x}} - am\nabla\Phi\frac{\partial}{\partial\mathbf{p}}\right) f(\tau, \mathbf{x}, \mathbf{p}) \quad \text{(Vlasov)}$$

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- Moments

$$m\int d^3p\, f = \rho = \bar{\rho}(1+\delta), \qquad m\int d^3p \frac{\mathbf{p}}{am} f = \rho\mathbf{u}$$

# Basic formalism for large scale structure

- Moments of Vlasov eq.

$$\begin{aligned} \frac{\partial \delta(\mathbf{x},\tau)}{\partial \tau} + \nabla \cdot \{(1+\delta(\mathbf{x},\tau)\mathbf{u}(\mathbf{x},\tau)\} &= 0 \quad \text{(continuity)} \\ \frac{\partial \mathbf{u}(\mathbf{x},\tau)}{\partial \tau} + \mathcal{H}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla\Phi - \frac{1}{\rho}\nabla_j(\sigma_{ij}\rho) \quad \text{(Euler)} \end{aligned}$$

- stress tensor $\sigma_{ij} = m \int d^3p \frac{p_i p_j}{a^2 m^2} f - \rho u_i u_j$
- vorticity $\mathbf{w} = \nabla \times \mathbf{u}$, divergence $\theta = \nabla \cdot \mathbf{u}$

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- solution of linearized eqs for $\sigma_{ij} = 0$ (pressureless perfect fluid PPF)

$$\begin{aligned}\begin{pmatrix}\delta_{lin}(\mathbf{x},\tau)\\ \theta_{lin}(\mathbf{x},\tau)/\mathcal{H}\end{pmatrix} &= [D_+(\tau)A + D_-(\tau)B]\cdot\begin{pmatrix}\delta_0(\mathbf{x})\\ \theta_0(\mathbf{x})/\mathcal{H}_0\end{pmatrix}\\ \mathbf{w}_{lin}(\mathbf{x},\tau) &= \mathbf{w}_0(\mathbf{x})\cdot a_0/a(\tau)\end{aligned}$$

with k-indep. growing/decaying mode ($D_+ \sim a$, $D_- \sim a^{-3/2}$ in EdS)

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with k-indep. growing/decaying mode ($D_+ \sim a, D_- \sim a^{-3/2}$ in EdS)

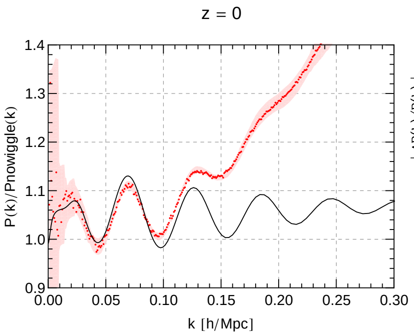
$$\Rightarrow \quad P_{lin}(k,\tau) = D_+(\tau)^2 P(k,\tau_0) + \mathcal{O}(D_-)$$

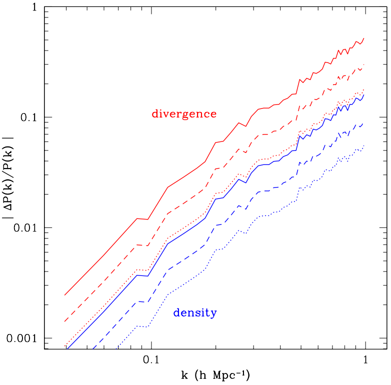
$(D_+'' + \mathcal{H}D_+' = \frac{3}{2}\Omega_m\mathcal{H}^2 D_+)$

# Validity of the linear and PPF approximations

Linear vs N-body

PPF vs non-ideal fluid




black=linear, red=Horizon Run 2

*N-body data from Kim et al 1112.1754*

corr. to $P_{\delta\delta}$ ($P_{\theta\theta}$) from non-zero $\sigma_{ij}$

*Pueblas, Scoccimarro 0809.4606*

# Non-linear corrections

- Continuity, Euler, Poisson for PPF

$$\partial_\eta \Psi_a(k,\eta) + \Omega_{ab}\Psi_b = \int_{k_1,k_2} \delta^{(3)}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\gamma_{abc}(k_1,k_2)\Psi_b(k_1,\eta)\Psi_c(k_2,\eta)$$

where $\eta = \ln a$, $\Psi = \begin{pmatrix} \delta \\ -\theta/\mathcal{H} \end{pmatrix}$, $\gamma_{121} = \frac{\mathbf{k}\cdot\mathbf{k}_1}{\mathbf{k}_1^2}$, $\gamma_{222} = \frac{\mathbf{k}^2\mathbf{k}_1\cdot\mathbf{k}_2}{2\mathbf{k}_1^2\mathbf{k}_2^2}$, $\Omega = \begin{pmatrix} 0 & -1 \\ -3/2 & 1/2 \end{pmatrix}$

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- Linear propagator

$$g(\eta,\eta_0) = \Theta(\eta-\eta_0)\exp\left(-\int_{\eta_0}^{\eta}\Omega\right) = \Theta(\eta-\eta_0)\left(\underbrace{e^{\eta-\eta_0}}_{D_+}A + \underbrace{e^{-3/2(\eta-\eta_0)}}_{D_-}B\right)$$

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$\Rightarrow$ iterative solution in powers of initial condition $\Psi(k,\eta_0)$

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# Loop expansion of the power spectrum

- Classical-statistical average over IC's yields perturbative expansion of $\langle\Psi(k,\eta)\Psi(-k,\eta)\rangle$ in terms of $n$-point fctns $\langle\Psi(k_1,\eta_0)\cdots\Psi(k_n,\eta_0)\rangle$

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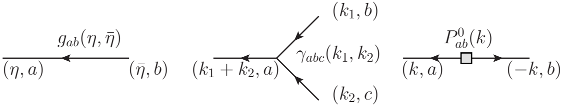

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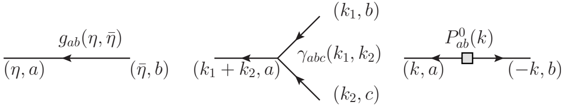
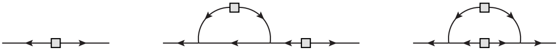



- 'Loop' expansion (only retarded prop, arrows=time-flow)



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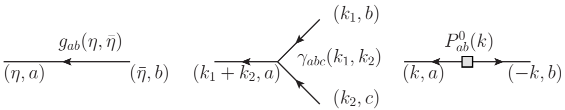


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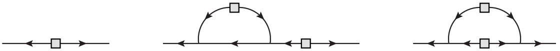
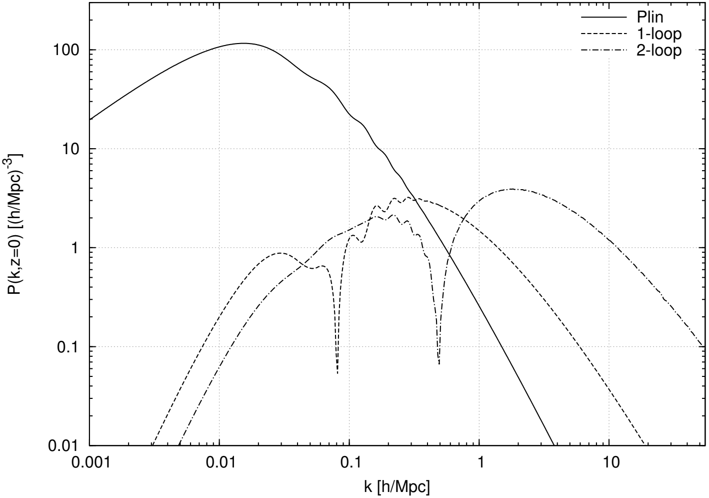


- Fastest growing contributions $P_{L-loop} \propto [D_+(z)^2 P_0(q_i)]^{L+1}$

$$
\begin{aligned}
P(k,z) &= \overbrace{D_+(z)^2 P_0(k)}^{P_{lin}} + \overbrace{D_+(z)^4(2P_{13}+P_{22})}^{P_{1-loop}} \\
&\quad + D_+(z)^6(2P_{15}+2P_{24}+P_{33}) + \dots
\end{aligned}
$$

e.g. $P_{22}(k) = 2\int d^3q\, F_2^s(\mathbf{q},\mathbf{k}-\mathbf{q})^2\, P_0(q)P_0(|\mathbf{k}-\mathbf{q}|)$, $F_2 = \frac{5}{7}\gamma_{121} + \frac{2}{7}\gamma_{222}$

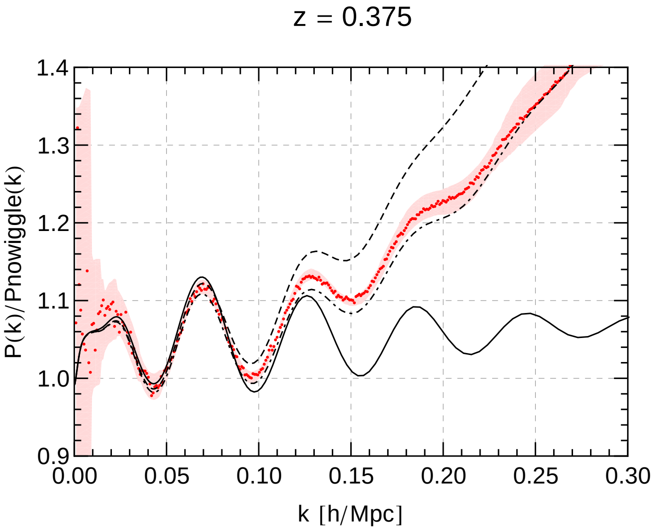
# Two loop



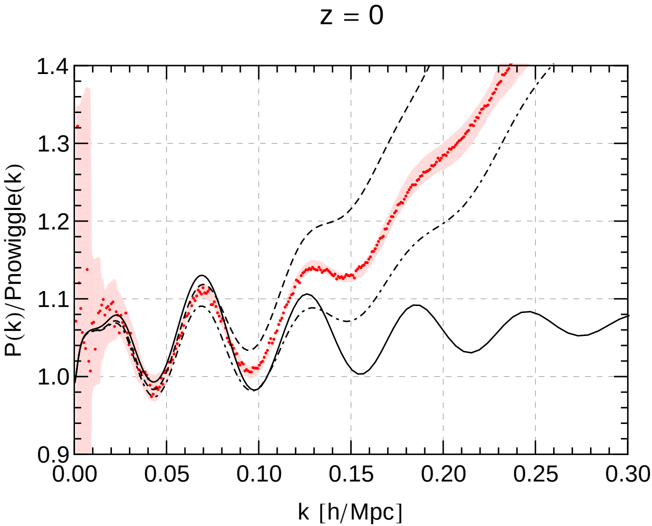

SPT breaks down at small scales / late times $[P_{L-loop} \propto D_+^{2L+2} \sim \left(\frac{1}{1+z}\right)^{2L+2}]$

initial spectrum from CAMB (ΛCDM - WMAP5)

# Two loop




black=linear, 1L(dashed), 2L(dotdashed), red=Horizon Run 2

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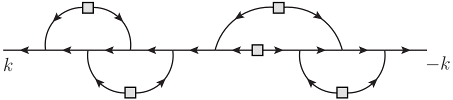
# Expansion parameter ?

- Enhancement from soft loops, vertex $\gamma \propto k \cdot q_i / q_i^2$ for soft $q_i \ll k$

$$k^2\sigma_d^2(z) \equiv \int d^3q \left(\frac{k\cdot q}{q^2}\right)^2 P_{lin}(q,z) \sim \mathcal{O}(10)\left(\frac{1}{1+z}\right)^2\left(\frac{k}{k_{NL}}\right)^2$$

- At L-loop $\sim (k^2\sigma_d^2)^\ell$ with $\ell \leq L$ (resummation $\rightarrow$ RPT *Crocce, Scoccimarro 05*)



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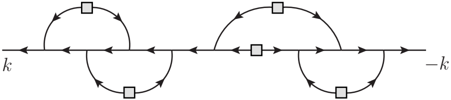
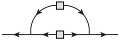
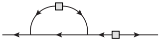



- Power spectrum at 1-loop, for large $k$

 $\to k^2\sigma_d^2(z)P_{lin}(k,z)$,  $\to -\frac{1}{2}k^2\sigma_d^2(z)P_{lin}(k,z)$

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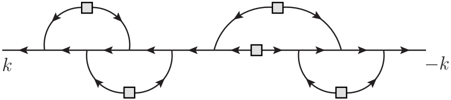


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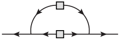
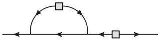
 $\to k^2\sigma_d^2(z)P_{lin}(k,z)$,  $\to -\frac{1}{2}k^2\sigma_d^2(z)P_{lin}(k,z)$

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- Expected from Galilean invariance *Frieman,Scoccimarro 95; Pietroni,Peloso 13*
- Subleading terms cancel as well ($1 \leq \ell \leq L$) *Blas, MG, Konstandin 13*

$\Rightarrow$ $k^2\sigma_d^2$ spurious scale for equal-time correlators *Sugiyama, Spergel 13*

# Expansion parameter

$$\sigma_l^2(k,z) \equiv 4\pi \int_0^k dq\, q^2 P_{lin}(q,z) \to \mathcal{O}(1)\left(\frac{1}{1+z}\right)^2 \ln^3(e + k/k_{NL})$$

Large $k$

$$P_{1-loop}(k) \sim \sigma_l^2(k,z)\left(1.14 - 0.55k\partial_k + 0.1[k\partial_k]^2\right) P_{lin}(k,z)$$

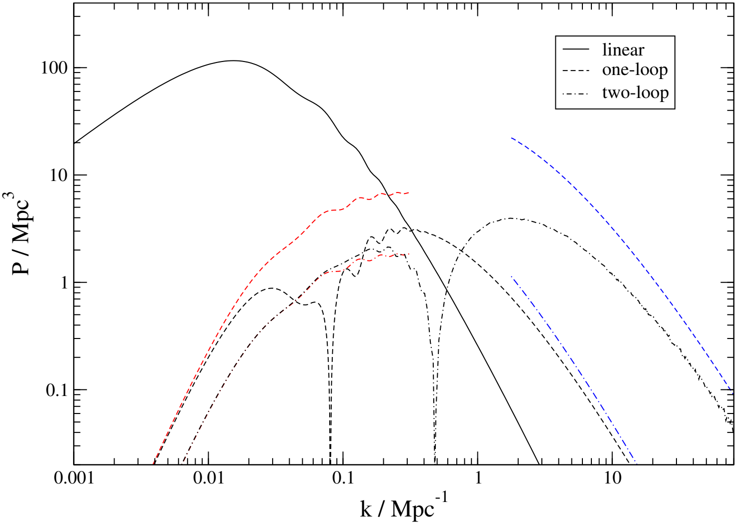
$$P_{2-loop}(k) \sim \sigma_l^4(k,z)\Big(2.14 - 1.62k\partial_k + 0.55[k\partial_k]^2 - 0.082[k\partial_k]^3 + 0.005[k\partial_k]^4\Big) P_{lin}(k,z)$$

Small $k$

$$P_{1-loop}(k) \to -\frac{61}{105} k^2 P_{lin}(k,z) \frac{4\pi}{3} \int_0^\infty dq P_{lin}(q,z) \sigma_l^0(q,z)$$

$$P_{2-loop}(k) \to -\frac{44764}{143325} k^2 P_{lin}(k,z) \frac{4\pi}{3} \int_0^\infty dq P_{lin}(q,z) \underbrace{J(q)}_{\sim\sigma_l^2(q,z)}$$

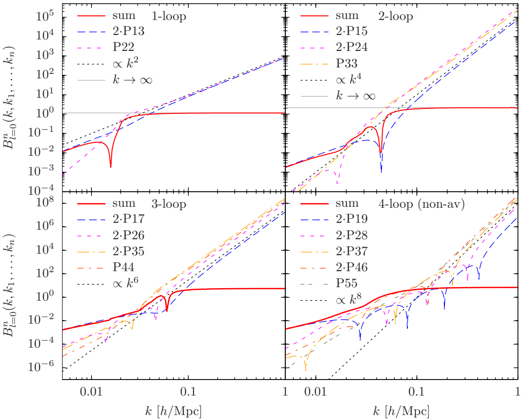
# Two loop



linear
one-loop
two-loop
100
10
1
0.1
P / Mpc$^3$
0.001
0.01
0.1
1
10
k / Mpc$^{-1}$

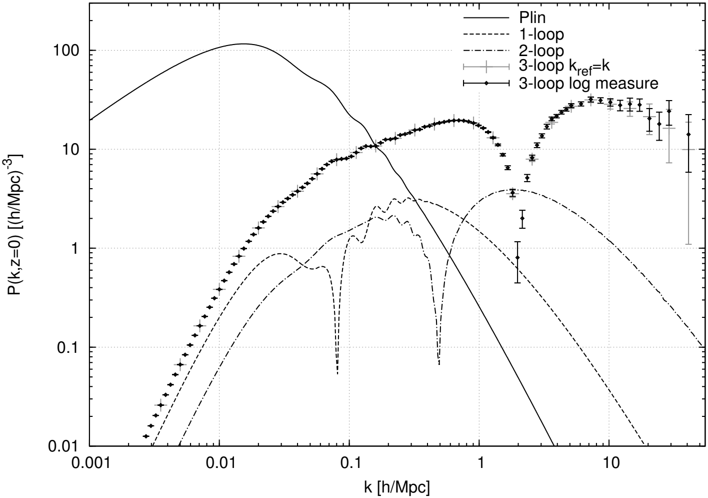
- Cancellation can be made manifest for loop integrand

*Blas, MG, Konstandin 13; Carrasco, Foreman, Green, Senatore 13*



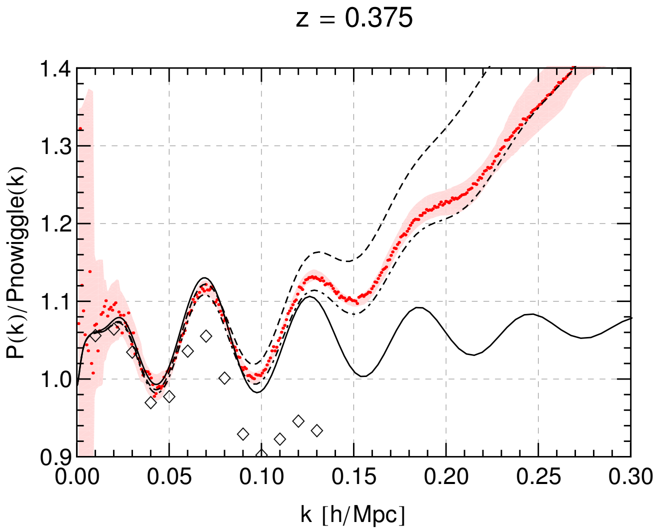

$\Rightarrow$ very helpful for Monte-Carlo integration (2L: 5d, 3L: 8d)

# Three loop




*Diego Blas, MG, Thomas Konstandin 1309.3308 (power spectrum); see also Bernardeau, Taruya, Nishimichi, 1211.1571 (propagator)*

# Three loop




black=linear, 1L(dashed), 2L(dotdashed), 3L(diamonds), red=Horizon Run 2

# Loop expansion of PS in the limit of small $k$

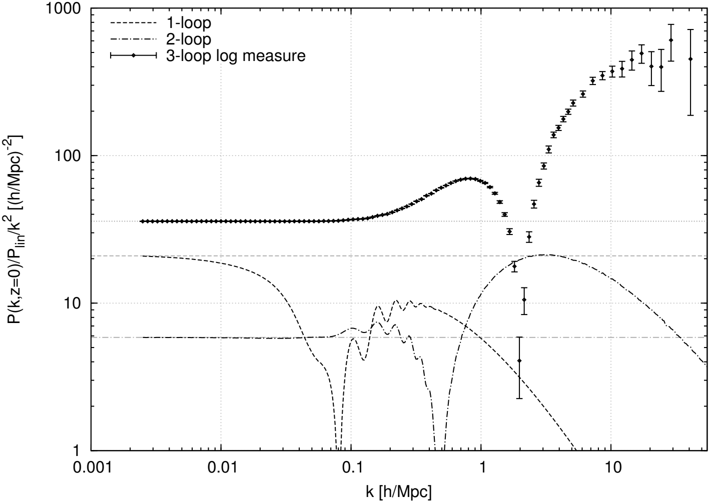
For small $k$ *Fry AJ421 (1994) 21*

$$P_{L-loop}(k,z) \quad \rightarrow \quad \frac{(2L+1)!}{2^{L-1}L!}\,P_{lin}(k,z)\int_{q_1}\cdots\int_{q_L} F^s_{2L+1}(\mathbf{k},\mathbf{q}_1,-\mathbf{q}_1,\ldots,\mathbf{q}_L,-\mathbf{q}_L) \times P_{lin}(q_1,z)\cdots P_{lin}(q_L,z)$$

Using property $F^s_{2L+1} \propto k^2$ for $k \ll q_i$ *Goroff, Grinstein, Rey, Wise AJ311 (1986) 6*

$$P_{L-loop}(k,z) \propto k^2 P_{lin}(k,z) \qquad \text{for small } k$$

# One, two and three loop normalized to $k^2 P_{lin}(k, z)$




$P_{L-loop}(k, z) \propto k^2 P_{lin}(k, z)$

up to $k = 0.003, 0.06, 0.08$ $h/\text{Mpc}$ for $1, 2, 3$-loop at %-level

# Loop expansion of PS in the limit of small $k$

Using property $F^s_{2L+1} \propto k^2/q^2$ for $k \ll q_i$ and $q = \text{max}(q_i)$

$$P_{L-loop} \to P^{small-k}_{L-loop} = -\frac{244\pi}{325} k^2 P_{lin}(k,z) \times C_L \times \int_0^\infty dq \underbrace{P_{lin}(q,z)}_{\to \sim q^{-3}} \sigma_l^{2L-2}(q,z)$$

with coeff. $C_L$ ($C_1 = 1$, $C_2 \simeq 0.71$, $C_3 \simeq 1.05$) and scale-dep. variance

$$\sigma_l^2(q,z) \equiv 4\pi \int_0^q dp\, p^2 P_{lin}(p,z) \to\sim D_+(z)^2 \ln^3(q/q_0)$$

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Estimate for Eisenstein-Hu spectrum with $n_s \simeq 1$

$$P^{small-k}_{L-loop} \propto k^2 P_{lin}(k,z) \times C_L \times \frac{(3L-1)!}{2^{3L}} D_+(z)^{2L}$$

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- Terms decrease up to a certain order $L_{max}(z)$, then increase
- Typical behaviour of an asymptotic series (e.g. loop exp. in QED)

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- Terms decrease up to a certain order $L_{max}(z)$, then increase
- Typical behaviour of an asymptotic series (e.g. loop exp. in QED)
- Partial sum up to smallest term yields best result, with error of order the smallest term (e.g. $P_{2-loop}/P_{lin} \simeq 6\%$ at $z = 0$, $k = 0.1h/$Mpc)

# Padé ansatz

Goal: improve convergence to go to %-accuracy
Idea: resummation in small-$k$ limit

$$P_{small-k}(k,z) = -\frac{244\pi}{315}\, k^2 P_{lin}(k,z) \times \int_0^\infty dq P_{lin}(q,z)\, K(\sigma_l^2(q,z))$$

where the integrand kernel $K$ is given by a series in $x \equiv \sigma_l^2(q,z)$,

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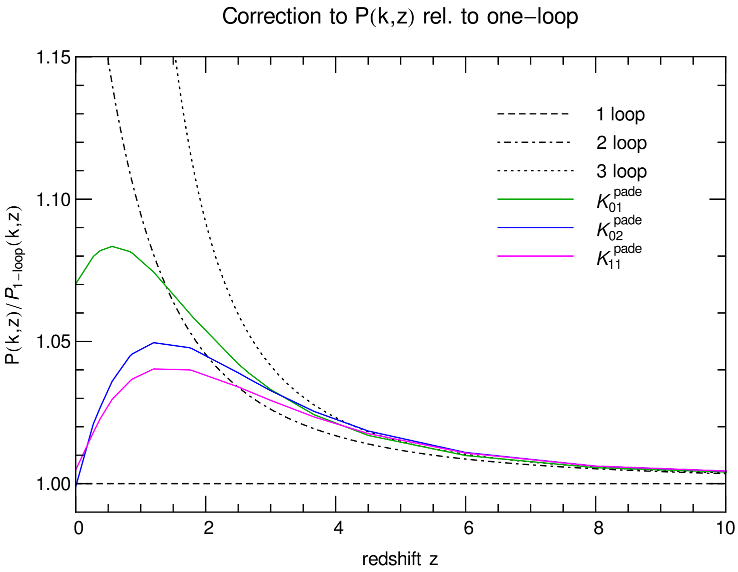
Padé ansatz

$$K_{nm}^{pade}(x) \equiv \frac{1+\sum_{i=1}^{n} a_i x^i}{1+\sum_{j=1}^{m} b_j x^j}$$

Match for small $x$, using coeff. up to three loop $C_1 = 1$, $C_2 \simeq 0.71$, $C_3 \simeq 1.05$

- Two-loop matching (using $C_1, C_2$): $n, m = 0, 1$
- Three-loop matching: either $n, m = 0, 2$ or $n, m = 1, 1$

# Result for Padé resummed small-k limit




black=SPT, solid=Padé resummed result

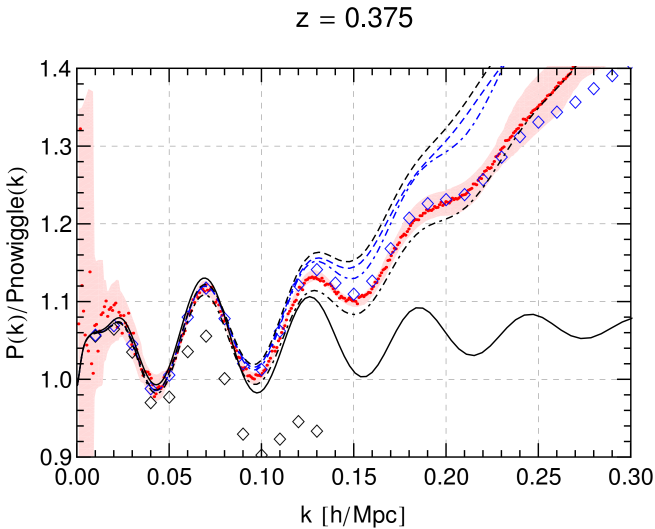
# Padé improved PT

$$\begin{aligned} P(k,z) &= P_{lin}(k,z) + P^{pade}_{small-k}(k,z) \\ &\quad + P^{sub}_{1-loop}(k,z) + P^{sub}_{2-loop}(k,z) + P^{sub}_{3-loop}(k,z) + \dots\,, \end{aligned}$$

where

$$P^{sub}_{L-loop}(k,z) \equiv P_{L-loop}(k,z) - P^{small-k}_{L-loop}(k,z)$$

# Padé improved PT vs N-body




black=SPT, blue=Padé improved PT, red=N-body Horizon Run 2

(solid=linear, dashed=1L, dotdashed=2L, diamonds=3L)

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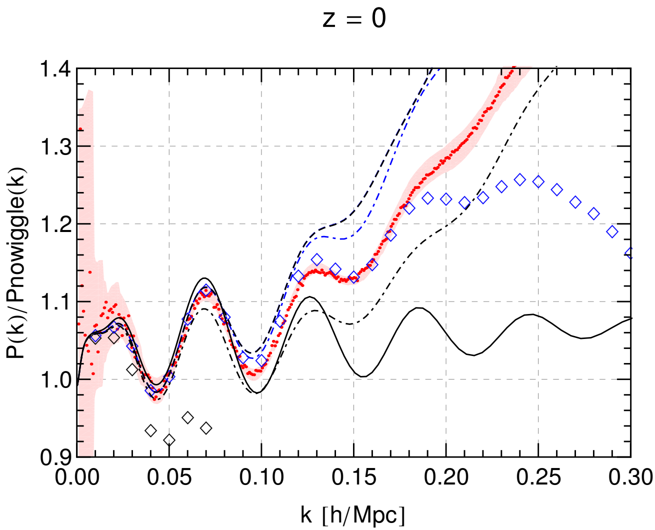


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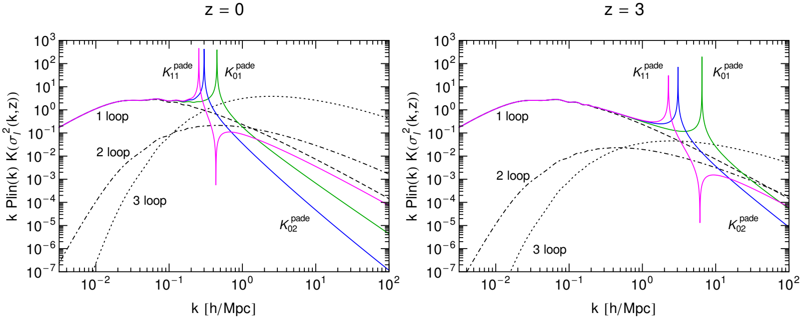
# What is going on?




Padé damps UV-sensitivity

# Conclusion

- Three loop larger than one loop at $z = 0$ at all scales
- Expected for realistic ($\sim \Lambda CDM$) spectrum
- Loop expansion exhibits behaviour of asymptotic series
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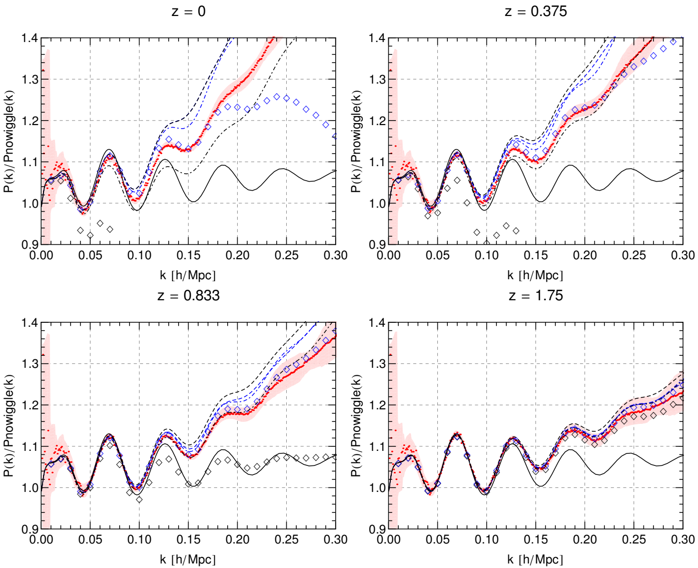
Many open questions

- Justification of Padé ansatz
- Origin of asymptotic behaviour
- Corrections from non-ideal fluid
- ...

# Padé improved PT vs N-body



# Standard Perturbation Theory

- Expand solution in $\delta_0(\mathbf{k}) = \delta(\mathbf{k}, z_0 \sim 10^3)$, for EdS / growing mode

$$\begin{aligned} \delta(\mathbf{k}, z) &= \sum_{n=1}^{\infty} (D_+(z))^n \int_{q_i} \delta^{(3)}(\mathbf{k} - \sum \mathbf{q}_i) F_n(\mathbf{q}_1, \ldots, \mathbf{q}_n) \delta_0(\mathbf{q}_1) \cdots \delta_0(\mathbf{q}_n) \\ &= D_+(z)\delta_0(\mathbf{k}) + \ldots \end{aligned}$$

e.g. $F_2^s(\mathbf{q}_1, \mathbf{q}_2) = \frac{5}{7} + \frac{1}{2} \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7} \frac{(\mathbf{q}_1 \cdot \mathbf{q}_2)^2}{q_1^2 q_2^2}$

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- Power spectrum $\langle \delta(\mathbf{k}, z)\delta(\mathbf{k}', z)\rangle = \delta^{(3)}(\mathbf{k} + \mathbf{k}')P(k, z)$ is obtained using Wick theorem in terms of $P_0(q) = P(q, z_0)$ for Gaussian IC

$$\begin{aligned}
P(k, z) &= \overbrace{D_+(z)^2 P_0(k)}^{P_{lin}} + \overbrace{D_+(z)^4 (2P_{13} + P_{22})}^{P_{1-loop}} \\
&\quad + D_+(z)^6 (2P_{15} + 2P_{24} + P_{33}) + \ldots
\end{aligned}$$

e.g. $P_{22}(k) = 2 \int d^3q \, F_2^s(\mathbf{q}, \mathbf{k} - \mathbf{q})^2 \, P_0(q) P_0(|\mathbf{k} - \mathbf{q}|)$